Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 3, 355–366

Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals

İmdat İşcan

Abstract. In this paper, firstly we have established Hermite–Hadamard-Fejér inequality for fractional integrals. Secondly, an integral identity and some Hermite-Hadamard-Fejér type integral inequalities for the fractional integrals have been obtained. The some results presented here would provide extensions of those given in earlier works.

Mathematics Subject Classification (2010): 26A51, 26A33, 26D10.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejér inequality, Riemann-Liouville fractional integral, convex function.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [4].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [3], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [1, 5, 6, 7, 12, 16].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.2. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \; x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt \text{ and } J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x).$$

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [2, 8, 9, 10, 14, 15, 17, 18].

In [14], Sarıkaya et. al. represented Hermite–Hadamard's inequalities in fractional integral forms as follows.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha > 0$.

In [14] some Hermite-Hadamard type integral inequalities for fractional integral were proved using the following lemma.

Lemma 1.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$ then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \tag{1.4}$$

$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)\,dt.$$

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$ then the following inequality for fractional integrals holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \tag{1.5}$$
$$\leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}}\right) \left[|f'(a)| + |f'(b)| \right].$$

Lemma 1.6 ([11, 18])**.** *For $0 < \alpha \leq 1$ and $0 \leq a < b$, we have*

$$|a^{\alpha} - b^{\alpha}| \leq (b-a)^{\alpha}.$$

In this paper, we firstly represented Hermite-Hadamard-Fejér inequality in fractional integral forms which is the weighted generalization of Hermite-Hadamard inequality (1.3). Secondly, we obtained some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for the fractional integrals.

2. Main results

Throughout this section, let $\|g\|_{\infty} = \sup_{t\in[a,b]} |g(x)|$, for the continuous function $g : [a,b] \to \mathbb{R}$.

Lemma 2.1. *If $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $(a+b)/2$ with $a < b$, then*

$$J_{a+}^{\alpha} g(b) = J_{b-}^{\alpha} g(a) = \frac{1}{2} \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a) \right]$$

with $\alpha > 0$.

Proof. Since g is symmetric to $(a+b)/2$, we have $g(a+b-x) = g(x)$, for all $x \in [a,b]$. Hence, in the following integral setting $x = a+b-t$ and $dx = -dt$ gives

$$\begin{aligned} J_{a+}^{\alpha} g(b) &= \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} g(x) dx \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} g(a+b-t) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} g(t) dt = J_{b-}^{\alpha} g(a). \end{aligned}$$

This completes the proof. □

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be convex function with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a+b)/2$, then the following* *inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \leq \left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \tag{2.1}$$

with $\alpha>0$.

Proof. Since f is a convex function on $[a,b]$, we have for all $t\in[0,1]$

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) \leq \frac{f(ta+(1-t)b)+f(tb+(1-t)a)}{2}. \tag{2.2}$$

Multiplying both sides of (2.2) by $2t^{\alpha-1}g(tb+(1-t)a)$ then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}g(tb+(1-t)a)\,dt \\ \leq\ &\int_0^1 t^{\alpha-1}\left[f(ta+(1-t)b)+f(tb+(1-t)a)\right]g(tb+(1-t)a)\,dt \\ =\ &\int_0^1 t^{\alpha-1}f(ta+(1-t)b)\,g(tb+(1-t)a)\,dt \\ &+\int_0^1 t^{\alpha-1}f(tb+(1-t)a)\,g(tb+(1-t)a)\,dt. \end{aligned}$$

Setting $x=tb+(1-t)a$, and $dx=(b-a)\,dt$ gives

$$\begin{aligned} &\frac{2}{(b-a)^{\alpha}}f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\alpha-1}g(x)\,dx \\ \leq\ &\frac{1}{(b-a)^{\alpha}}\left\{\int_a^b (x-a)^{\alpha-1}f(a+b-x)\,g(x)\,dx+\int_a^b (x-a)^{\alpha-1}f(x)\,g(x)\,dx\right\} \\ =\ &\frac{1}{(b-a)^{\alpha}}\left\{\int_a^b (b-x)^{\alpha-1}f(x)\,g(a+b-x)\,dx+\int_a^b (x-a)^{\alpha-1}f(x)\,g(x)\,dx\right\} \\ =\ &\frac{1}{(b-a)^{\alpha}}\left\{\int_a^b (b-x)^{\alpha-1}f(x)\,g(x)\,dx+\int_a^b (x-a)^{\alpha-1}f(x)\,g(x)\,dx\right\}. \end{aligned}$$

