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Continuous wavelet transform in variable Lebesgue spaces

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Abstract. In the present note we investigate norm and almost everywhere convergence of the inverse continuous wavelet transform in the variable Lebesgue space.

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1. Introduction

The topic of variable Lebesgue spaces is a new chapter of mathematics and it is studied intensively nowadays. Instead of the classical L_p-norm, the variable $L_{p(\cdot)}$-norm is defined by

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^d} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

and the variable $L_{p(\cdot)}$ spaces contains all measurable functions f, for which $\|f\|_{p(\cdot)} < \infty$. The variable Lebesgue spaces have a lot of common property with the classical Lebesgue spaces (see Kováčik and Rákosník [12], Cruz-Uribe and Fiorenza [4], Diening, Hästö and Růžička [6], Cruz-Uribe, Firorenza and Neugebauer [3], Cruz-Uribe, Fiorenza, Martell and Pérez [2]).

The so called θ-summation method is studied intensively in the literature (see e.g. Butzer and Nessel [1], Trigub and Belinsky [15], Gát [9], Goginava [10], Simon [14] and Weisz [17, 18]). This summability is generated by a single function θ and

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includes the well known Fejér, Riesz, Weierstrass, Abel, etc. summability methods. The θ-summation is defined by

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-t) T^d \theta(Tt)\, dt.$$

Feichtinger and Weisz [7, 8, 16] have proved that the θ-means $\sigma_T^\theta f$ converge to f almost everywhere and in norm as $T \to \infty$, whenever f is in the $L_p(\mathbb{R}^d)$ space or in a Wiener amalgam space. The points of the set of almost everywhere convergence are characterized as the Lebesgue points.

Some similar results are known in the variable Lebesgue spaces (see e.g. Cruz-Uribe and Fiorenza [4]). Under some conditions on the exponent function $p(\cdot)$ and θ, the θ-means of f converge to f almost everywhere and in norm for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$ as $T \to \infty$.

The continuous wavelet transform of f with respect to a wavelet g is defined by $W_g f(x,s) = \langle f, T_x D_s g\rangle$ ($x \in \mathbb{R}^d, s \in \mathbb{R}, s \neq 0$), where D_s is the dilation operator and T_x is the translation operator. The inversion formula holds for all $f \in L_2(\mathbb{R}^d)$ (in case g and γ are suitable):

$$\int_0^\infty \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \,\frac{dxds}{s^{d+1}} = C_{g,\gamma} f,$$

where the equality is understood in a vector-valued weak sense (see Daubechies [5] and Gröchenig [11]).

Recently Li and Sun [13] have proved that if g and γ are radial, both have a radial majorant φ such that $\varphi(\cdot)\ln(2+|\cdot|) \in L_1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d}(g^* * \gamma)(x)\,dx = 0$, then for any $f \in L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$

$$\lim_{S\to 0_+, T\to\infty} \int_S^T \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \,\frac{dxds}{s^{d+1}} = C'_{g,\gamma} f \tag{1.1}$$

at every Lebesgue point of f, where $C'_{g,\gamma}$ is a constant depending on g and γ. If $1 < p < \infty$, or if $1 \le p < \infty$ and $T = \infty$, then the convergence holds in the $L_p(\mathbb{R}^d)$-norm for all $f \in L_p(\mathbb{R}^d)$. Under some other conditions Weisz [19] has proved similar results.

In this paper we will investigate the norm and almost everywhere convergence of (1.1) in variable Lebesgue spaces. We lead back the problem to the summability of Fourier transforms, more exactly, we show that the integral on the left hand side of (1.1) can be formulated as $\sigma_{1/S}^\theta f - \sigma_{1/T}^\theta f$, where θ is a given function depending on g and γ.

2. θ-summability on the classical Lebesgue spaces

Let us fix $d \ge 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. The space $L_p(\mathbb{R}^d)$ equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p\, dx\right)^{1/p} \qquad (1 \le p \le \infty),$$

is the classical Lebesgue space. We use the notation $|I|$ for the Lebesgue measure of the set I. The set of *locally integrable functions* is denoted by $L_1^{loc}(\mathbb{R}^d)$.

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \le p, q \le \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k\in\mathbb{Z}^d} \|f(\cdot + k)\|_{L_p[0,1)^d}^q \right)^{1/q} < \infty$$

with the usual modification for $q = \infty$. Note that for all $1 \le p \le \infty$, $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and $L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d)$.

Let $\theta \in L_1(\mathbb{R}^d)$ be a radial function. The *θ-means* of $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ are defined by

$$\sigma_T^\theta f(x) := (f * \theta_T)(x) = \int_{\mathbb{R}^d} f(x-t)\theta_T(t)\, dt,$$

where

$$\theta_T(x) := T^d \theta(Tx) \qquad (x \in \mathbb{R}^d, T > 0).$$

It is known that $\theta(t) = \widehat{\chi}_{B(0,1)}(t)$ implies $\sigma_T^\theta f = s_T f$, where $s_T f$ is the Dirichlet integral of the Fourier transform of f,

$$s_T f(x) := \int_{\{\|u\|_2 < T\}} \widehat{f}(u) e^{2\pi\imath x\cdot u}\, du \qquad (T > 0)$$

and

$$B(a, \delta) := \{x \in \mathbb{R}^d : \|x - a\|_2 < \delta\}.$$

Similarly, if $\theta(t) = \widehat{F}(t)$, where $F(t) = \max(1 - \|t\|_2, 0)$, then we obtain the Fejér means of f.

