

# Fully developed mixed convection through a vertical porous channel with an anisotropic permeability: case of heat flux

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**Abstract.** The effect of anisotropy on the steady fully developed mixed convection flow in a vertical porous channel is analytically studied. The side walls of the channel are prescribed by a constant heat flux and the flow at the entrance is upward, so that natural convection aids the forced flow. It is shown that the anisotropy parameter has a significant effect of the flow and heat transfer characteristics. We extend the study in [9] with the case of opposing flow by using Computer Algebra software.

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## 1. Introduction

This work is an extension of the paper [9], which considers all the cases of assisting and opposing flow.

Convective heat transfer in a saturated porous medium has attracted considerable interest in recent years, due to its frequent occurrence in industrial and technological applications. Examples of these applications include geothermal reservoirs, thermal insulation, enhanced oil recovery, drying of porous solids, packed-bed catalytic reactors, volcanic eruption, electronic circuits, and many others, see, for example the books [13], [10], [15], [18, 19], [12], [11], and [17].

The aim of this paper is to study the effects of anisotropy on the fully developed mixed convection flow through a vertical channel filled with a porous medium. Such studies were performed by [16], [14], [8], [7], [4], [5], [6], [20], etc. It was found by these authors that the effect of the anisotropy ratio parameter on the flow characteristics was significant.

We have followed here also the papers by [2], and [3].

## 2. The basic equations

We consider the problem of steady fully developed flow in a vertical porous channel bounded by two parallel walls at a distance  $L$ , which are maintained at uniform and equal wall heat fluxes  $q_w$  (Figure 1). The channel has a rectangular cross-section infinitely long in the  $z$ -direction. The porous medium is assumed to be anisotropic in permeability with its principal axes of the porous matrix denoted by  $K_1$  and  $K_2$ .

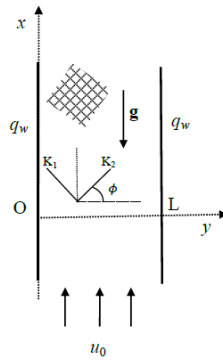


FIGURE 1. The geometry of the problem

The anisotropy of the porous medium is then characterized by the anisotropy ratio  $K^* = \frac{K_1}{K_2}$  and the orientation angle  $\phi$ , defined as the angle between the horizontal and  $K_2$ . It is also assumed that the flow is uniform with the characteristic velocity at the entrance of the channel denoted by  $u_0$ .

Under these assumptions, along with the Boussinesq approximation, the basic equations governing the steady conservation of mass, momentum (Brinkman-Darcy's law) and energy can be written as follows ([6])

$$\nabla \cdot \mathbf{v} = 0 \quad (2.1)$$

$$\mathbf{v} = \frac{\overline{\overline{K}}}{\mu} (-\nabla p - \mu \nabla^2 \mathbf{v} + \rho [1 - \beta (T - T_0)] \mathbf{g}) \quad (2.2)$$

$$\nabla (vT) = \alpha_m \nabla^2 T \quad (2.3)$$

Here  $\overline{\overline{K}}$  is the symmetrical second-order permeability tensor, which is defined as

$$\overline{\overline{K}} = \begin{bmatrix} K_1 \cos^2 \phi + K_2 \sin^2 \phi & (K_1 - K_2) \sin \phi \cos \phi \\ (K_1 - K_2) \sin \phi \cos \phi & K_1 \cos^2 \phi + K_2 \sin^2 \phi \end{bmatrix} \quad (2.4)$$

Equation (2.1) can be also written in the form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.5)$$

Assuming that the flow is fully developed it results in that  $v = 0$  and  $u = u(y)$ . Thus, the governing equations (2.2) and (2.3) can be written in reduced form as follow

$$\mu \frac{d^2 u}{dy^2} - \frac{a\mu}{K_1} u + \rho g \beta (T - T_0) = \frac{\partial p}{\partial x} \quad (2.6)$$

$$\frac{\partial p}{\partial x} = 0 \quad (2.7)$$

$$u \frac{\partial T}{\partial x} = \alpha_m \frac{\partial^2 T}{\partial y^2}, \quad (2.8)$$

where the constant  $a$  is given by

$$a = \cos^2 \phi + K^* \sin^2 \phi. \quad (2.9)$$

The boundary conditions for Eqs. (2.6) and (2.8) are

$$u(0) = u(L) = 0, \quad \left. \frac{\partial T}{\partial y} \right|_{y=0} = \frac{q}{k}, \quad \left. \frac{\partial T}{\partial y} \right|_{y=L} = -\frac{q}{k} \quad (2.10)$$

along with the mass flux ([2]) and thermal ([3]) conditions

$$\int_0^L u dy = Q_0, \quad T_0 = \frac{1}{L} \int_0^L T dy, \quad (2.11)$$

where  $Q_0$  is the mass flux across the channel and  $T_0$  is the mean temperature in the horizontal direction of the channel, and is chosen as the reference temperature.