Therefore, by Lemma 2.1 we have

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if f is a convex function, then, for all $t\in[0,1]$, it yields

$$f(ta+(1-t)b)+f(tb+(1-t)a)\leq f(a)+f(b). \tag{2.3}$$

Then multiplying both sides of (2.3) by $2t^{\alpha-1}g(tb+(1-t)a)$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} f(ta+(1-t)b)\, g(tb+(1-t)a)\, dt \\ & \quad + \int_0^1 t^{\alpha-1} f(tb+(1-t)a)\, g(tb+(1-t)a)\, dt \\ \leq & \quad [f(a)+f(b)] \int_0^1 t^{\alpha-1} g(tb+(1-t)a)\, dt \end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]$$

The proof is completed. □

Remark 2.3. In Theorem 2.2,
(i) if we take $\alpha = 1$, then inequality (2.1) becomes inequality (1.2) of Theorem 1.1.
(ii) if we take $g(x) = 1$, then inequality (2.1) becomes inequality (1.3) of Theorem 1.3.

Lemma 2.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $(a+b)/2$ then the following equality for fractional integrals holds*

$$\begin{aligned} & \left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \\ = & \quad \frac{1}{\Gamma(\alpha)}\int_a^b\left[\int_a^t (b-s)^{\alpha-1}g(s)ds-\int_t^b (s-a)^{\alpha-1}g(s)ds\right]f'(t)dt \qquad (2.4) \end{aligned}$$

with $\alpha > 0$.

Proof. It suffices to note that

$$\begin{aligned} I &= \int_a^b\left[\int_a^t (b-s)^{\alpha-1}g(s)ds-\int_t^b (s-a)^{\alpha-1}g(s)ds\right]f'(t)dt \\ &= \int_a^b\left(\int_a^t (b-s)^{\alpha-1}g(s)ds\right)f'(t)dt+\int_a^b\left(-\int_t^b (s-a)^{\alpha-1}g(s)ds\right)f'(t)dt \\ &= I_1+I_2. \end{aligned}$$

By integration by parts and Lemma 2.1 we get

$$\begin{aligned} I_1 &= \left(\int_a^t (b-s)^{\alpha-1} g(s)ds\right) f(t)\Big|_a^b - \int_a^b (b-t)^{\alpha-1} g(t)f(t)dt \\ &= \left(\int_a^b (b-s)^{\alpha-1} g(s)ds\right) f(b) - \int_a^b (b-t)^{\alpha-1} (fg)(t)dt \\ &= \Gamma(\alpha)\left[f(b)J_{a+}^{\alpha}g(b) - J_{a+}^{\alpha}(fg)(b)\right] \\ &= \Gamma(\alpha)\left[\frac{f(b)}{2}\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - J_{a+}^{\alpha}(fg)(b)\right] \end{aligned}$$

and similarly

$$\begin{aligned} I_2 &= \left(-\int_t^b (s-a)^{\alpha-1} g(s)ds\right) f(t)\Big|_a^b - \int_a^b (t-a)^{\alpha-1} g(t)f(t)dt \\ &= \left(\int_a^b (s-a)^{\alpha-1} g(s)ds\right) f(a) - \int_a^b (t-a)^{\alpha-1} (fg)(t)dt \\ &= \Gamma(\alpha)\left[\frac{f(a)}{2}\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - J_{b-}^{\alpha}\left(fg\right)(a)\right]. \end{aligned}$$

Thus, we can write

$$\begin{aligned} I &= I_1 + I_2 \\ &= \Gamma(\alpha)\left\{\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right\}. \end{aligned}$$

Multiplying the both sides by $(\Gamma(\alpha))^{-1}$ we obtain (2.4) which completes the proof. □

Remark 2.5. In Lemma 2.4, if we take $g(x) = 1$, then equality (2.4) becomes equality (1.4) of Lemma 1.4.

