Stud. Univ. Babeş-Bolyai Math. 60(2015), No. 2, 267–275

On convergence of nonlinear singular integral operators with non-isotropic kernels

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Abstract. Here we give some approximation theorems concerning pointwise convergence and rate of pointwise convergence of nonlinear singular integral operators with non-isotropic kernels of the form

$$T_{w,\lambda}(f)(s) = \int\limits_{\mathbb{R}^n} K_w\left(|s-t|_\lambda, f(t)\right) dt,$$

where the kernel function satisfies Lipschitz condition and some singularity assumptions. Here Λ is a non-empty set of indices, 0 is an accumulation point of Λ and $|s-t|_\lambda$ denotes the non-isotropic distance between the points $s, t \in \mathbb{R}^n$.

Mathematics Subject Classification (2010): 41A35, 41A25, 47G10.

Keywords: Nonlinear singular integral, non-isotropic distance, Lipschitz condition.

1. Introduction

The theory of approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [12] and widely developed in [4]. He considers nonlinear integral operators, replacing linearity assumption by Lipschitz condition for kernel functions generating the operators and satisfying suitable singularity assumptions. After this discovery, several matematicians have undertaken the program of extending approximation by nonlinear integral operators in many ways and to several settings, including modular function spaces, pointwise and uniform convergence of operators, Korovkin type theorems, abstract function spaces, sampling series and so on. Especially, this kind of operators were extensively studied by C. Bardaro, J. Musielak and G. Vinti [5],[6] in connection with the modular space. Operators of type

$$T_w(f)(x) = \int_a^b K_w\left(x-t, f(t)\right) dt, x \in (a,b) \tag{1.1}$$

This paper was presented at the third edition of the International Conference on Numerical Analysis and Approximation Theory (NAAT 2014), Cluj-Napoca, Romania, September 17-20, 2014.

and its special cases were studied by Swiderski and Wachnicki [15], Karsli [9], [10] and Karsli-Ibikli [11] in some Lebesgue spaces. Such developments delineated a theory which is nowadays referred to as the theory of approximation by nonlinear integral operators.

The kernel of the operator of type (1.1) depends on Euclidean distance, so it holds the properties of isotropy. As an extension of the isotropic distance, non-isotropy was defined by Besov and Nikolsky 1975 in [7] . It is useful to mention that, non-isotropy were studied on linear singular integral operators [2], [3], [16], and on potential theory [8], [14]. In this paper we assume that the kernel of the operator depends on non-isotropic distance.

The present paper concerns with pointwise convergence of families of nonlinear singular integral operators $T_{w,\lambda}(f)(s)$ of the form

$$T_{w,\lambda}(f)(s) = \int_{\mathbb{R}^n} K_w\left(|s-t|_\lambda, f(t)\right) dt. \tag{1.2}$$

The convergence of the family of operators of type 1.2 is proved for some points with the suitable assumptions and properties. The next section contains some definitions, notations, assumptions and properties.

2. Preminilaries

Definition 2.1. [7] Let λ_1, λ_2, ...,λ_n be positive real numbers. The non-isotropic λ-distance of $x \in \mathbb{R}^n$ to the origin is

$$|x|_\lambda = \left(|x_1|^{\frac{1}{\lambda_1}} + |x_2|^{\frac{1}{\lambda_2}} + ... + |x_n|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}}$$

where

$$\begin{aligned} x &= (x_1, x_2, ..., x_n), \\ \lambda &= (\lambda_1, \lambda_2, ..., \lambda_n), \\ |\lambda| &= \lambda_1 + \lambda_2 + ... + \lambda_n. \end{aligned}$$

In the case $\lambda_k = \frac{1}{2}(k = 1, ...n)$ we have the well-known Euclidean norm. Note also that for any $t > 0$

$$\left(\left|t^{\lambda_1} x_1\right|^{\frac{1}{\lambda_1}} + \left|t^{\lambda_2} x_2\right|^{\frac{1}{\lambda_2}} + ... + \left|t^{\lambda_n} x_n\right|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}} = t^{\frac{|\lambda|}{n}} |x|_\lambda .$$

It means that non-isotropic λ-distance has the homogeneity of the degree $\frac{|\lambda|}{n}$, and also λ-distance has the following properties,

a) $|x|_\lambda = 0 \Leftrightarrow x = 0$,

b) $\left|p^\lambda x\right|_\lambda = p^{\frac{|\lambda|}{n}} |x|_\lambda$,

c) $|x+y|_\lambda \leq 2^{(1+\frac{1}{\lambda_{\min}})\frac{|\lambda|}{n}} (|x|_\lambda + |y|_\lambda)$.

