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A direct approach for proving Wallis ratio estimates and an improvement of Zhang-Xu-Situ inequality

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Abstract. In time, inequalities about Wallis ratio and related functions were presented by many mathematicians. In this paper, we show how estimates on the Wallis ratio can be obtained using the asymptotic series. Finally, an improvement of an inequality due to X.-M. Zhang, T.-Q. Xu and L.-B. Situ [Geometric convexity of a function involving gamma function and application to inequality theory, J. Inequal. Pure Appl. Math. 8 (1) (2007) Art. 17, 9 pp.] is presented.

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1. Introduction and motivation

Wallis ratio

$$P_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}, \quad n = 1, 2, 3 \cdots$$

plays a main role in mathematics and other branches of science. This expression is closely related to the Euler gamma function defined for all real $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

We have:

$$P_n = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n+1)}. \tag{1.1}$$

For further details, we recommend the basic monograph [1]. Many mathematicians were preoccupied to give estimates for P_n and other expressions related to gamma function.We refer for example to the following recent titles: Chen and Qi [2]-[3], Hirschhorn [5], Lin, Deng and Chen [7], Mortici [9]-[13], Păltănea [25].

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Chronologically, we mention the following inequalities for every integer $n \geq 1$ due to Wallis [29]

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi n}}, \tag{1.2}$$

Kazarinoff [6]

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \tag{1.3}$$

Hirschhorn [5]

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{8}{3}}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{7}{3}}\right)}} \tag{1.4}$$

or Panaitopol [24]

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{1.5}$$

Chen and Qi [2] proposed the following inequality

$$\frac{1}{\sqrt{\pi\left(n+A\right)}} \leq P_n < \frac{1}{\sqrt{\pi\left(n+B\right)}}, \quad n \geq 1, \tag{1.6}$$

where the constants $A = \frac{4}{\pi} - 1$ and $B = \frac{1}{4}$ are sharp. Zhao [27] proved

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < P_n \leq \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{3}}\right)}} \tag{1.7}$$

and Zhang et al. [28] showed:

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} < P_n \leq \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}}. \tag{1.8}$$

The above estimates were obtained using a various of methods such as mean inequality, Jensen inequality, monotonicity of some sequences, or monotonicity and complete monotonicity of some functions. In this work, we exploit some inequalities obtained by truncation of certain asymptotic series.

As a new result, in the last section of this work we present an improvement of an inequality due to X.-M. Zhang, T.-Q. Xu and L.-B. Situ stated in [28].

2. The asymptotic series of P_n

The following inequalities were presented by Slavić [26], for every real $x > 0$ and integers $m, l \geq 1$:

$$\sqrt{x}\exp\left(a_m(x)\right) < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x}\exp\left(b_l(x)\right), \tag{2.1}$$

where

$$a_m(x) = \sum_{k=1}^{2m} \frac{\left(1 - 2^{-2k}\right) B_{2k}}{k\,(2k-1)\,x^{2k-1}}$$

and

$$b_l(x) = \sum_{k=1}^{2l-1} \frac{\left(1 - 2^{-2k}\right) B_{2k}}{k\,(2k-1)\,x^{2k-1}}.$$

Here B_j are the Bernoulli numbers given by the generating function

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}.$$

The first Bernoulli numbers are $B_0 = 1$, $B_1 = 1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, while $B_{2m+1} = 0$, for every integer $m \geq 1$. See, *e.g.* [1].

As a direct consequence of Slavić inequalities (2.1), the following asymptotic formula holds true as $x \to \infty$:

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\exp\left\{\sum_{k=1}^{\infty} \frac{\left(1 - 2^{-2k}\right) B_{2k}}{k\,(2k-1)\,x^{2k-1}}\right\}.$$

Using (1.1), we get:

$$P_n \sim \frac{1}{\sqrt{n\pi}}\exp\left\{-\sum_{k=1}^{\infty} \frac{\left(1 - 2^{-2k}\right) B_{2k}}{k\,(2k-1)\,n^{2k-1}}\right\}, \quad n \to \infty, \tag{2.2}$$

while inequalities (2.1) can be rewritten as

$$\frac{1}{\sqrt{n\pi}}\exp\left\{\alpha_l(n)\right\} < P_n < \frac{1}{\sqrt{n\pi}}\exp\left\{\beta_m(n)\right\}, \tag{2.3}$$

where $\alpha_l(n) = -b_l(n)$ and $\beta_m(n) = -a_m(n)$. The first truncations are the following:

$$\begin{aligned}
\alpha_1(n) &= -\frac{1}{8n}, \\
\alpha_2(n) &= -\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5}, \ldots
\end{aligned}$$

and

$$\begin{aligned}
\beta_1(n) &= -\frac{1}{8n} + \frac{1}{192n^3}, \\
\beta_2(n) &= -\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5} + \frac{17}{14336n^7}, \ldots .
\end{aligned}$$

As examples, we show complete arguments of our method for proving Kazarinoff's inequality and Panaitopol's inequality. All other inequalities on Wallis ratio presented in the first part of this work can be similarly proven, as we indicate in the next section.