The *classical Hardy-Littlewood maximal operator* is defined by

$$M(f)(x) := \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f|\, d\lambda,$$

where $f \in L_1^{loc}(\mathbb{R}^d)$ and the supremum is taken over all cube $Q \subset \mathbb{R}^d$ with sides parallel to the axes. It is known that

$$\|Mf\|_p \le C\,\|f\|_p \tag{2.1}$$

for all $f \in L_p(\mathbb{R}^d)$ $(1 < p \le \infty)$ and

$$\sup_{t>0} t\lambda\left(x \in \mathbb{R}^d : Mf(x) > t\right) \le C\,\|f\|_1$$

for all $f \in L_1(\mathbb{R}^d)$.

A point $x \in \mathbb{R}^d$ is called a *Lebesgue point* of $f \in L_1^{loc}(\mathbb{R}^d)$ if

$$\lim_{h\to 0_+} \left(\frac{1}{|B(0,h)|} \int_{B(0,h)} |f(x+u) - f(x)|\, du \right) = 0.$$

It is known that if $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ $(1 \le p \le \infty)$, then almost every $x \in \mathbb{R}^d$ is a Lebesgue point of f (see Feichtinger and Weisz [7, 8]).

We say that η is a *radial majorant* of f if η is radial, non-increasing as a function on $(0,\infty)$, non-negative, bounded, $|f| \le \eta$ and $\eta \in L_1(\mathbb{R}^d)$. If in addition $\eta(\cdot)\ln(|\cdot|+2) \in L_1(\mathbb{R}^d)$, then we say that η is a *radial log-majorant* of f.

The following results were proved in Feichtinger and Weisz [7] and [8].

Theorem 2.1. *Suppose that θ has a radial majorant η. Then for all $T>0$*

$$|\sigma_T^\theta f(x)| \le C\|\eta\|_1 Mf(x) \qquad (x \in \mathbb{R}^d).$$

Theorem 2.2. *Suppose that θ has a radial majorant. Then*

(i) *for all $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = \int_{\mathbb{R}^d} \theta(y)\,dy \cdot f(x)$$

at each Lebesgue points of f.

(ii) *for all $f \in L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$*

$$\lim_{T\to 0_+} \sigma_T^\theta f(x) = 0$$

for all $x \in \mathbb{R}^d$.

Proof. The proof of the first statement can be found in Feichtinger and Weisz [8].

Consider (ii). Since θ has a radial majorant, $\theta \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Therefore $\theta \in L_p(\mathbb{R}^d)$ $(1 \le p \le \infty)$. Let q the conjugate exponent of p i.e., $1/p + 1/q = 1$. Using Hölder's inequality

$$\begin{aligned}
\left|\sigma_T^\theta f(x)\right| &\le T^d \int_{\mathbb{R}^d} |f(x-t)|\,|\theta(Tt)|\,dt \\
&\le T^d \left(\int_{\mathbb{R}^d} |\theta(Tt)|^q\,dt\right)^{1/q} \left(\int_{\mathbb{R}^d} |f(x-t)|^p\,dt\right)^{1/p} \\
&= T^d \left(\int_{\mathbb{R}^d} |\theta(y)|^q T^{-d}\,dy\right)^{1/q} \|f\|_p \\
&= T^{d(1-1/q)} \|\theta\|_q \|f\|_p \to 0,
\end{aligned}$$

as $T \to 0_+$, because of $d(1-1/q) > 0$. $\square$

Almost every point is a Lebesgue point of $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$, so (i) holds almost everywhere.

The next Theorem can be found in Feichtinger and Weisz [7].

Theorem 2.3. *Suppose that $\theta \in L_1(\mathbb{R}^d)$. Then*

(i) *for all $f \in L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$*

$$\lim_{T\to\infty} \sigma_T^\theta f = \int_{\mathbb{R}^d} \theta(x)\,dx \cdot f \qquad \textit{in the } L_p(\mathbb{R}^d)\textit{-norm.}$$

(ii) *If in addition θ has a radial majorant, then for all $f \in L_p(\mathbb{R}^d)$ $(1 < p < \infty)$*

$$\lim_{T\to 0_+} \sigma_T^\theta f = 0 \qquad \textit{in the } L_p(\mathbb{R}^d)\textit{-norm.}$$

Proof. For the proof of (i) see Feichtinger and Weisz [7].

(ii) follows from Theorem 2.2 (ii), Theorem 2.1, (2.1) and Lebesgue dominated convergence theorem. □

The next lemma can be found in Li and Sun [13].

