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Circular mappings with minimal critical sets

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Abstract. We provide classes of manifolds M satisfying the relation $\varphi_{S^1}(M) = \varphi(M)$, we discuss the situation $\varphi_{S^1}(M) = 1$, and we formulate a circular version of the Ganea conjecture.

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1. Introduction

The systematic study of the smooth circular functions defined on a manifold was initiated by E.Pitcher in the articles [23],[24]. His goal was to extend in this context the classical Morse theory for real-valued functions. The importance of this study was pointed out by Novikov in the early 1980s. The Morse - Novikov theory is now a large and actively developing domain of Differential Topology, with applications and connections to many geometrical problems (see the monographs [11] and [21]).

The *φ-category* of a manifold M is $\varphi(M) = \min\{\mu(f) : f \in \mathcal{C}^\infty(M, \mathbb{R})\}$, and it represents the φ-category of the pair $(M, \mathbb{R})$.

The *circular φ-category* of a manifold M was introduced in the paper [4]. It is defined as the φ-category of the pair (M, S^1) corresponding to the family $C^\infty(M, S^1)$, where S^1 is the unit circle. That is

$$\varphi_{S^1}(M) = \min\{\mu(f) : f \in \mathcal{C}^\infty(M, S^1)\},$$

where $\mu(f)$ denotes the cardinality of the critical set of mapping $f : M \to S^1$.

If we restrict the class of smooth functions to its subclass of Morse functions, then we obtain, in the real case, the *Morse-Smale characteristic*

$$\gamma(M) = \min\{\mu(f) : f \in \mathcal{C}^\infty(M, \mathbb{R}),\ f - \text{ Morse}\},$$

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and the *circular Morse-Smale characteristic*

$$\gamma_{S^1}(M) = \min\{\mu(f) : f \in \mathcal{C}^\infty(M, S^1),\ f -\ \text{circular Morse function}\}$$

in the circular case. For the Morse-Smale characteristic of the closed surfaces we refere the reader to [5]. The inequalities

$$\varphi_{S^1}(M) \leq \varphi(M), \quad \gamma_{S^1}(M) \leq \gamma(M) \tag{1.1}$$

rely on the property $C(\exp \circ g) = C(g)$ which is quite obvious due to the property of the exponential map to be a local diffeomorphism. Thus, the quality of a real valued function $g : M \longrightarrow \mathbb{R}$ to be Morse is transmited to the function $\exp \circ g$ and the second inequality of (1.1) is also justified. On the other hand, the inequalities

$$\varphi(M) \leq \gamma(M), \quad \varphi_{S^1}(M) \leq \gamma_{S^1}(M) \tag{1.2}$$

are obvious.

One of the main goals of this paper is to provide classes of manifolds M satisfying (1.1) with equality, i.e. $\varphi_{S^1}(M) = \varphi(M)$ and $\gamma_{S^1}(M) = \gamma(M)$. In the last section we discuss the situation $\varphi_{S^1}(M) = 1$ and we formulate a circular version of the Ganea conjecture.

2. Manifolds with $\varphi_{S^1}(M) = \varphi(M)$ and $\gamma_{S^1}(M) = \gamma(M)$

Let us first observe that the inequality $\varphi_{S^1}(M) \leq \varphi(M)$ ensured by (1.1) can be strict. Indeed, the m-dimensional torus $T^m = S^1 \times \cdots \times S^1$ (m times) has, according to [1, Example 3.6.16], the φ-category $\varphi(T^m) = m + 1$. On the other hand, every projection $T^m \to S^1$ is a trivial differentiable fibration, hence it has no critical points, implying $\varphi_{S^1}(T^m) = 0$. This example is part of the following more general remark. For a closed manifold M we have $\varphi_{S^1}(M) = 0$ if and only if there is a differentiable fibration $M \to S^1$. Indeed, the existence of a differentiable fibration $M \to S^1$ ensures the equality $\varphi_{S^1}(M) = 0$, as the fibration itself has no critical points at all. Conversely, the equality $\varphi_{S^1}(M) = 0$ ensures the existence of a submersion $M \to S^1$, which is also proper, as its inverse images of the compact sets in S^1 are obviously compact. Thus, by the well-known Ehresmann's fibration theorem (see for instance the reference [10, p. 15]) one can conclude that our submersion is actually a locally trivial fibration. Note that this property works for arbitrary closed target manifolds, not just for the circle S^1.