Definition 2.2. [13] Let $g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function and f be a one-variable function, defined almost everywhere on $[0, \infty)$. g is called radial function, if there is a representation such that

$$g(x_1, x_2, ..., x_n) = f\left(\sqrt{(x_1)^2 + (x_2)^2 + ... + (x_n)^2}\right).$$

Definition 2.3. [1] Let $f \in L_p(\mathbb{R}^n)$ $1 \le p \le +\infty$ be a function and a point $x \in \mathbb{R}^n$ is called (λ, p)-Lebesgue point of the function f, if

$$\lim_{h\to 0}\left(\frac{1}{h^{2|\lambda|}}\int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}}\le h}|f(x-t)-f(x)|^p\,dt\right)^{\frac{1}{p}} = 0\ .$$

Definition 2.4. [1] λ-spherical coordinates are given by

$$\begin{aligned}
x_1 &= (u\cos\theta_1)^{2\lambda_1},\\
x_2 &= (u\sin\theta_1\cos\theta_2)^{2\lambda_2},\\
&\vdots\\
x_{n-1} &= (u\sin\theta_1\sin\theta_1...\sin\theta_{n-2}\cos\theta_{n-1})^{2\lambda_{n-1}},\\
x_n &= (u\sin\theta_1\sin\theta_1...\sin\theta_{n-2})^{2\lambda_n}
\end{aligned}$$

where

$$0 \le \theta_1, \theta_2, ..., \theta_{n-2} \le \pi, 0 \le \theta_{n-1} \le 2\pi, u \ge 0.$$

Denoting the Jacobian of this transformation by $J_\lambda(u, \theta_1, \theta_2, ..., \theta_{n-1})$, we obtain

$$J_\lambda(u, \theta_1, \theta_2, ..., \theta_{n-1}) = u^{2|\lambda|-1}\Omega_\lambda(\theta),$$

where

$$\Omega_\lambda(\theta) = 2^n\lambda_1...\lambda_n\prod_{j=1}^{n-1}(\cos\theta_j)^{2\lambda_j-1}(\sin\theta_j)^{\sum\limits_{k=j}^{j+1}2\lambda_k-1}$$

and we can easily see that

$$\omega_{\lambda,n-1} = \int\limits_{S^{n-1}}\Omega_\lambda(\theta)d\theta$$

is finite, where $S^{n-1}$is the unit sphere in $\mathbb{R}^n$.

Definition 2.5. $K_w(|.|_\lambda, .)$ belongs to λ-class, if the following conditions are satisfied

(A) There exists a summable function $L_w(|.|_\lambda) : \mathbb{R}^n \to \mathbb{R}$ such that

$$|K_w(|t|_\lambda, u) - K_w(|t|_\lambda, v)| \le L_w(|t|_\lambda)|u - v|$$

for any $w \in W$, $t \in \mathbb{R}^n$, $u, v \in \mathbb{R}$,

(B) For every $t \in \mathbb{R}^n$ and $u \in \mathbb{R}$

$$\lim_{w\to 0}\left|\int\limits_{\mathbb{R}^n}K_w(|t|_\lambda, u)\,dt - u\right| = 0.$$

(C) $L_w(|t|_\lambda) = w^{-|\lambda|}L\left(\left|\frac{t}{w^\lambda}\right|_\lambda\right)$ for any $w \in W$, $t \in \mathbb{R}^n$,

(D) $\lim_{w\to 0} R_q^\delta(w) = 0$, where

$$R_q^\delta(w) = \begin{cases} \left(\int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} (L_w(|t|_\lambda))^q dt \right)^{\frac{1}{q}} & 1 \le q < +\infty \\ \operatorname*{ess\,sup}\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w(|t|_\lambda) & q = +\infty. \end{cases}$$

3. Convergence results

Theorem 3.1. *Let $f \in L_p(\mathbb{R}^n)$, $1 \le p \le +\infty$. Assume that the kernel of the family of operators of type* 1.2 *is in λ-class, then*

$$|T_{w,\lambda}(f)(s) - f(s)| \to 0 \quad \text{as } w \to 0$$

when s is the continuity point of the function f.