Lemma 2.4. *If g and γ have radial log-majorants, then $(g * \gamma)\ln(|\cdot|) \in L_1(\mathbb{R}^d)$ and $(|g| * |\gamma|)\ln(|\cdot|) \in L_1(\mathbb{R}^d)$.*

3. θ-summability on the variable Lebesgue spaces

For the variable Lebesgue spaces we can state similar theorems. A function $p(\cdot)$ belongs to $\mathcal{P}(\mathbb{R}^d)$ if $p : \mathbb{R}^d \to [1, \infty]$ and $p(\cdot)$ is measurable. Then we say that $p(\cdot)$ is an exponent function. Let

$$p_- := \inf\{p(x) : x \in \mathbb{R}^d\} \qquad \text{and} \qquad p_+ := \sup\{p(x) : x \in \mathbb{R}^d\}.$$

Set

$$\Omega_\infty := \{x \in \mathbb{R}^d : p(x) = \infty\}.$$

The modular generated by $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^d \setminus \Omega_\infty} |f(x)|^{p(x)} \, dx + \|f\|_{L_\infty(\Omega_\infty)},$$

where f is a measurable function. A measurable function f belongs to the $L_{p(\cdot)}(\mathbb{R}^d)$ space if there exists $\lambda > 0$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$. We can see that the modular $\varrho_{p(\cdot)}$ is not a norm. Define the $L_{p(\cdot)}(\mathbb{R}^d)$-norm by

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

Then $\|\cdot\|_{p(\cdot)}$ is a norm and the space $(L_{p(\cdot)}(\mathbb{R}^d), \|\cdot\|_{p(\cdot)})$ is a normed space. In case $p(\cdot) = p$ is a constant, then we get back the usual $L_p(\mathbb{R}^d)$ spaces.

We say that $r(\cdot)$ is locally log-Hölder continuous if there exists a constant C_0 such that for all $x, y \in \mathbb{R}^d$, $0 < \|x - y\|_2 < 1/2$,

$$|r(x) - r(y)| \le \frac{C_0}{-\log(\|x - y\|_2)}.$$

We denote this set by $LH_0(\mathbb{R}^d)$.

We say that $r(\cdot)$ is log-Hölder continuous at infinity if there exist constants C_∞ and r_∞ such that for all $x \in \mathbb{R}^d$

$$|r(x) - r_\infty| \le \frac{C_\infty}{\log(e + \|x\|_2)}.$$

We write briefly $r(\cdot) \in LH_\infty(\mathbb{R}^d)$. Let

$$LH(\mathbb{R}^d) := LH_0(\mathbb{R}^d) \cap LH_\infty(\mathbb{R}^d).$$

The following two results were proved in Cruz-Uribe and Fiorenza [4, p.27, p.35].

Theorem 3.1 (Hölder's inequality). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(x) + 1/q(x) = 1$. Then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$ and $g \in L_{q(\cdot)}(\mathbb{R}^d)$, $fg \in L_1(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} |f(x)g(x)|\, dx \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

Lemma 3.2. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $K \subset \mathbb{R}^d$, $|K| < \infty$, then $\chi_K \in L_{p(\cdot)}(\mathbb{R}^d)$ and*

$$\|\chi_K\|_{p(\cdot)} \leq |K| + 1.$$

We need also the next statement.

Theorem 3.3. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, then $L_{p(\cdot)} \subset W(L_1, \ell_\infty)(\mathbb{R}^d)$.*

Proof. Let $f \in L_{p(\cdot)}(\mathbb{R}^d)$ and $q(\cdot)$ the conjugate function of $p(\cdot)$. Then by Theorem 3.1 and Lemma 3.2,

$$\int_{[n,n+1)} |f(x)|\, dx \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \|\chi_{[n,n+1)}\|_{q(\cdot)} \leq 2C_{p(\cdot)} \|f\|_{p(\cdot)},$$

for $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, where $n + 1 = (n_1 + 1, \ldots, n_d + 1)$. Hence

$$\|f\|_{W(L_1,\ell_\infty)} \leq 2C_{p(\cdot)} \|f\|_{p(\cdot)} < \infty,$$

which implies the theorem. □

The following three theorems can be found in Cruz-Uribe and Fiorenza [4, p.56, p.44, p.42]

Theorem 3.4. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p_+ < \infty$, then the set of bounded functions with compact support is dense in $L_{p(\cdot)}(\mathbb{R}^d)$.*

Theorem 3.5. *If $p \in \mathcal{P}(\mathbb{R}^d)$ and $p_+(\mathbb{R}^d \setminus \Omega_\infty) < \infty$, then the following properties are equivalent:*

(i) *convergence in norm,*
(ii) *convergence in modular.*

Theorem 3.6. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, then*

$$L_{p(\cdot)}(\mathbb{R}^d) \subset L_{p_+}(\mathbb{R}^d) + L_{p_-}(\mathbb{R}^d),$$

i.e., for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$ there exist $f_1 \in L_{p_-}(\mathbb{R})$ and $f_2 \in L_{p_+}(\mathbb{R})$ such that $f = f_1 + f_2$.

The next theorem says that $\sigma_T^\theta f(x)$ converges at every Lebesgue point.

Theorem 3.7. *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and θ has a radial majorant, then*

(i) for all Lebesgue points of $f \in L_{p(\cdot)}(\mathbb{R}^d)$,

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = \int_{\mathbb{R}^d} \theta(y)\, dy \cdot f(x).$$

(ii) If in addition $p_+ < \infty$, then

$$\lim_{T\to 0_+} \sigma_T^\theta f(x) = 0$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$ and for all $x \in \mathbb{R}^d$.