Theorem 2.6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\ &\le \frac{(b-a)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \end{aligned} \tag{2.5}$$

with $\alpha > 0$.

Proof. From Lemma 2.4 we have

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}\left(fg\right)(b) + J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_a^b \left|\int_a^t (b-s)^{\alpha-1} g(s)ds - \int_t^b (s-a)^{\alpha-1} g(s)ds\right| \left|f'(t)\right| dt. \end{aligned} \tag{2.6}$$

Since $|f'|$ is convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f'(t)|=\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|\le\frac{b-t}{b-a}\left|f'\left(a\right)\right|+\frac{t-a}{b-a}\left|f'\left(b\right)\right|, \tag{2.7}$$

and since $g:[a,b]\to\mathbb{R}$ is symmetric to $(a+b)/2$ we write

$$\int_t^b\left(s-a\right)^{\alpha-1}g(s)ds=\int_a^{a+b-t}\left(b-s\right)^{\alpha-1}g(a+b-s)ds=\int_a^{a+b-t}\left(b-s\right)^{\alpha-1}g(s)ds,$$

then we have

$$\begin{aligned}&\left|\int_a^t\left(b-s\right)^{\alpha-1}g(s)ds-\int_t^b\left(s-a\right)^{\alpha-1}g(s)ds\right|\\=\;&\left|\int_t^{a+b-t}\left(b-s\right)^{\alpha-1}g(s)ds\right|\\\le\;&\begin{cases}\int_t^{a+b-t}\left|\left(b-s\right)^{\alpha-1}g(s)\right|ds, & t\in\left[a,\frac{a+b}{2}\right]\\\int_{a+b-t}^{t}\left|\left(b-s\right)^{\alpha-1}g(s)\right|ds, & t\in\left[\frac{a+b}{2},b\right]\end{cases}.\end{aligned} \tag{2.8}$$

A combination of (2.6), (2.7) and (2.8), we get

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right|\\\le\;&\frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|\left(b-s\right)^{\alpha-1}g(s)\right|ds\right)\left(\frac{b-t}{b-a}\left|f'\left(a\right)\right|+\frac{t-a}{b-a}\left|f'\left(b\right)\right|\right)dt\\&+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|\left(b-s\right)^{\alpha-1}g(s)\right|ds\right)\left(\frac{b-t}{b-a}\left|f'\left(a\right)\right|+\frac{t-a}{b-a}\left|f'\left(b\right)\right|\right)dt\\\le\;&\frac{\|g\|_{\infty}}{(b-a)\,\Gamma(\alpha+1)}\left\{\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]\left((b-t)\left|f'\left(a\right)\right|+(t-a)\left|f'\left(b\right)\right|\right)dt\right.\\&\left.+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]\left((b-t)\left|f'\left(a\right)\right|+(t-a)\left|f'\left(b\right)\right|\right)dt\right\}\end{aligned} \tag{2.9}$$

Since

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right](b-t)\,dt\\=\;&\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right](t-a)\,dt\\=\;&\frac{(b-a)^{\alpha+2}}{(\alpha+1)}\left(\frac{\alpha+1}{\alpha+2}-\frac{1}{2^{\alpha+1}}\right)\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} [(b-t)^\alpha - (t-a)^\alpha](t-a)\,dt \\
= &\int_{\frac{a+b}{2}}^b [(t-a)^\alpha - (b-t)^\alpha](b-t)\,dt \\
= &\frac{(b-a)^{\alpha+2}}{(\alpha+1)}\left(\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}}\right) \qquad (2.11)
\end{aligned}$$

Hence, if we use (2.10) and (2.11) in (2.9), we obtain the desired result. This completes the proof. □

Remark 2.7. In Theorem 2.6, if we take $g(x) = 1$, then equality (2.5) becomes equality (1.5) of Theorem 1.5.