Proof. We can easily observe that

$$\int\limits_{\mathbb{R}^n} K_w\left(|s-t|_\lambda, f(s)\right) dt = \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt$$

then

$$\begin{aligned} \left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt - f(s) \right| &= \left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt - \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt \right. \\ &\quad \left. + \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt - f(s) \right| \\ &\le \int\limits_{\mathbb{R}^n} \left| K_w\left(|t|_\lambda, f(s+t)\right) dt - K_w\left(|t|_\lambda, f(s)\right) \right| dt \\ &\quad + \left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt - f(s) \right| \\ &= I_1 + I_2. \end{aligned}$$

from the condition (B) of the λ-class $I_2 \to 0$ whenever $w \to 0$.

Now we will prove $I_1 \to 0$ whenever $w \to 0$. From the condition (A) of the λ-class, there exists a function which is in $L_q(\mathbb{R}^n)$ with

$$I_1 \le \int\limits_{\mathbb{R}^n} L_w(|t|_\lambda) \, |f(s+t) - f(s)| \, dt.$$

Due to the continuity of the function f at $t = s, \forall \varepsilon > 0$ we can find a $\delta > 0$ such that

$$|f(s+t) - f(s)| < \varepsilon \quad \text{whenever} \quad |t|_\lambda^{\frac{n}{2|\lambda|}} \le \delta$$

$$\begin{aligned} I_1 &\leq \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} \leq \delta} L_w(|t|_\lambda)\,|f(s+t)-f(s)|\,dt + \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w(|t|_\lambda)\,|f(s+t)-f(s)|\,dt \\ &= J_1 + J_2 \end{aligned}$$

from the property of continuity of the function $f(x)$ and the summability of the function $L_w(|t|_\lambda)$, $J_1 \to 0$ whenever $w \to 0$, and

$$\begin{aligned} J_2 &= \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w(|t|_\lambda)\,|f(s+t)-f(s)|\,dt \\ &\leq R_q^\delta(w) \left(\int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} (|f(s+t)-f(s)|)^p dt \right)^{\frac{1}{p}} \end{aligned}$$

from the condition (D), $J_2 \to 0$ whenever $w \to 0$.

Theorem 3.2. *Let $f \in L_p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$. Assume that the kernel of the family of operators of type* 1.2 *is in λ-class, then*

$$|T_{w,\lambda}(f)(s) - f(s)| \to 0 \quad \text{as } w \to 0$$

when s is the (λ, p)-Lebesgue point of the function f.

Proof. We can easily observe that

$$\int\limits_{\mathbb{R}^n} K_w\left(|s-t|_\lambda, f(s)\right) dt = \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt$$

then

$$\begin{aligned} \left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt - f(s) \right| &= \left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s+t)\right) dt - \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt \right. \\ &\quad \left. + \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt - f(s) \right| \\ &\leq \int\limits_{\mathbb{R}^n} \left| K_w\left(|t|_\lambda, f(s+t)\right) dt - K_w\left(|t|_\lambda, f(s)\right) \right| dt \\ &\quad + \left| \int\limits_{\mathbb{R}^n} K_w\left(|t|_\lambda, f(s)\right) dt - f(s) \right| \\ &= I_1 + I_2 \end{aligned}$$

from the condition (B) of the λ-class $I_2 \to 0$ whenever $w \to 0$.

Now we will prove $I_1 \to 0$ whenever $w \to 0$. From the condition (A) of the λ-class there exists a function which is in $L_q(\mathbb{R}^n)$ with

$$I_1 \le \int\limits_{\mathbb{R}^n} L_w(|t|_\lambda)\,|f(s+t)-f(s)|\,dt.$$

We give some inequalities to prove this part.

Define a function,

$$\psi_\lambda(x) = \underset{|t|_\lambda \ge |x|_\lambda}{ess\sup} \int\limits_{\mathbb{R}^n} L_w(|t|_\lambda)dt.$$