Proof. To prove (i), let $f \in L_{p(\cdot)}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ be a Lebesgue point of f. Using Theorem 3.3, we have $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. By Theorem 2.2 we get that

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = \int_{\mathbb{R}^d} \theta(y)\, dy \cdot f(x).$$

Consider (ii). Let $f \in L_{p(\cdot)}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ arbitrary. Then by Theorem 3.6 there exist $f_1 \in L_{p_-}(\mathbb{R}^d)$ and $f_2 \in L_{p_+}(\mathbb{R}^d)$ such that $f = f_1 + f_2$. Since $p_+ < \infty$ we can use Theorem 2.2 to obtain

$$\lim_{T\to 0_+} \sigma_T^\theta f(x) = \lim_{T\to 0_+} \sigma_T^\theta f_1(x) + \lim_{T\to 0_+} \sigma_T^\theta f_2(x) = 0,$$

which proves the theorem. □

Of course, the convergence in (i) holds almost everywhere (see also Cruz-Uribe and Fiorenza [4, p.197]). The first and the second statement of the next theorem can be found in Cruz-Uribe and Fiorenza [4, p.199].

Theorem 3.8. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p_+ < \infty$, $1/p(x) + 1/q(x) = 1$. If θ has a radial majorant and the maximal operator is bounded on $L_{q(\cdot)}(\mathbb{R}^d)$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$*

(i)

$$\left\|\sigma_T^\theta f\right\|_{p(\cdot)} \le C \left\|f\right\|_{p(\cdot)} \qquad (T > 0). \tag{3.1}$$

(ii)

$$\lim_{T\to\infty} \sigma_T^\theta f = \int_{\mathbb{R}^d} \theta(x)\, dx \cdot f \quad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm.}$$

(iii) *If in addition $p_- > 1$, then*

$$\lim_{T\to 0_+} \sigma_T^\theta f = 0 \quad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

Proof. To prove (iii), fix $\varepsilon > 0$. By Theorem 3.4 there exists a bounded function g with compact support, such that $\|f - g\|_{p(\cdot)} < \varepsilon$. Using (3.1) we have

$$\left\|\sigma_T^\theta f\right\|_{p(\cdot)} \le \left\|\sigma_T^\theta (f-g)\right\|_{p(\cdot)} + \left\|\sigma_T^\theta g\right\|_{p(\cdot)} < C\varepsilon + \left\|\sigma_T^\theta g\right\|_{p(\cdot)}.$$

So it is enough to show that

$$\lim_{T\to 0_+} \left\|\sigma_T^\theta g\right\|_{p(\cdot)} = 0.$$

Since $p_+ < \infty$, then by Theorem 3.5 we have to show that

$$\lim_{T\to 0_+} \int_{\mathbb{R}^d} |\sigma_T^\theta g(x)|^{p(x)}\, dx = 0.$$

Let

$$g_0(x) := \frac{g(x)}{\|\theta\|_1 \|g\|_\infty}.$$

Then $\|g_0\|_\infty \le 1/\|\theta\|_1$ and

$$|\sigma_T^\theta g_0(x)| = |(g_0 * \theta_T)(x)| \le \|g_0\|_\infty \|\theta_T\|_1 = \|g_0\|_\infty \|\theta\|_1 \le 1.$$

Therefore

$$\begin{aligned}\lim_{T\to 0_+}\int_{\mathbb{R}^d}|\sigma_T^\theta g(x)|^{p(x)}\,dx &= \lim_{T\to 0_+}\int_{\mathbb{R}^d}(\|\theta\|_1\|g\|_\infty)^{p(x)}|\sigma_T^\theta g_0(x)|^{p(x)}\,dx\\ &\le (\|\theta\|_1\|g\|_\infty+1)^{p_+}\lim_{T\to 0_+}\int_{\mathbb{R}^d}|\sigma_T^\theta g_0(x)|^{p_-}\,dx.\end{aligned}$$

Here $1<p_-<\infty$ and $g_0\in L_{p_-}(\mathbb{R}^d)$, therefore by Theorem 2.3 we get that

$$\lim_{T\to 0_+}\int_{\mathbb{R}^d}|\sigma_T^\theta g_0(x)|^{p_-}\,dx=0,$$

which proves the theorem. □

The next theorem about the boundedness of the classical Hardy-Littlewood maximal operator in variable Lebesgue spaces can be found in Cruz-Uribe and Fiorenza [4, p.89].

Theorem 3.9. *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^d)$ and $1/p(\cdot)\in LH(\mathbb{R}^d)$.*

(i) *Then for all $f\in L_{p(\cdot)}(\mathbb{R}^d)$ and $t>0$*

$$\left\|t\chi_{\{x\in\mathbb{R}^d:Mf(x)>t\}}\right\|_{p(\cdot)}\le C\left\|f\right\|_{p(\cdot)}.$$

(ii) *If in addition $p_->1$, then for all $f\in L_{p(\cdot)}(\mathbb{R}^d)$*

$$\left\|Mf\right\|_{p(\cdot)}\le C\left\|f\right\|_{p(\cdot)}.$$

Remark 3.10. If $1/p(\cdot)\in LH(\mathbb{R}^d)$ and $p_+<\infty$, then $1/q(\cdot)\in LH(\mathbb{R}^d)$ and $q_->1$ so the maximal operator is bounded on $L_{q(\cdot)}(\mathbb{R}^d)$. Therefore if $1/p(\cdot)\in LH(\mathbb{R}^d)$, $p_+<\infty$ and θ has a radial majorant, then the hypotheses of Theorem 3.8 remain true.