### 3. The dimensionless equations

Introducing the dimensionless variables defined as

$$\begin{aligned} X &= \frac{\alpha_m x}{u_0 L^2}, \quad Y = \frac{y}{L}, \quad U(Y) = \frac{u}{u_0} \\ \theta(X, Y) &= \frac{T - T_0}{\frac{q_w L}{K}}, \quad P(X) = \frac{\alpha_m p}{\mu u_0^2 L}, \end{aligned} \quad (3.1)$$

equations (2.6) and (2.7) become

$$\frac{d^2 U}{dY^2} - \zeta^2 U + \lambda \theta + \gamma = 0 \quad (3.2)$$

$$U \frac{\partial \theta}{\partial X} = \frac{\partial^2 \theta}{\partial Y^2} \quad (3.3)$$

Here  $\zeta$  is the *anisotropic parameter*,  $\gamma$  is the *constant pressure gradient parameter* and  $\lambda$  is the *mixed convection parameter*, which are defined as

$$\zeta^2 = \sqrt{\frac{a}{\text{Da}}}, \quad \gamma = -\frac{\partial P}{\partial X} = -\frac{\partial P}{\partial x}, \quad \lambda = \frac{\text{Gr}}{\text{Re}} \quad (3.4)$$

where  $\text{Da} = K_1/L$  is the Darcy number,  $\text{Gr} = g\beta(q_w \frac{L}{k}) \frac{L^3}{\nu^2}$  is the Grashof number based on the heat flux  $q_w$  and  $\text{Re} = u_0 L/w$  is the Reynolds number.

The boundary conditions (2.10) become

$$U(0) = U(1) = 0, \quad \frac{\partial \theta}{\partial Y} \Big|_{Y=0} = -1, \quad \frac{\partial \theta}{\partial Y} \Big|_{Y=1} = 1 \quad (3.5)$$

and the mass flux and thermal conditions (2.11) reduces to

$$\int_0^1 U dY = 1, \quad \int_0^1 \theta dY = 0 \quad (3.6)$$

where we took  $Q_0 = u_0 L$ .

By integrating Eq. (3.3) over the channel cross-section, and making use the boundary conditions (3.5) for  $\theta$ , it can be shown that

$$\frac{\partial \theta}{\partial X} = 2 \quad (3.7)$$

Thus, Eq. (3.3) becomes

$$\frac{\partial^2 \theta}{\partial Y^2} = 2U \quad (3.8)$$

Note that Al-Hadrhrami et al. [1] have considered

$$\theta(y) = \bar{\theta}(y) + 2x$$

Thus, it results that

$$\frac{\partial^2 \bar{\theta}}{\partial Y^2} = 2U \quad (3.9)$$

If we integrate this equation twice, we get

$$\bar{\theta}(Y) = 2 \int \left( \int U(Y) dy + C_5 \right) dY + C_6$$

where  $C_5$  and  $C_6$  are constants of integration. Thus, we can obtain  $\theta$  from  $U$ .

**The physical quantities of interest** are the *wall skin friction* or *wall shear stress coefficient*  $C_f$  and the *Nusselt number*  $Nu$ , which are defined as

$$C_f = \frac{\tau_w}{\rho u_0^2}, \quad Nu = \frac{qL}{k(T_w - T_0)} \quad (3.10)$$

where

$$\tau_w = \mu \frac{\partial u}{\partial y} \Big|_{y=0,L}$$

Using (2.11) and (3.8) we get

$$\text{Re } C_f = \frac{\partial U}{\partial Y} \Big|_{Y=0,1}, \quad Nu = \frac{1}{\theta} \Big|_{Y=0,1}.$$