Assume that every smooth (Morse) circle valued function $f : M \longrightarrow S^1$ can be lifted to a smooth (Morse) real valued function $\tilde{f} : M \longrightarrow \mathbb{R}$, i.e. we have $\exp \circ \tilde{f} = f$. Since the universal cover $\exp : \mathbb{R} \longrightarrow S^1$ is a local diffeomorphism, it follows that $\mu(f) = \mu(\tilde{f}) \geq \varphi(M)$, for every smooth function $f : M \longrightarrow S^1$. This shows that the inequalities $\varphi_{S^1}(M) \geq \varphi(M)$, $\gamma_{S^1}(M) \geq \gamma(M)$ hold, which combined to the general inequalities (1.1), leads to the following result.

Proposition 2.1. ([6]) *Let M be a connected smooth manifold. If M satisfies the lifting property* $\mathrm{Hom}\,(\pi(M), \mathbb{Z}) = 0$*, then $\varphi_{S^1}(M) = \varphi(M)$ and $\gamma_{S^1}(M) = \gamma(M)$. In particular $\varphi_{S^1}(M) = \varphi(M)$ and $\gamma_{S^1}(M) = \gamma(M)$ whenever the fundamental group of M is a torsion group.*

2.1. On the categories of some Grassmann manifolds

Proposition 2.2. *If $n \geq 2$ is an integer, then $\varphi_{S^1}(S^n) = \varphi(S^n) = \gamma_{S^1}(S^n) = \gamma(S^n) = 2$ and*

$$\begin{aligned} \varphi_{S^1}(\mathbb{RP}^n) &= \varphi(\mathbb{RP}^n) = \gamma_{S^1}(\mathbb{RP}^n) = \gamma(\mathbb{RP}^n) = \operatorname{cat}(\mathbb{RP}^n) = \\ \varphi_{S^1}(\mathbb{CP}^n) &= \varphi(\mathbb{CP}^n) = \gamma_{S^1}(\mathbb{CP}^n) = \gamma(\mathbb{CP}^n) = \operatorname{cat}(\mathbb{CP}^n) = n+1, \end{aligned}$$

where $\operatorname{cat}(\mathbb{CP}^n)$ *stands for the Lusternik-Schnirelmann category of the complex projective space $\mathbb{CP}^n$.*

Proof. We shall only justify the equalities

$$\varphi_{S^1}(\mathbb{CP}^n) = \varphi(\mathbb{CP}^n) = \gamma(\mathbb{CP}^n) = \gamma_{S^1}(\mathbb{CP}^n) = \operatorname{cat}(\mathbb{CP}^n) = n+1,$$

as the other equalities have been already proved in [6]. The equalities $\varphi_{S^1}(\mathbb{CP}^n) = \varphi(\mathbb{CP}^n)$ and $\gamma_{S^1}(\mathbb{CP}^n) = \gamma(\mathbb{CP}^n)$ follow from Proposition 2.1 taking into account the simply-connectedness of the complex projective space $\mathbb{CP}^n$. On the other hand the inequality $\varphi(\mathbb{CP}^n) \leq \gamma(\mathbb{CP}^n)$ follow from the general inequality (1.2). Therefore $\varphi_{S^1}(\mathbb{CP}^n) = \varphi(\mathbb{CP}^n) \leq \gamma(\mathbb{CP}^n) = \gamma_{S^1}(\mathbb{CP}^n)$. In order to prove the equalities $\gamma(\mathbb{CP}^n) = \operatorname{cat}(\mathbb{CP}^n) = n+1$ we observe that

$$\gamma(\mathbb{CP}^n) \leq \mu(f) = \operatorname{card}(C(f)) = n+1,$$

as the function

$$f : \mathbb{CP}^n \longrightarrow \mathbb{R}, \quad f([z_1, \ldots, z_{n+1}]) = \frac{|z_1|^2 + 2|z_2|^2 + \cdots + n|z_n|^2 + (n+1)|z_{n+1}|^2}{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2}.$$

is a Morse function with the $n+1$ critical points

$$[1, 0, \ldots, 0], [0, 1, \ldots, 0], \ldots, [0, 0, \ldots, 1] \in \mathbb{CP}^n \ \text{[19, p. 89]}.$$