4. The continuous wavelet transform

The continuous wavelet transform of f with respect to a wavelet g is defined by

$$W_gf(x,s):=|s|^{-d/2}\int_{\mathbb{R}^d}f(t)\overline{g(s^{-1}(t-x))}\,dt=\langle f,T_xD_sg\rangle,$$

$(x\in\mathbb{R}^d, s\in\mathbb{R}, s\neq 0)$ when the integral does exist. We suppose that
$g,\gamma\in L_2(\mathbb{R}^d)$ and

$$\int_0^\infty|\widehat{g}(s\omega)||\widehat{\gamma}(s\omega)|\,\frac{ds}{s}<\infty$$

for almost $\omega\in\mathbb{R}^d$ with $\|\omega\|_2=1$. If

$$C_{g,\gamma}:=\int_0^\infty\overline{\widehat{g}}(s\omega)\widehat{\gamma}(s\omega)\,\frac{ds}{s}$$

is independent of ω, then the inversion formula holds for all $f\in L_2(\mathbb{R}^d)$:

$$\int_0^\infty\int_{\mathbb{R}^d}W_gf(x,s)T_xD_s\gamma\,\frac{dxds}{s^{d+1}}=C_{g,\gamma}\cdot f,$$

where the equality is understood in a vector-valued weak sense. Consider the operators

$$\rho_S f := \int_S^\infty \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \, \frac{dxds}{s^{d+1}}$$

and

$$\rho_{S,T} f := \int_S^T \int_{\mathbb{R}^d} W_g f(x,s) T_x D_s \gamma \, \frac{dxds}{s^{d+1}},$$

where $0 < S < T < \infty$. Let

$$C'_{g,\gamma} := - \int_{\mathbb{R}^d} (g^* * \gamma)(x) \ln(|x|) \, dx,$$

where $g^*(x) = \overline{g(-x)}$ is the involution operator. If g and γ both have radial log-majorants, then $C'_{g,\gamma}$ is finite by Lemma 2.4.

Li and Sun [13] proved that if g and γ radial, $\int_{\mathbb{R}^d} (g^* * \gamma)(x) \, dx = 0$, and both have a radial log-majorant, then for any $f \in L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$

$$\lim_{S \to 0_+, T \to \infty} \rho_{S,T} f(x) = \lim_{S \to 0_+} \rho_S f(x) = C'_{g,\gamma} f(x)$$

at every Lebesgue point of f. Moreover, if $1 < p < \infty$, then the convergence holds in the $L_p(\mathbb{R}^d)$-norm for all $f \in L_p(\mathbb{R}^d)$. If $p = 1$, then

$$\lim_{S \to 0_+} \rho_S f = C'_{g,\gamma} f \qquad \text{in the } L_1(\mathbb{R}^d)\text{-norm}$$

for all $f \in L_1(\mathbb{R}^d)$. Under some similar conditions Weisz [19] proved similar results. In this paper we investigate the same questions on the variable Lebesgue spaces and we will prove similar theorems. Of course, $C_{g,\gamma} = C'_{g,\gamma}$ under some conditions (see Li and Sun [13]).

5. Convergence of ϱ_S and $\varrho_{S,T}$

We will denote the surface area of the unit ball in $\mathbb{R}^d$ by ω_{d-1}. The next theorem plays central role in this chapter. Under some conditions we lead back $\varrho_S f$ to a θ-summation.

Theorem 5.1. *Suppose that g, γ have radial log-majorants and*

$$\int_{\mathbb{R}^d} (g^* * \gamma)(x) \, dx = 0.$$

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p_+ < \infty$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$

$$\varrho_S f = \sigma^\theta_{1/S} f \quad (S > 0),$$

where θ is defined later in the proof.

Proof. Let $y \in \mathbb{R}^d$ be arbitrary and decompose $\varrho_S f(y)$

$$\begin{aligned}
\varrho_S f(y) &= \int_S^\infty \int_{\mathbb{R}^d} \frac{1}{s^{2d+1}} \int_{\mathbb{R}^d} f(t)\overline{g\left(\frac{t-x}{s}\right)}\gamma\left(\frac{y-x}{s}\right) dt dx ds \\
&= \int_S^\infty \int_{\|y-t\|_2<S} \frac{1}{s^{2d+1}} \int_{\mathbb{R}^d} f(t)\overline{g\left(\frac{t-x}{s}\right)}\gamma\left(\frac{y-x}{s}\right) dx dt ds \\
&- \int_0^S \int_{\|y-t\|_2 \geq S} \frac{1}{s^{2d+1}} \int_{\mathbb{R}^d} f(t)\overline{g\left(\frac{t-x}{s}\right)}\gamma\left(\frac{y-x}{s}\right) dx dt ds \\
&+ \int_0^\infty \int_{\|y-t\|_2 \geq S} \frac{1}{s^{2d+1}} \int_{\mathbb{R}^d} f(t)\overline{g\left(\frac{t-x}{s}\right)}\gamma\left(\frac{y-x}{s}\right) dx dt ds \\
&=: I - II + III.
\end{aligned}$$