#### 4. Analysis and modeling

In order to obtain  $U$ , we consider Eq. (3.2) and differentiate it twice, by taking into account Eq. (3.8). We obtain the following equations

$$\frac{d^4U}{dY^4} - \zeta^2 \frac{d^2U}{dY^2} + 2\lambda U = 0 \quad (4.1)$$

$$\frac{d^2\theta}{dY^2} = 2U(Y) \quad (4.2)$$

with boundary conditions

$$\begin{aligned} U(0) &= 0, \quad U(1) = 0 \\ \theta'(0) &= -1, \quad \theta'(1) = 1 \\ \int_0^1 \theta(Y) dY &= 0. \end{aligned} \quad (4.3)$$

The form of the solutions of (4.1)+(4.2)+(4.3) depends on the roots of the characteristic equations

$$r^4 - \zeta^2 r^2 + 2\lambda = 0.$$

Formally, these roots are

$$\pm \frac{1}{2} \sqrt{2\zeta^2 \pm 2\sqrt{\zeta^4 - 8\lambda}}.$$

Let

$$\Delta = \zeta^4 - 8\lambda.$$

According to the sign of  $\Delta$  and  $\lambda$ , there will be several cases to be considered.

1.  $\lambda > 0$  (assisting flow)
  - (a)  $\Delta < 0$ , four complex roots
  - (b)  $\Delta = 0$ , two double real roots,  $\pm \frac{\sqrt{2}}{2}\zeta$
  - (c)  $\Delta > 0$ , four real roots
2.  $\lambda = 0$ , the roots are  $0, 0, \zeta, -\zeta$
3.  $\lambda < 0$ , (opposing flow) two real and two complex roots

For the solution of (4.1)+(4.2)+(4.3) and solutions plot we used the computer algebra system Maple.

#### 5. Results and discussion

We present several graphs which illustrate the influence of  $\lambda$  and  $\zeta$  on the velocity and temperature profile.

It can be seen from Figs. 2(a), 3(a) and 4(a) that the effect of an increasing anisotropic parameter  $\zeta$  leads to a decrease of the dimensionless fluid velocity next to left wall of the channel and to an increase of the dimensionless velocity profiles near the right wall of the channel, for all three cases considered  $\Delta < 0$ ,  $\Delta = 0$  and  $\Delta < 0$ . This is true for opposing flow for small values of  $\lambda$  (see Figures 6 and 7).

However, Figs. 2(b) and 4(b) shows that in these cases of  $\Delta$  the dimensionless temperature increases with the increase of the anisotropic parameter  $\zeta$ .

It should also be mentioned that the dimensionless velocity and temperature profiles illustrated in Figs. 2(a) to 5(b) resemble the same shapes as in the paper [3].

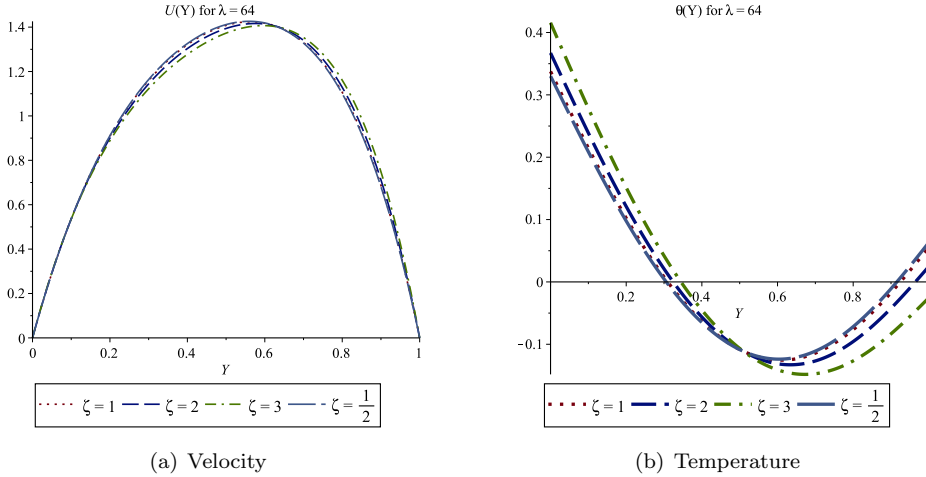


FIGURE 2. Velocity (left) and temperature profile for  $\lambda = 64$  and various values of  $\zeta$  when  $\Delta < 0$