Thus $\varphi(\mathbb{CP}^n) \leq \gamma(\mathbb{CP}^n) \leq n+1$. Finally, we use the well-known inequality $\varphi(\mathbb{CP}^n) \geq \operatorname{cat}(\mathbb{CP}^n)$ and the relation $\operatorname{cat}(\mathbb{CP}^n) = n+1$ [9, p. 3, pp. 7-13]. □

Note that the equalities $\varphi_{S^1}(\mathbb{RP}^n) = \varphi(\mathbb{RP}^n) = \operatorname{cat}(\mathbb{RP}^n) = n+1$ are being similarly proved in [6] by using the $\mathbb{Z}_2$ structure of the fundamental group of $\mathbb{RP}^n$, the Morse function

$$F_n : \mathbb{RP}^n \longrightarrow \mathbb{R}, \ F_n([x_1, \ldots, x_{n+1}]) = \frac{x_1^2 + 2x_2^2 + \cdots + nx_n^2 + (n+1)x_{n+1}^2}{x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2},$$

whose critical set is $C(F_n) = \{[1, 0, \ldots, 0], [0, 1, \ldots, 0], \ldots, [0, 0, \ldots, 1]\}$, and the well-known relations $\varphi(\mathbb{RP}^n) \geq \operatorname{cat}(\mathbb{RP}^n) = n+1$ [22, pp. 190-192].

Proposition 2.3. *If $n \geq 3$ and $1 \leq k \leq n-1$, then*

$$\varphi_{S^1}(G_{k,n}) = \varphi(G_{k,n}) \leq \gamma(G_{k,n}) = \gamma_{S^1}(G_{k,n}) \leq \binom{n+k}{k},$$

where $G_{k,n}$ stands for the Grassmann manifold of all k-dimensional subspaces of the space $\mathbb{R}^{n+k}$.

Proof. The equalities $\varphi_{S^1}(G_{k,n}) = \varphi(G_{k,n})$ and $\gamma_{S^1}(G_{k,n}) = \gamma(G_{k,n})$ follow due to Proposition 2.1 and the $\mathbb{Z}_2$ structure of the fundamental group of $G_{k,n}$. Thus $\varphi_{S^1}(G_{k,n}) = \varphi(G_{k,n}) \leq \gamma(G_{k,n}) = \gamma_{S^1}(G_{k,n})$. Recall that $G_{k,n}$ can be embedded into the projective space $\mathbb{RP}^{n+k-1}$ via the Plücker embedding

$$p : G_{k,n} \hookrightarrow P\left(\Lambda^k(\mathbb{R}^{n+k})\right) = \mathbb{RP}^{d(n,k)-1},\ p(W) = [w_1 \wedge \cdots \wedge w_k],$$

where $\{w_1, \ldots, w_k\}$ is an arbitrary basis of W and $d(n,k)$ stands for the dimension of $\Lambda^k(\mathbb{R}^{n+k})$, i.e.

$$d(k,n) = \begin{pmatrix} n+k \\ k \end{pmatrix}.$$

The composed function $F_{d(k,n)-1} \circ p : G_{k,n} \longrightarrow \mathbb{R}$ is, according to Hangan [15], a Morse function with $d(k,n)$ critical points and show that $\gamma(G_{k,n}) \leq \mu\left(F_{d(k,n)-1} \circ p\right) = d(k,n)$. □

Corollary 2.4. *If $n = 1$ or $k = 1$ or ($n = 2$ and $k = 2p-1$ for some p) or ($n = 2p-1$ and $k = 2$), then* $nk \leq \varphi_{S^1}(G_{k,n}) = \varphi(G_{k,n}) \leq \gamma_{S^1}(G_{k,n}) = \gamma(G_{k,n}) \leq \begin{pmatrix} n+k \\ k \end{pmatrix}$.