Theorem 2.8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|^q, q > 1$, is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right]\right| \qquad (2.12)$$

$$\leq \frac{2(b-a)^{\alpha+1}\|g\|_\infty}{(b-a)^{1/q}(\alpha+1)\Gamma(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}$$

where $\alpha > 0$ and $1/p + 1/q = 1$.

Proof. Using Lemma 2.4, Hölder's inequality, (2.8) and the convexity of $|f'|^q$, it follows that

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right] - \left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right]\right|$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^b \left|\int_t^{a+b-t} (b-s)^{\alpha-1} g(s)ds\right| dt\right)^{1-1/q} \\
&\times \left(\int_a^b \left|\int_t^{a+b-t} (b-s)^{\alpha-1} g(s)ds\right| |f'(t)|^q\,dt\right)^{1/q} \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \left|(b-s)^{\alpha-1} g(s)\right| ds\right) dt\right. \\
&\left.+ \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \left|(b-s)^{\alpha-1} g(s)\right| ds\right) dt\right]^{1-1/q} \\
&\times \left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \left|(b-s)^{\alpha-1} g(s)\right| ds\right) |f'(t)|^q\,dt\right.
\end{aligned}$$

$$+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)|f'(t)|^{q}\,dt\Bigg]^{1/q}$$

$$\leq\frac{2^{1-1/q}\|g\|_{\infty}}{(b-a)^{1/q}\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{1-1/q}$$

$$\times\left\{\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]\left((b-t)|f'(a)|^{q}+(t-a)|f'(b)|^{q}\right)dt\right.$$

$$\left.+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]\left((b-t)|f'(a)|^{q}+(t-a)|f'(b)|^{q}\right)dt\right\}^{1/q} \tag{2.13}$$

where it is easily seen that

$$\int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{a+b-t}(b-s)^{\alpha-1}\,ds\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}(b-s)^{\alpha-1}\,ds\right)dt$$

$$=\frac{2(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left[1-\frac{1}{2^{\alpha}}\right].$$

Hence, if we use (2.10) and (2.11) in (2.13), we obtain the desired result. This completes the proof. □

We can state another inequality for $q>1$ as follows:

Theorem 2.9. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^{q}, q>1$, is convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals hold:*

(i) $$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq\frac{2^{1/p}\|g\|_{\infty}(b-a)^{\alpha+1}}{(\alpha p+1)^{1/p}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)^{1/p}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{1/q} \tag{2.14}$$

with $\alpha>0$.

(ii) $$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq\frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{(\alpha p+1)^{1/p}\Gamma(\alpha+1)}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{1/q} \tag{2.15}$$

for $0<\alpha\leq 1$, where $1/p+1/q=1$.

Proof. (i) Using Lemma 2.4, Hölder's inequality, (2.8) and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)ds\right|^p dt\right)^{1/p}\left(\int_a^b|f'(t)|^q dt\right)^{1/q}. \\
\leq\ & \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^\alpha-(t-a)^\alpha\right]^p dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^\alpha-(b-t)^\alpha\right]^p dt\right)^{1/p} \\
&\times\left(\int_a^b\left(\frac{b-t}{b-a}\left|f'(a)\right|^q+\frac{t-a}{b-a}\left|f'(b)\right|^q\right)dt\right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
=\ & \frac{\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^\alpha-t^\alpha\right]^p dt+\int_{\frac{1}{2}}^1\left[t^\alpha-(1-t)^\alpha\right]^p dt\right)^{1/p} \\
&\times\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{1/q} \qquad (2.16)
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right]dt\right)^{1/p} \\
&\times\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{1/q}
\end{aligned}$$

$$\leq\frac{\|g\|_\infty(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\frac{2}{\alpha p+1}\left[1-\frac{1}{2^{\alpha p}}\right]\right)^{1/p}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{1/q}.$$

Here we use

$$\left[(1-t)^\alpha-t^\alpha\right]^p\leq(1-t)^{\alpha p}-t^{\alpha p}$$

for $t\in[0,1/2]$ and

$$\left[t^\alpha-(1-t)^\alpha\right]^p\leq t^{\alpha p}-(1-t)^{\alpha p}$$

for $t\in[1/2,1]$, which follows from

$$(A-B)^q\leq A^q-B^q,$$

for any $A\geq B\geq 0$ and $q\geq 1$. Hence the inequality (2.14) is proved.