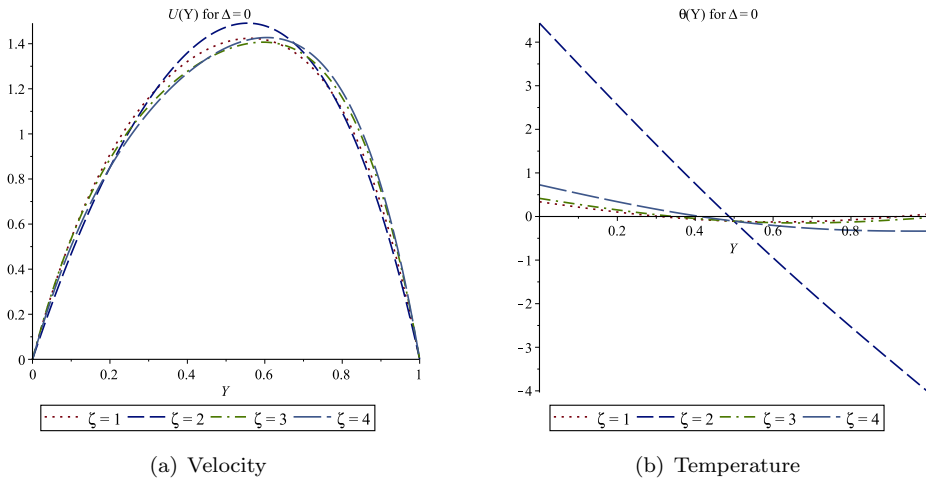


FIGURE 3. Velocity (left) and temperature profile for  $\Delta = 0$  and various values of  $\zeta$

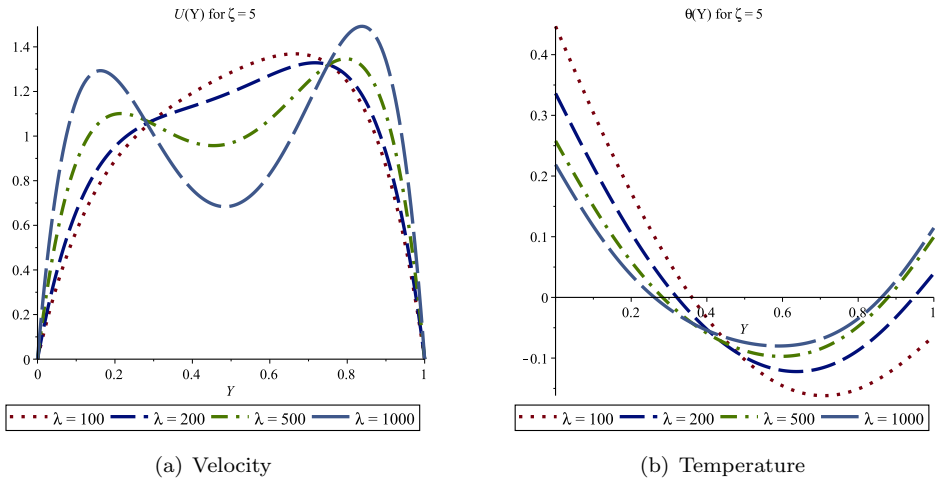


FIGURE 4. Velocity (left) and temperature profile for  $\zeta = 5$  and various values of  $\lambda$  when  $\Delta < 0$

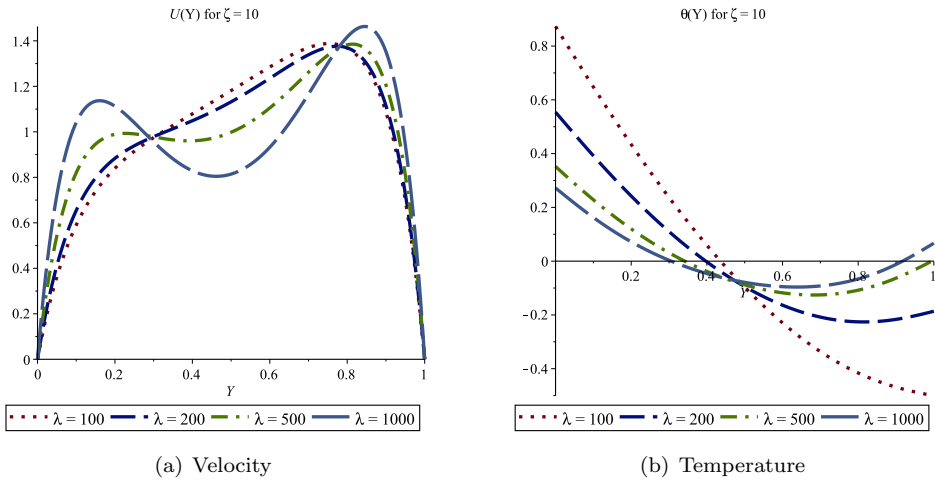


FIGURE 5. Velocity (left) and temperature profile for  $\zeta = 10$  and various values of  $\lambda$  when  $\Delta > 0$