Proof. We only need to use the inequality $\varphi(G_{k,n}) \geq \operatorname{cat}(G_{k,n})$ and the equalities $\operatorname{cat}(G_{k,n}) = nk$, proved by Berstein [8], whenever $n = 1$ or $k = 1$ or ($n = 2$ and $k = 2p-1$ for some p) or ($n = 2p-1$ and $k = 2$). □

2.2. On the categories of some classical Lie groups

Proposition 2.5. *If $n \geq 3$, then the following relations hold*

$$\varphi_{S^1}(SO(n)) = \varphi(SO(n)) \leq \gamma(SO(n)) = \gamma_{S^1}(SO(n)) \leq 2^{n-1}.$$

Proof. The equalities $\varphi_{S^1}(SO(n)) = \varphi(SO(n))$ and $\gamma(SO(n)) = \gamma_{S^1}(SO(n))$ follow from Proposition 2.1 by using the fundamental group of $SO(n)$ which is $\mathbb{Z}_2$. Thus $\varphi_{S^1}(SO(n)) = \varphi(SO(n)) \leq \gamma(SO(n)) = \gamma_{S^1}(SO(n))$. In order to prove the inequality $\gamma(SO(n)) \leq 2^{n-1}$ we observe that

$$\gamma(SO(n)) \leq \mu(f) = \operatorname{card}(C(f)) = 2^{n-1},$$

where $f : SO(n) \longrightarrow \mathbb{R}$, $f([a_{ij}]_{n\times n}) = a_{11} + 2a_{22} + \cdots + na_{nn}$ is a Morse function. The critical set of f consists in all diagonal matrices D with ± 1 as diagonal entries and $\det(D) = 1$ [19, p. 92]. In other words, $C(f)$ is the collection of all diagonal matrices D with an even number of -1 on the main diagonal. The number of such diagonal matrices is $\begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} + \cdots = 2^{n-1}$, i.e. $\mu(f) = 2^{n-1}$. □

Remark 2.6. *If $n \geq 3$, then the following relations hold*

$$\varphi_{S^1}(Spin(n)) = \varphi(Spin(n)) \leq \gamma(Spin(n)) = \gamma_{S^1}(Spin(n)) \leq 2^n.$$

Moreover, $\varphi(Spin(9)) \geq cat(Spin(9)) = 9$ [17]. We only need to justify the inequality $\gamma_{S^1}(Spin(n)) \leq 2^n$, as the other ones rely on the general inequalities (1.2) and the simply connectedness of $Spin(n)$. The inequlity $\gamma_{S^1}(Spin(n)) \leq 2^n$ follows from the *general inequality* $\gamma_{S^1}(\tilde{M}) \leq k \cdot \gamma_{S^1}(M)$*, where* $\tilde{M}$ *is a* k*-fold cover of* M [5, Proposition 1.5]*, taking into account that the universal cover* $Spin(n) \longrightarrow SO(n)$ *is a* 2*-fold cover.*

Corollary 2.7. $9 \leq \varphi(SO(5)) = \varphi_{S^1}(SO(5)) \leq \gamma_{S^1}(SO(5)) = \gamma(SO(5)) \leq 16.$

Proof. The relations $\varphi(SO(5)) = \varphi_{S^1}(SO(5)) \leq \gamma_{S^1}(SO(5)) = \gamma(SO(5)) \leq 16$ follow from Proposition 2.5 and the left hand side inequality follows by means of the following well-known relations $\varphi(SO(5)) \geq \text{cat}(SO(5))$ and $\text{cat}(SO(5)) = 9$ [9, p. 279], [18]. □

Unfortunately, we do not know at this moment the precise values of these categories among the values $9, 10, \ldots, 16$.