It is seen that $\psi_\lambda(x)$ is radial function of the non-isotropic distance, so

$$\psi_\lambda(x) = \psi_\lambda(|x|).$$

If $|x|_\lambda^{\frac{n}{2|\lambda|}} = r$, then $\psi_\lambda(r) = \psi_\lambda(x)$. So, from the definition of the function $\psi_\lambda(x)$, it is a monotone decreasing function with respect to r. Hence,

$$\begin{aligned}
\int\limits_{\frac{r^{\frac{1}{2}}}{2}<|t|_\lambda^{\frac{n}{2|\lambda|}}<r^{\frac{1}{2}}} \psi_\lambda(t)dt &\ge \psi_\lambda(r^{\frac{1}{2}}) \int\limits_{\frac{r^{\frac{1}{2}}}{2}<|t|_\lambda^{\frac{n}{2|\lambda|}}<r^{\frac{1}{2}}} dt \\
&= \psi_\lambda(r^{\frac{1}{2}}) \int\limits_{\frac{r^{\frac{1}{2}}}{2}}^{r^{\frac{1}{2}}} \int\limits_{s^{n-1}} \Omega_\lambda(\theta)\rho^{2|\lambda|-1}d\theta d\rho \\
&= \psi_\lambda(r^{\frac{1}{2}}) \left(\frac{r^{|\lambda|}}{2\,|\lambda|} - \frac{r^{|\lambda|}}{2\,|\lambda|\,2^{2|\lambda|}}\right)\omega_{\lambda,n-1} \\
&= \psi_\lambda(r^{\frac{1}{2}})r^{|\lambda|} \left(\frac{1}{2\,|\lambda|} - \frac{1}{2\,|\lambda|\,2^{2|\lambda|}}\right)\omega_{\lambda,n-1}.
\end{aligned}$$

Since $\psi_\lambda(r^{\frac{1}{2}})r^{|\lambda|} \to 0$ whenever $r \to 0$ and $r \to \infty$, so we can find a constant $A > 0$ such that

$$\psi_\lambda(r^{\frac{1}{2}})r^{|\lambda|} \le A \qquad 0 < r < \infty$$

and to obtain the second inequality, we use the property of (λ, p)-Lebesgue point,

$$\lim_{h\to 0}\left(\frac{1}{h^{2|\lambda|}} \int\limits_{|t|_\lambda^{\frac{n}{2|\lambda|}} \le h} \left\{\int\limits_{s^{n-1}} \left|f(s+(\theta\rho)^{2|\lambda|})-f(s)\right|^p \Omega_\lambda(\theta)d\theta\right\}\rho^{2|\lambda|-1}d\rho\right)^{\frac{1}{p}}$$

$$g_\lambda(\rho) := \int\limits_{s^{n-1}} \left|f((s+\theta\rho)^{2|\lambda|})-f(s)\right|^p \Omega_\lambda(\theta)d\theta$$

$$\lim_{h\to 0}\left(\frac{1}{h^{2|\lambda|}}\int\limits_{|t|_{\lambda}^{\frac{n}{2|\lambda|}}\le h} g_{\lambda}(\rho)\rho^{2|\lambda|-1}d\rho\right)^{\frac{1}{p}}=0$$

$$G_{\lambda}(\rho):=\int\limits_{0}^{\rho} g_{\lambda}(\eta)\eta^{2|\lambda|-1}d\eta$$

so it is obvious that

$$G_{\lambda}(\rho)\le\varepsilon^{p}\rho^{2|\lambda|}$$

then

$$\begin{aligned}
I_1 &\le \int\limits_{\mathbb{R}^n} L_w(|t|_{\lambda})\left|f(s+t)-f(s)\right|dt\\
&= \int\limits_{|t|_{\lambda}^{\frac{n}{2|\lambda|}}\le\delta} L_w(|t|_{\lambda})\left|f(s+t)-f(s)\right|dt+\int\limits_{|t|_{\lambda}^{\frac{n}{2|\lambda|}}>\delta} L_w(|t|_{\lambda})\left|f(s+t)-f(s)\right|dt\\
&= J_1+J_2.
\end{aligned}$$

$$\begin{aligned}
J_1 &= \int\limits_{|t|_{\lambda}^{\frac{n}{2|\lambda|}}\le\delta} w^{-|\lambda|}L\left(\left|\frac{t}{w^{\lambda}}\right|_{\lambda}\right)\left|f(s+t)-f(s)\right|dt\\
&= \left(\int\limits_{0}^{\delta}\left\{\int\limits_{S^{n-1}}\left|f(s+(\theta\rho)^{2|\lambda|})-f(s)\right|^{p}\Omega_{\lambda}(\theta)d\theta\right\}\rho^{2|\lambda|-1}w^{-|\lambda|}\psi_{\lambda}\left(\frac{\rho}{w^{\frac{1}{2}}}\right)d\rho\right)^{\frac{1}{p}}\\
&= \left(\int\limits_{0}^{\delta} g_{\lambda}(\rho)\rho^{2|\lambda|-1}w^{-|\lambda|}\psi_{\lambda}\left(\frac{\rho}{w^{\frac{1}{2}}}\right)d\rho\right)^{\frac{1}{p}}.
\end{aligned}$$