## 6. Conclusions

The paper presents an analytical study of the fully developed assisting mixed convection flow through a porous channel with an anisotropic permeability when the walls of the channels are kept at constant heat fluxes. The Brinkman-Darcy model has been used.

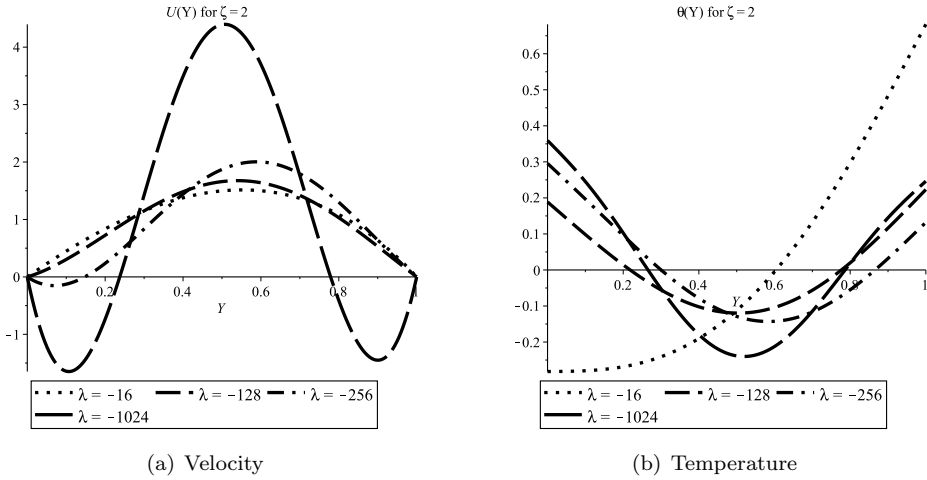


FIGURE 6. Velocity (left) and temperature profile for  $\zeta = 2$  and various values of  $\lambda < 0$  (opposing flow)

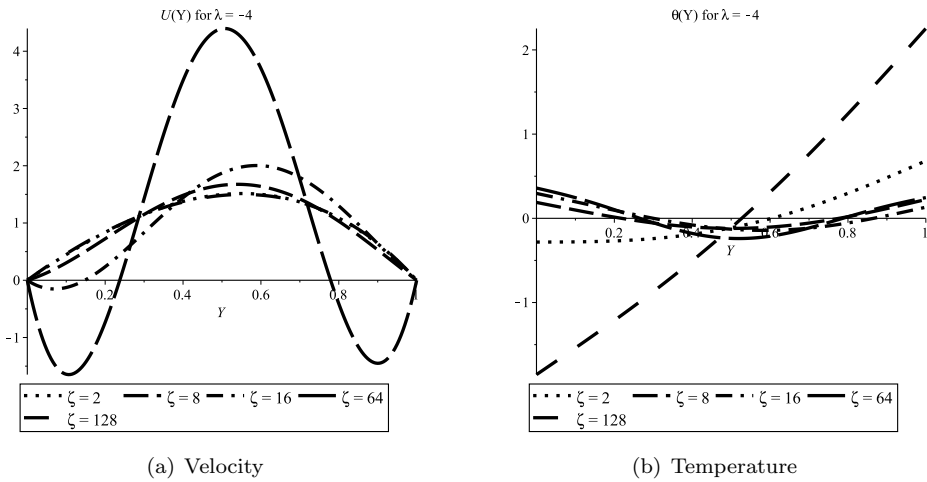


FIGURE 7. Velocity (left) and temperature profile for  $\lambda = -2$  and various values of  $\zeta$  (opposing flow)

The following conclusions can be drawn:

1. The effect of anisotropy on the dimensionless velocity profiles is substantial, especially for large values of the mixed convection parameter  $\lambda$ ;
2. The effect of anisotropy is less important for the dimensionless temperature profiles when the mixed convection parameter  $\lambda$  increases.



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