We can write I and II as a convolution, similarly as in Li and Sun [13]:

$$I = (f * \varphi_{1/S})(y),$$

where

$$\varphi(t) := \int_1^\infty H\left(\frac{t}{u}\right) \frac{1}{u^{d+1}} \chi_{B(0,1)}(t)\, du$$

and $H := g^* * \gamma$. Since γ has radial log-majorant, $\gamma \in L_\infty(\mathbb{R}^d)$ and since $g \in L_1(\mathbb{R}^d)$, $H \in L_\infty(\mathbb{R}^d)$. Therefore if $t \in B(0,1)$, then

$$|\varphi(t)| \leq \|H\|_\infty \int_1^\infty \frac{1}{u^{d+1}}\, du = C\|H\|_\infty < \infty.$$

If $t \notin B(0,1)$, then $\varphi(t) = 0$. Thus $C\|H\|_\infty \chi_{B(0,1)}$ is a radial majorant of φ.

$$II = (f * \psi_{1/S})(y),$$

where

$$\psi(t) := \int_0^1 H\left(\frac{t}{u}\right) \frac{1}{u^{d+1}} \chi_{\mathbb{R}^d \setminus B(0,1)}(t)\, du.$$

Let $G := |g| * |\gamma|$. Then $H \leq G$, and since g, γ have radial log-majorants, Lemma 2.4 implies that H, G have radial log-majorants, too. Since G is radial, there exists η such that $G(x) = \eta(\|x\|_2)$. If $t \in B(0,1)$, then $\psi(t) = 0$. If $t \in \mathbb{R}^d \setminus B(0,1)$, then

$$\begin{aligned}
|\psi(t)| &\leq \int_0^1 G\left(\frac{t}{u}\right) \frac{1}{u^{d+1}}\, du = \int_0^1 \eta\left(\frac{\|t\|_2}{u}\right) \frac{1}{u^{d+1}}\, du \\
&= \frac{1}{\|t\|_2^d} \int_{\|t\|_2}^\infty \eta(s) s^{d-1}\, ds =: \zeta(t)
\end{aligned}$$

and let

$$\zeta(t) := \int_1^\infty \eta(s) s^{d-1}\, ds \leq \frac{1}{\omega_{d-1}} \|G\|_1 < \infty \qquad (t \in B(0,1)).$$

It is easy to see that $|\psi| \leq \zeta$, ζ is radial, non-increasing on $(0,\infty)$ and bounded. Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^d} \zeta(t)\,dt &= \int_{B(0,1)} \zeta(t)\,dt + \int_{\mathbb{R}^d\setminus B(0,1)} \zeta(t)\,dt \\
&= C + \int_{\mathbb{R}^d\setminus B(0,1)} \frac{1}{\|t\|_2^d} \int_{\|t\|_2}^{\infty} \eta(s)s^{d-1}\,dsdt \\
&\leq C + \omega_{d-1} \int_1^{\infty} \frac{1}{r} \int_r^{\infty} \eta(s)s^{d-1}\,dsdr \\
&= C + \omega_{d-1} \int_1^{\infty} \eta(s)s^{d-1} \int_1^{s} \frac{1}{r}\,drds \\
&= C + \int_{\mathbb{R}^d\setminus B(0,1)} G(t)\ln(|t|)\,dt < \infty,
\end{aligned}$$

i.e., ζ is integrable so ζ is a radial majorant of ψ.

We will show that $III = 0$. To apply Fubini's theorem we will verify that

$$\int_0^{\infty} \int_{\|y-t\|_2 \geq S} \frac{1}{s^{d+1}} |f(t)| G\left(\frac{y-t}{s}\right)\,dtds < \infty. \tag{5.1}$$

Since G is radial

$$\begin{aligned}
&\int_{\|y-t\|_2 \geq S} |f(t)| \int_0^{\infty} \frac{1}{s^{d+1}} \eta\left(\frac{\|y-t\|_2}{s}\right)\,dsdt \\
&= \int_{\|y-t\|_2 \geq S} |f(t)| \int_0^{\infty} \frac{1}{\|y-t\|_2^d} \eta(u)u^{d-1}\,dudt \\
&= \frac{1}{\omega_{d-1}} \|G\|_1 \int_{\|y-t\|_2 \geq S} |f(t)| \frac{1}{\|y-t\|_2^d}\,dt. \qquad (5.2)
\end{aligned}$$