Applying integration by parts,

$$\begin{aligned}
&= \left(G_{\lambda}(\rho)w^{-|\lambda|}\psi_{\lambda}\left(\frac{\rho}{w^{\frac{1}{2}}}\right)\Big|_0^{\delta}-\int\limits_{0}^{\delta}G_{\lambda}(\rho)d\left(w^{-|\lambda|}\psi_{\lambda}\left(\frac{\rho}{w^{\frac{1}{2}}}\right)\right)\right)^{\frac{1}{p}}\\
&\le \left(\varepsilon^{p}\rho^{2|\lambda|}w^{-|\lambda|}\psi_{\lambda}\left(\frac{\rho}{w^{\frac{1}{2}}}\right)\Big|_0^{\delta}-\int\limits_{0}^{\frac{\delta}{w^{\frac{1}{2}}}}G_{\lambda}(w^{\frac{1}{2}}u)w^{-|\lambda|}d\left(\psi_{\lambda}(u)\right)\right)^{\frac{1}{p}}\\
&= \left(\varepsilon^{p}\left(\frac{\delta^2}{w}\right)\psi_{\lambda}\left(\frac{\delta}{w^{\frac{1}{2}}}\right)-\int\limits_{0}^{\frac{\delta}{w^{\frac{1}{2}}}}\varepsilon^{p}w^{|\lambda|}u^{2|\lambda|}w^{-|\lambda|}d\left(\psi_{\lambda}(u)\right)\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leq \varepsilon \left(A - \int_0^\infty u^{2|\lambda|} d\left(\psi_\lambda(u)\right) \right)^{\frac{1}{p}}.$$

Using integration by parts to calculate the last integral,

$$\begin{aligned} -\int_0^\infty u^{2|\lambda|} d\left(\psi_\lambda(u)\right) &= \lim_{r\to\infty} -r^{|\lambda|}\psi_\lambda(r) + 2|\lambda| \int_0^\infty u^{2|\lambda|-1}\psi_\lambda(r)du \\ &= \frac{2|\lambda|}{\omega_{\lambda,n-1}} \int_{\mathbb{R}^n} \psi_\lambda(x)dx < B. \end{aligned}$$

Here B is a constant, hence, $J_1 < \varepsilon B$. Now we investigate J_2 whenever $w \to 0$,

$$\begin{aligned} J_2 &= \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w(|t|_\lambda) \left| f(s+t) - f(s) \right| dt \\ &\leq \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} \left| f(s+t)) \right| L_w(|t|_\lambda) dt + \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} \left| f(s) \right| L_w(|t|_\lambda) dt. \end{aligned}$$

Let ψ_δ is the characteristic function of the set $\left\{ t \in \mathbb{R}^n : |t|^{\frac{n}{2|\lambda|}} > \delta \right\}$, then

$$\begin{aligned} &= \int_{\mathbb{R}^n} \left| f(s+t) \right| L_w(|t|_\lambda)\psi_\delta dt + \int_{\mathbb{R}^n} \left| f(s) \right| L_w(|t|_\lambda)\psi_\delta dt \\ &= \left\| f \right\|_p \left\| \psi_\delta L_w\left(|t|_\lambda\right) \right\|_q + \left| f(s) \right| \left\| \psi_\delta L_w\left(|t|_\lambda\right) \right\|_1. \end{aligned}$$

In view of the conditions (D),$\left\| \psi_\delta L_w\left(|t|_\lambda\right) \right\|_q$ goes to zero whenever $w \to 0$. From (C) of λ-class, one has

$$\begin{aligned} \left\| \psi_\delta L_w\left(|t|_\lambda\right) \right\|_1 &= \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L_w\left(|t|_\lambda\right) dt = \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \delta} w^{-|\lambda|} L\left(\left| \frac{t}{w^\lambda} \right|_\lambda \right) dt \\ &= \int_{|tw^\lambda|_\lambda^{\frac{n}{2|\lambda|}} > \delta} L(|t|_\lambda)dt = \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} > \frac{\delta}{\sqrt{w}}} L(|t|_\lambda)dt \end{aligned}$$

and this part also goes to zero whenever $w \to 0$.

Acknowledgment. The authors would like to thank in advance to the referee for his/her critical comments and remarks, which have definitely improved the quality of this study.

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