3. Kazarinoff's inequality

We start with Kazarinoff's inequality (1.3). In his proof, Kazarinoff used the Legendre's formula for digamma function:

$$\psi(x) = -\gamma + \int_0^1 \frac{t^x - 1}{t-1} dt$$

($\gamma = 0.577215\cdots$ is the Euler-Mascheroni constant) and the inequality

$$[\ln \phi(t)]'' - \left\{[\ln \phi(t)]'\right\}^2 > 0,$$

where

$$\phi(t) = \int_0^1 \sin^t x \; dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{t+2}{2}\right)}.$$

Chen and Qi [2] rediscovered the right-hand side of Wallis' inequality using the monotonicity of the sequence

$$Q_n = \left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 - n,$$

with $\lim_{n\to\infty} Q_n = 1/4$.

Our idea for proving Kazarinoff's inequality using the asymptotic series (2.2) is to consider as many as necessary terms α_l and β_m such that

$$\begin{aligned} \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} &< \frac{1}{\sqrt{n\pi}} \exp\left(\alpha_l(n)\right) \\ &< P_n \\ &< \frac{1}{\sqrt{n\pi}} \exp\left(\beta_m(n)\right) < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \end{aligned} \tag{3.1}$$

Individual tryings we made showed that already first truncations α_1 and β_1 make inequalities (3.1) true, namely

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{1}{\sqrt{n\pi}} \exp\left(-\frac{1}{8n}\right) \tag{3.2}$$

and

$$\frac{1}{\sqrt{n\pi}} \exp\left(-\frac{1}{8n} + \frac{1}{192n^3}\right) < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{3.3}$$

By taking the logarithm, the inequalities (3.2)-(3.3) are equivalent to

$$-\frac{1}{8n} - \frac{1}{2}\ln n + \frac{1}{2}\ln\left(n+\frac{1}{2}\right) > 0$$

and

$$-\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{2}\ln n + \frac{1}{2}\ln\left(n+\frac{1}{4}\right) < 0,$$

for every integer $n \geq 1$. It suffices $f > 0$ and $g < 0$ on $[1, \infty)$, where

$$f(x) = -\frac{1}{8x} - \frac{1}{2}\ln x + \frac{1}{2}\ln\left(x + \frac{1}{2}\right)$$

and

$$g(x) = -\frac{1}{8x} + \frac{1}{192x^3} - \frac{1}{2}\ln x + \frac{1}{2}\ln\left(x + \frac{1}{4}\right).$$

As

$$f'(x) = -\frac{(2x-1)}{8(2x+1)x^2} < 0, \quad g'(x) = \frac{8x^2 - 4x - 1}{64x^4(4x+1)} > 0,$$

the function f is strictly decreasing and g is strictly increasing on $[1, \infty)$. But $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, so $f > 0$ and $g < 0$ on $[1, \infty)$ and our assertion is proved.

4. Panaitopol's inequality

Panaitopol [24] improved the left-hand side of Wallis' inequality as

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32n}\right)}} < P_n.$$

As above, we search a truncation of asymptotic series such that

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32n}\right)}} < \frac{1}{\sqrt{n\pi}}\exp\left(\alpha_2(n)\right) < P_n. \tag{4.1}$$

Remark that in this case, the second truncation should be selected:

$$\alpha_2(n) = -\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5}.$$

The first inequality (4.1) is equivalent to $h > 0$ on $[1, \infty)$, where

$$h(x) = -\frac{1}{8x} + \frac{1}{192x^3} - \frac{1}{640x^5} - \frac{1}{2}\ln x + \frac{1}{2}\ln\left(x + \frac{1}{4} + \frac{1}{32x}\right).$$

As

$$h'(x) = \frac{1}{8x^2(32x^2 + 4x + 1)} > 0,$$

the function h is strictly increasing.

Then $h(x) \geq h(1) = \frac{1}{2}\ln\frac{41}{32} - \frac{233}{1920} = 0.00256\cdots > 0$, for every $x \in [1, \infty)$ and the assertion follows.

5. Further examples

All the other inequalities presented in the first section can be similarly proven. More precisely, we reduced Zhao De Jun's inequality (1.7) to

$$\begin{aligned}\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} &< \frac{1}{\sqrt{n\pi}}\exp\left(\alpha_1(n)\right)\\ &< P_n\\ &< \frac{1}{\sqrt{n\pi}}\exp\left(\beta_1(n)\right) < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{3}}\right)}},\end{aligned}$$

Hirschhorn's inequality (1.4) to

$$\begin{aligned}\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{8}{3}}\right)}} &< \frac{1}{\sqrt{n\pi}}\exp\left(\alpha_2(n)\right)\\ &< P_n\\ &< \frac{1}{\sqrt{n\pi}}\exp\left(\beta_1(n)\right) < \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{7}{3}}\right)}}\end{aligned}$$

and Zhang et al. inequality (1.8) to

$$\begin{aligned}\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} &< \frac{1}{\sqrt{n\pi}}\exp\left(\alpha_1(n)\right)\\ &< P_n\\ &< \frac{1}{\sqrt{n\pi}}\exp\left(\beta_1(n)\right) < \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}}.\end{aligned}$$

The great advantage of the asymptotic series method we present here is that all computations are reduced to some elementary functions involving polynomial functions and logarithmic functions. In consequence, the monotonicity, or positivity, of such functions can be easily stated.