Proposition 2.8. *The following relations hold:*

1. $n \leq \varphi(U(n)) \leq \gamma(U(n)) \leq 2^n$.
2. $n - 1 \leq \varphi_{S^1}(SU(n)) = \varphi(SU(n)) \leq \gamma(SU(n)) = \gamma_{S^1}(SU(n)) \leq 2^{n-1}$.

Proof. (1) In order to prove the inequality $\gamma(U(n)) \leq 2^n$ we recall that

$$\gamma(U(n)) \leq \mu(f) = \text{card}(C(f)) = 2^n,$$

where $f : U(n) \longrightarrow \mathbb{R}$, $f([z_{ij}]_{n \times n}) = \text{Re}(z_{11} + 2z_{22} + \cdots + nz_{nn})$, which is a Morse function and its critical set consists in all diagonal matrices D with ± 1 as diagonal entries [19, p. 98]. The number of such diagonal matrices is obviously 2^n. For the left-hand-side inequality we have $\varphi(U(n)) \geq \text{cat}(U(n))$ and $\text{cat}(U(n)) = n$ [25].

(2) The equalities $\varphi_{S^1}(SU(n)) = \varphi(SU(n))$ and $\gamma_{S^1}(SU(n)) = \gamma(SU(n))$ follows from Proposition 2.1 by using the simply conectedness of $SU(n)$. Consequently $\varphi_{S^1}(SU(n)) = \varphi(SU(n)) \leq \gamma(SU(n)) = \gamma_{S^1}(SU(n))$. In order to prove the inequality $\gamma(SU(n)) \leq 2^{n-1}$ we observe that

$$\gamma(SU(n)) \leq \mu\left(f\big|_{SU(n)}\right) = \text{card}\left(C\left(f\big|_{SU(n)}\right)\right) = 2^{n-1},$$

as the restricted function $f\big|_{SU(n)}$ is also a Morse function and its critical set consists in all diagonal matrices D with ± 1 as diagonal entries and $\det(D) = 1$ [19, p. 99]. In other words, $C\left(f\big|_{SU(n)}\right)$ is the collection of all diagonal matrices D with an even number of -1 on the main diagonal. The number of such diagonal matrices is $\begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} + \cdots = 2^{n-1}$, i.e. $\mu(f) = 2^{n-1}$. The left-hand-side inequality follows by means of the relations $\varphi(SU(n)) \geq \text{cat}(SU(n))$ and $\text{cat}(SU(n)) = n - 1$ [25]. □

Remark 2.9. *The inequality* $\varphi(U(n)) \leq \varphi_{S^1}(U(n))$ *might be strict as the unitary group is diffeomorphic (but not isomorphic) to the product* $SU(n) \times S^1$ [19, p. 103] *and Proposition* 2.1 *does not apply, since the fundamental group of* $U(n)$ *is therefore* $\mathbb{Z}$*.*

2.3. On the categories of some products and connected sums

In this subsection we shall rehearse several computations of (circular) φ-category proved in the previous work [6].

If $k, l, m_1, \ldots, m_k \geq 2$, are integers, then the following relations hold:

1. $\varphi_{S^1}(S^{m_1} \times \cdots \times S^{m_k}) = \varphi(S^{m_1} \times \cdots \times S^{m_k}) = k + 1$.
2. $\varphi_{S^1}(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) = \varphi(\mathbb{RP}^{m_1} \times \cdots \times \mathbb{RP}^{m_k}) \leq m_1 + m_2 + \cdots + m_k + 1$.
3. $\varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = \varphi_{S^1}(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = 5$, where $L(r,s)$ is the lens space of dimension 3 of type (r,s).
4. $\varphi_{S^1}(\mathbb{RP}^k \times S^l) = \varphi(\mathbb{RP}^k \times S^l) \leq k + 2$.