(ii) The inequality (2.15) is easily proved using (2.16) and Lemma 1.6. □

Remark 2.10. In Theorem 2.9, if we take $\alpha=1$, then equality (2.15) becomes equality in [18, Corollary 13].

References

[1] Bombardelli, M., Varošanec, S., *Properties of h-convex functions related to the Hermite Hadamard Fejér inequalities*, Computers and Mathematics with Applications, **58**(2009), 1869-1877.

[2] Dahmani, Z., *On Minkowski and Hermite-Hadamard integral inequalities via fractional via fractional integration*, Ann. Funct. Anal. **1**(1)(2010), 51-58.

[3] Fejér, L., *Uberdie Fourierreihen, II, Math.*, Naturwise. Anz Ungar. Akad. Wiss, **24**(1906), 369-390, (in Hungarian).

[4] Hadamard, J., *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58**(1893), 171-215.

[5] İşcan, İ., *New estimates on generalization of some integral inequalities for (α, m)-convex functions*, Contemp. Anal. Appl. Math., **1**(2)(2013), 253-264.

[6] İşcan, İ., *New estimates on generalization of some integral inequalities for s-convex functions and their applications*, Int. J. Pure Appl. Math., **86**(4)(2013), 727-746.

[7] İşcan, İ., *Some new general integral inequalities for h-convex and h-concave functions*, Adv. Pure Appl. Math., **5**(1)(2014), 21-29, doi: 10.1515/apam-2013-0029.

[8] İşcan, İ., *Generalization of different type integral inequalitiesfor s-convex functions via fractional integrals*, Applicable Analysis, **93**(9)(2014), 1846-1862.

[9] İşcan, İ., *New general integral inequalities for quasi-geometrically convex functions via fractional integrals*, J. Inequal. Appl., **2013**(491)(2013), 15 pages.

[10] İşcan, İ., *On generalization of different type integral inequalities for s-convex functions via fractional integrals*, Mathematical Sciences and Applications E-Notes, **2**(1)(2014), 55-67.

[11] Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I., *Integral and series. In: Elementary Functions*, vol. 1, Nauka, Moscow, 1981.

[12] Sarıkaya, M.Z., *On new Hermite Hadamard Fejér type integral inequalities*, Stud. Univ. Babeş-Bolyai Math. **57**(2012), no. 3, 377-386.

[13] Sarıkaya, M.Z., Ogunmez, H., *On new inequalities via Riemann-Liouville fractional integration*, Abstract an Applied Analysis, **2012**(2012), 10 pages, Article ID 428983. doi:10.1155/2012/428983

[14] Sarıkaya, M.Z., Set, E., Yaldız, H., Başak, N., *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, **57**(9)(2013), 2403-2407.

[15] Set, E., *New inequalities of Ostrowski type for mapping whose derivatives are s-convex in the second sense via fractional integrals*, Computers and Math. with Appl., **63**(2012), 1147-1154.

[16] Tseng, K.-L., Yang, G.-S., Hsu, K.-C., *Some inequalities for differentiable mappings and applications to Fejér inequality and weighted trapezoidal formula*, Taiwanese journal of Mathematics, **15**(4)(2011), 1737-1747.

[17] Wang, J., Li, X., Fečkan, M., Zhou, Y., *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Appl. Anal., **92**(11)(2012), 2241-2253. doi:10.1080/00036811.2012.727986

[18] Wang, J., Zhu, C., Zhou, Y., *New generalized Hermite-Hadamard type inequalities and applications to special means*, J. Inequal. Appl., **2013**(325)(2013), 15 pages.

İmdat İşcan
Giresun University,
Faculty of Sciences and Arts,
Department of Mathematics,
Giresun, Turkey
e-mail: imdat.iscan@giresun.edu.tr