6. An improvement of Zhang-Xu-Situ inequality

In this section, motivated by Zhang-Xu-Situ inequality (1.8), we propose the following better approximation

$$P_n \approx \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}+\frac{1}{48n^2}-\frac{1}{2880n^3}}. \tag{6.1}$$

This approximation is obtained firstly by considering the following class of approximations

$$P_n \approx \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{an+\frac{b}{n}+\frac{c}{n^2}+\frac{d}{n^3}}, \tag{6.2}$$

where a, b, c, $d \in \mathbb{R}$ are any real parameters. In order to find the values of a, b, c, d that provide the most accurate approximation (6.2), we use a method first introduced by Mortici in [8]. This method was proven to be a strong tool for constructing asymptotic expansions, or for accelerating some convergences. See, *e.g.* [14]-[23].

Let us define the relative error sequence w_n by the following formulas for every integer $n \geq 1$:

$$P_n = \frac{1}{\sqrt{e\pi n}} \left(1 + \frac{1}{2n}\right)^{an + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3}} \exp w_n .$$

We consider an approximation (6.2) better when the speed of convergence of w_n to zero is higher. But w_n is faster convergent together to the difference $w_n - w_{n+1}$. Using a Maple software, we get

$$\begin{aligned} w_n - w_{n+1} \quad = \quad & \left(\frac{1}{8}a - \frac{1}{8}\right)\frac{1}{n^2} + \left(-\frac{5}{24}a - b + \frac{1}{8}\right)\frac{1}{n^3} \\ & + \left(\frac{19}{64}a + \frac{15}{8}b - \frac{3}{2}c - \frac{7}{64}\right)\frac{1}{n^4} \\ & + \left(-\frac{197}{480}a - \frac{35}{12}b + \frac{7}{2}c - 2d + \frac{3}{32}\right)\frac{1}{n^5} \\ & + \left(\frac{217}{384}a + \frac{815}{192}b - \frac{155}{24}c + \frac{45}{8}d - \frac{31}{384}\right)\frac{1}{n^6} \\ & + O\left(\frac{1}{n^7}\right). \end{aligned}$$

The fastest convergence is obtained when the first four coefficients vanish of n^{-k}, that is for the values

$$a = 1, \; b = -\frac{1}{12}, \; c = \frac{1}{48}, \; d = -\frac{1}{2880}.$$

Now the approximation (6.1) is completely justified. We are now in a position to improve the upper bound of Zhang-Xu-Situ inequality as follows.

Theorem 6.1. *The following inequality is valid, for every integer $n \geq 1$:*

$$P_n \leq \frac{1}{\sqrt{e\pi n}} \left(1 + \frac{1}{2n}\right)^{n - \frac{1}{12n} + \frac{1}{48n^2} - \frac{1}{2880n^3}} . \tag{6.3}$$

Proof. It suffices to show that

$$\frac{1}{\sqrt{n\pi}} \exp\left(\beta_2(n)\right) < \frac{1}{\sqrt{e\pi n}} \left(1 + \frac{1}{2n}\right)^{n - \frac{1}{12n} + \frac{1}{48n^2} - \frac{1}{2880n^3}} ,$$

or equivalently

$$\begin{aligned} & -\frac{1}{8n} + \frac{1}{192n^3} - \frac{1}{640n^5} + \frac{17}{14336n^7} + \frac{1}{2} \\ < & \left(n - \frac{1}{12n} + \frac{1}{48n^2} - \frac{1}{2880n^3}\right) \ln\left(1 + \frac{1}{2n}\right). \end{aligned}$$

We have to prove that $\varphi < 0$ for every $x \in [1, \infty)$, where

$$\varphi(x) = \frac{-\frac{1}{8x} + \frac{1}{192x^3} - \frac{1}{640x^5} + \frac{17}{14336x^7} + \frac{1}{2}}{\left(x - \frac{1}{12x} + \frac{1}{48x^2} - \frac{1}{2880x^3}\right)} - \ln\left(1 + \frac{1}{2x}\right).$$

But

$$\varphi'(x) = \frac{P(x-1)}{56x\,(2x+1)\,(60x - 240x^2 + 2880x^4 - 1)^2} > 0,$$

with

$$\begin{aligned} P(x) \;=\; & 784\,374\,011 + 2985\,594\,595x + 4717\,628\,082x^2 \\ & +4035\,936\,400x^3 + 2026\,365\,656x^4 \\ & +598\,429\,920x^5 + 96\,371\,520x^6 + 6531\,840x^7, \end{aligned}$$

so the function φ is strictly increasing on $[1, \infty)$. But $\lim_{x\to\infty} \varphi(x) = 0$, so $\varphi(x) < 0$ for all real $x \geq 1$. The proof is now completed. □

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