The proofs of the equalities

$$\varphi(S^{m_1} \times \cdots \times S^{m_k}) = k + 1$$
$$\varphi(L(7,1) \times S^4) = \varphi(L(7,1) \times S^4) = 5$$

have been done by C. Gavrilă [14, Proposition 4.6, Example 4.7] and the estimate $\varphi(\mathbb{RP}^k \times S^l) \leq k + 2$ relies on [14, Proposition 4.19].

An immediate consequence of Proposition 2.1 is the following

Corollary 2.10. *If $M_1^n, \ldots, M_r^n, n \geq 3$, are connected manifolds with torsion fundamental groups, then $\varphi_{S^1}(M_1 \# \cdots \# M_r) = \varphi(M_1 \# \cdots \# M_r)$. In particular the following equality $\varphi_{S^1}(r\mathbb{RP}^n) = \varphi(r\mathbb{RP}^n)$ holds, where $r\mathbb{RP}^n$ stands for the connected sum $\mathbb{RP}^n \# \cdots \# \mathbb{RP}^n$ of r copies of $\mathbb{RP}^n$.*

The following result is mentioned in the monograph [9, p. 221].

Lemma 2.11. *If M and N are closed manifolds, then the following inequality holds $\varphi(M \# N) \leq \max\{\varphi(M), \varphi(N)\}$. In particular $\varphi(X \# X) \leq \varphi(X)$ for every closed manifold X.*

Recall that P_g denotes the closed connected non-orientable surface $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ of genus g, and Σ_g stands for the closed connected orientable surface $T^2 \# \cdots \# T^2$ of genus g.

Based on Corollary 2.10 and Lemma 2.11 we were able to prove in [6] the following relations

- $\varphi(\Sigma_g) = \varphi(P_g) = 3, g \geq 1$;
- $2 \leq \varphi(r\mathbb{RP}^n) = \varphi_{S^1}(r\mathbb{RP}^n) \leq n + 1, \ r \geq 1, \ n \geq 3$.
- If $k, l \geq 2$ are positive integers, then

$$\varphi_{S^1}\left((S^k \times S^l) \# \cdots \# (S^k \times S^l)\right) = \varphi\left((S^k \times S^l) \# \cdots \# (S^k \times S^l)\right) = 3. \tag{2.1}$$

3. Manifolds with $\varphi_{S^1}(M) = 1$ and the circular version of the Ganea conjecture

We do not have any example of a closed manifold M such that $\operatorname{cat}(M) < \varphi(M)$, and also the equality $\operatorname{cat}(M) = \varphi(M)$ is proved only for some isolated classes of manifolds. An example in this respect is given by the connected sum

$(S^k \times S^l)\#\cdots\#(S^k \times S^l)$, $k, l \geq 2$, justified by equality in (2.1). In order to emphasize the difficulty of the above mentioned problem, assume that the equality $\text{cat}(M) = \varphi(M)$ holds for every closed manifold. Let us only look to the following particular situation: $\text{cat}(M) = \varphi(M) = 2$. From $\text{cat}(M) = 2$ one obtains that M is a homotopy sphere. Taking into account the well-known Reeb's result, from the equality $\varphi(M) = 2$ it follows that M is a topological sphere. Therefore, the equalities $\text{cat}(M) = \varphi(M) = 2$ are related to the Poincaré conjecture, proved by Perelmann, it follows for instance that for any closed manifold with $\text{cat}(M) = 2$ we have $\varphi(M) = 2$ and therefore $\text{cat}(M) = \varphi(M) = 2$.

Taking into account these comments, in the article [6] we have formulated the following Reeb type problem for circular functions : *Characterize the closed manifolds M^m with the property $\varphi_{S^1}(M) = 1$.*
When $m = 2$, one example of such a manifold, suggested to us by L. Funar, is given by the closed orientable surface Σ_g of genus $g \geq 2$, i.e. we have the following result :

Proposition 3.1. *The following relation holds : $\varphi_{S^1}(\Sigma_g) = 1, g \geq 2$.*