By Theorem 3.1

$$\int_{\|y-t\|_2 \geq S} |f(t)| \frac{1}{\|y-t\|_2^d}\,dt \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \left\| \frac{1}{\|y-\cdot\|_2^d} \chi_{\{\|y-\cdot\|_2 \geq S\}} \right\|_{q(\cdot)},$$

where $1/p(x) + 1/q(x) = 1$ $(x \in \mathbb{R}^d)$. Let $\lambda := 1/S^d$. Then

$$\frac{1}{\lambda\|y-t\|_2^d} \leq \frac{1}{\lambda S^d} = 1 \qquad \text{and} \qquad \left(\frac{1}{\lambda\|y-t\|_2^d}\right)^{q(t)} \leq \left(\frac{1}{\lambda\|y-t\|_2^d}\right)^{q_-}.$$

If $p_+ < \infty$, then $q_- > 1$ and

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\frac{\chi_{\{\|y-t\|_2 \geq S\}}}{\lambda\|y-t\|_2^d}\right)^{q(t)} dt &\leq \int_{\|y-t\|_2 \geq S} \left(\frac{1}{\lambda\|y-t\|_2^d}\right)^{q_-} dt \\
&= S^{dq_-} \int_{\|y-t\|_2 \geq S} \frac{1}{\|y-t\|_2^{dq_-}}\,dt \\
&= \omega_{d-1} S^{dq_-} \int_S^{\infty} u^{-dq_- + d - 1}\,du < \infty.
\end{aligned}$$

Moreover,

$$\frac{\frac{1}{\|y-t\|_2^d}\chi_{\{\|y-t\|_2\geq S\}}(t)}{\lambda} \leq \frac{1}{\lambda S^d} = 1,$$

in other words,

$$\left\| \frac{\frac{1}{\|y-\cdot\|_2^d}\chi_{\{\|y-\cdot\|_2\geq S\}}}{\lambda} \right\|_{L_\infty(\Omega_\infty)} \leq 1,$$

where

$$\Omega_\infty = \{x \in \mathbb{R}^d : q(x) = \infty\}.$$

So

$$\varrho_{q(\cdot)}\left(\frac{\frac{1}{\|y-\cdot\|_2^d}\chi_{\{\|y-\cdot\|_2\geq S\}}}{\lambda} \right) < \infty$$

and

$$\left\| \frac{1}{\|y-\cdot\|_2^d}\chi_{\{\|y-\cdot\|_2\geq S\}} \right\|_{q(\cdot)} < \infty.$$

We get that (5.2) is finite so we can apply Fubini's theorem. Since H is radial, there exists ν such that $H(x) = \nu(\|x\|_2)$ and

$$\begin{aligned}
III &= \int_0^\infty \int_{\|y-t\|_2\geq S} f(t)\frac{1}{s^{d+1}} H\left(\frac{y-t}{s}\right) dtds \\
&= \int_{\|y-t\|_2\geq S} f(t) \int_0^\infty \frac{1}{s^{d+1}} \nu\left(\frac{\|y-t\|_2}{s}\right) dsdt \\
&= \int_{\|y-t\|_2\geq S} f(t)\frac{1}{\|y-t\|_2^d} \int_0^\infty \nu(u)u^{d-1}\, dudt \\
&= \frac{1}{\omega_{d-1}} \int_{\|y-t\|_2\geq S} f(t)\frac{1}{\|y-t\|_2^d} \int_{\mathbb{R}^d} H(u)\, dudt = \\
&= \frac{1}{\omega_{d-1}} \int_{\|y-t\|_2\geq S} f(t)\frac{1}{\|y-t\|_2^d} \int_{\mathbb{R}^d} (g^* * \gamma)(u)\, dudt = 0.
\end{aligned}$$

We have that

$$\varrho_S f(y) = (f * \varphi_{1/S})(y) - (f * \psi_{1/S})(y) = \big(f * (\varphi_{1/S} - \psi_{1/S})\big)(y) =: \sigma^\theta_{1/S} f(y),$$

where

$$\theta(y) := \varphi(y) - \psi(y).$$

Since φ and ψ have radial majorants, θ has, too. □

Using the previous theorem we can prove the convergence of $\varrho_S f$ and $\varrho_{S,T} f$ at Lebesgue points, almost everywhere and in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm.

Theorem 5.2. *Suppose that g, γ have radial log-majorants and*

$$\int_{\mathbb{R}^d} (g^* * \gamma)(x)\, dx = 0.$$

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p_+ < \infty$, then for all Lebesgue points of $f \in L_{p(\cdot)}(\mathbb{R}^d)$,

(i)
$$\lim_{S\to 0_+} \varrho_S f(x) = C'_{g,\gamma} \cdot f(x).$$

(ii)
$$\lim_{S\to 0_+, T\to\infty} \varrho_{S,T} f(x) = C'_{g,\gamma} \cdot f(x).$$

Proof. By Theorem 5.1 and Theorem 3.7 we have

$$\lim_{S\to 0_+} \varrho_S f(x) = \lim_{S\to 0_+} \sigma^\theta_{1/S} f(x) = \int_{\mathbb{R}^d} \theta(y)\, dy \cdot f(x),$$

i.e., it is enough to prove that

$$\int_{\mathbb{R}^d} \theta(y)\, dy = C'_{g,\gamma}.$$

We have

$$\int_{\mathbb{R}^d} \theta(y)\, dy = \int_{\mathbb{R}^d} \varphi(y)\, dy - \int_{\mathbb{R}^d} \psi(y)\, dy.$$