Proof. We will construct a function with one critical point from Σ_g to S^1 by composing the projection $p : T^2 = S^1 \times S^1 \to S^1$, $p(x, y) = x$, with a map $f : \Sigma_g \to T^2$ having precisely one critical point. The existence of the map f is assured by [2] (see also [3] and [12]) as $\varphi(\Sigma_g, T^2) = 1$, and the projection p is a fibration, i.e. the critical set $C(p)$ is empty. Therefore, the composed function $p \circ f$ has at most one critical point as $C(p \circ f) \subseteq C(f)$ and $\text{card}(C(f)) = 1$. This shows that $\varphi_{S^1}(\Sigma_g) \leq 1$. For the opposite inequality, assume that $\varphi_{S^1}(\Sigma_g) = 0$ and consider a fibration $g : \Sigma_g \to S^1$, whose fiber F is a compact one dimensional manifold without boundary, i.e. a circle or a disjoint union of circles. By applying the product property of the Euler-Poincaré characteristic associated to the fibration $F \hookrightarrow \Sigma_g \xrightarrow{g} S^1$, one obtains $2-2g = \chi(\Sigma_g) = \chi(F)\chi(S^1) = 0$ as $\chi(S^1) = 0$, a contradiction with the initial assumption $g \geq 2$. □

In what follows we rely on the following relation

$$\varphi_{S^1}(M \times N) \leq \varphi_{S^1}(M) \cdot \varphi_{S^1}(N). \tag{3.1}$$

(see [6]) in order to produce other examples of closed manifolds X with $\varphi_{S^1}(X) = 1$. In fact, we will prove that the following class of closed manifolds

$$\mathcal{M}_1 := \{X - \text{closed manifold } : \varphi_{S^1}(X) = 1 \text{ and } \chi(X) \neq 0\}$$

is closed with respect to the cross product. More precisely, we have:

Proposition 3.2. *If $M, N \in \mathcal{M}_1$, then $M \times N \in \mathcal{M}_1$.*

Proof. If $M, N \in \mathcal{M}_1$, then, due to inequality 3.1, we conclude that $\varphi_{S^1}(M \times N) \leq \varphi_{S^1}(M) \cdot \varphi_{S^1}(N) = 1$. We now assume that $\varphi_{S^1}(M \times N) = 0$, i.e. there exists a fibration $F \hookrightarrow M \times N \longrightarrow S^1$. Since the Euler-Poincaré characteristic is multiplicative with respect to fibrations and vanishes on Lie groups, we deduce that $\chi(M \times N) = \chi(F) \cdot \chi(S^1)$, i.e. $\chi(M)\chi(N) = 0$, a contradiction with the initial assumption $\chi(M)$, $\chi(N) \neq 0$. □

The following example shows the existence of even dimensional manifolds X^{2k} with $\varphi_{S^1}(X) = 1, k = 1, 2, \ldots$.

Example 3.3. *If $g_1, \ldots, g_k \geq 2$, then $\varphi_{S^1}(\Sigma_{g_1} \times \cdots \times \Sigma_{g_k}) = 1$, where Σ_g stands for the closed oriented surface of genus g. Moreover, if M is a closed manifold, then*

$$\varphi_{S^1}(M \times \Sigma_{g_1} \times \cdots \times \Sigma_{g_k}) \leq \varphi_{S^1}(M).$$

Ganea's conjecture is a claim in Algebraic Topology, now disproved. It states that

$$\mathrm{cat}(X \times S^n) = \mathrm{cat}(X) + 1, n > 0,$$

where $\mathrm{cat}(X)$ is the Lusternik-Schnirelmann category of the topological space X, and S^n is the n-dimensional sphere. The conjecture was formulated by T. Ganea in 1971 (see the original reference [13]). Many particular cases of this conjecture were proved, till finally N. Iwase [16] gave a counterexample in 1998. The φ-category version of Ganea's conjecture has been studied by C. Gavrilă [14]. Now we formulate the φ_{S^1}-version of this conjecture :

Conjecture. *For every closed manifold N with $\varphi_{S^1}(N) = 1$, and for every closed manifold M, the following relation holds :*

$$\varphi_{S^1}(M \times N) = \varphi_{S^1}(M).$$

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