Here

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi(y)\, dy &= \int_{B(0,1)} \int_1^\infty H\left(\frac{y}{u}\right) \frac{1}{u^{d+1}}\, du dy \\
&= \omega_{d-1} \int_0^1 r^{d-1} \int_1^\infty \nu\left(\frac{r}{u}\right) \frac{1}{u^{d+1}}\, du dr \\
&= \omega_{d-1} \int_0^1 \frac{1}{r} \int_0^r \nu(t) t^{d-1}\, dt dr \\
&= \omega_{d-1} \int_0^1 t^{d-1} \nu(t) \int_t^1 \frac{1}{r}\, dr dt \\
&= -\int_{B(0,1)} H(t) \ln(|t|)\, dt \\
&= -\int_{B(0,1)} (g^* * \gamma)(t) \ln(|t|)\, dt
\end{aligned}$$

and we have similarly that

$$\int_{\mathbb{R}^d} \psi(y)\, dy = \int_{\mathbb{R}^d \setminus B(0,1)} (g^* * \gamma)(t) \ln(|t|)\, dt,$$

i.e.

$$\begin{aligned}
\int_{\mathbb{R}^d} \theta(y)\, dy &= -\int_{B(0,1)} (g^* * \gamma)(t) \ln(|t|)\, dt - \int_{\mathbb{R}^d \setminus B(0,1)} (g^* * \gamma)(t) \ln(|t|)\, dt \\
&= -\int_{\mathbb{R}^d} (g^* * \gamma)(t) \ln(|t|)\, dt = C'_{g,\gamma}.
\end{aligned}$$

To prove (ii), observe that

$$\varrho_{S,T} f(x) = \varrho_S f(x) - \varrho_T f(x).$$

Then use Theorem 5.1 and Theorem 3.7 to get

$$\lim_{S\to 0_+, T\to\infty} \varrho_{S,T} f(x) = \lim_{S\to 0_+} \varrho_S f(x) - \lim_{T\to\infty} \sigma^\theta_{1/T} f(x) = C'_{g,\gamma} \cdot f(x) - 0,$$

which proves the theorem. □

Corollary 5.3. *Suppose that g, γ have radial log-majorants and*

$$\int_{\mathbb{R}^d} (g^* * \gamma)(x)\, dx = 0.$$

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $p_+ < \infty$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$

(i)

$$\lim_{S\to 0_+} \varrho_S f = C'_{g,\gamma} f \qquad a.e.$$

(ii)

$$\lim_{S\to 0_+, T\to\infty} \varrho_{S,T} f = C'_{g,\gamma} f \qquad a.e.$$

Theorem 5.4. *Suppose that g, γ have radial log-majorants and*

$$\int_{\mathbb{R}^d} (g^* * \gamma)(x)\, dx = 0.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p_+ < \infty$, $1/p(x) + 1/q(x) = 1$. If the maximal operator is bounded on $L_{q(\cdot)}(\mathbb{R}^d)$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$,

(i)

$$\lim_{S\to 0_+} \varrho_S f = C'_{g,\gamma} \cdot f \qquad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm.}$$

(ii) *If in addition $p_- > 1$, then for all $f \in L_{p\cdot}(\mathbb{R}^d)$*

$$\lim_{S\to 0_+, T\to\infty} \varrho_{S,T} f = C'_{g,\gamma} \cdot f \qquad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm.}$$

Proof. To prove (i), use Theorem 5.1 and Theorem 3.8

$$\lim_{S\to 0_+} \varrho_S f = \lim_{S\to 0_+} \sigma^\theta_{1/S} f = \int_{\mathbb{R}^d} \theta(x)\, dx \cdot f \qquad \text{in the } L_{p(\cdot)}(\mathbb{R}^d)\text{-norm.}$$

We have seen, that $\int_{\mathbb{R}^d} \theta(x)\, dx = C'_{g,\gamma}$.

We can prove (ii) similarly. Use Theorem 5.1 and Theorem 3.8 to obtain

$$\lim_{S\to 0_+, T\to\infty} \varrho_{S,T} f = \lim_{S\to 0_+} \varrho_S f - \lim_{T\to\infty} \varrho_T f = C'_{g,\gamma} \cdot f$$

in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm. The proof of the theorem is complete. □

By Remark 3.10 if we suppose that $1/p(\cdot) \in LH(\mathbb{R}^d)$ and $p_+ < \infty$, then the maximal operator is bonded on $L_{q(\cdot)}(\mathbb{R}^d)$. Therefore we have

Corollary 5.5. *Suppose that g, γ have radial log-majorants and*

$$\int_{\mathbb{R}^d} (g^* * \gamma)(x)\, dx = 0.$$

If $1/p(\cdot) \in LH(\mathbb{R}^d)$, $p_+ < \infty$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$,

(i) $\lim_{S\to 0_+} \varrho_S f = C'_{g,\gamma} \cdot f$ *in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm.*

(ii) If in addition $p_- > 1$, then for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$

$$\lim_{S\to 0_+, T\to\infty} \varrho_{S,T} f = C'_{g,\gamma} \cdot f \qquad \textit{in the } L_{p(\cdot)}(\mathbb{R}^d)\textit{-norm.}$$

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