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On systems of semilinear hyperbolic functional equations

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Abstract. We consider a system of second order semilinear hyperbolic functional differential equations where the lower order terms contain functional dependence on the unknown function. Existence of solutions for $t \in (0,T)$ and $t \in (0,\infty)$, further, examples and some qualitative properties of the solutions in $(0,\infty)$ are shown.

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1. Introduction

In the present work we shall consider weak solutions of initial-boundary value problems of the form

$$u_j''(t) + Q_j(u(t)) + \varphi(x)D_j h(u(t)) + H_j(t,x;u) + G_j(t,x;u,u') = F_j, \tag{1.1}$$

$$t > 0, \quad x \in \Omega, \quad j = 1,...,N$$

$$u(0) = u^{(0)}, \quad u'(0) = u^{(1)} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and we use the notations $u(t) = (u_1(t),...,u_N(t))$, $u(t) = (u_1(t,x),...,u_N(t,x))$, $u' = (u_1',...,u_N') = D_t u = (D_t u_1,...,D_t u_N)$, $u'' = D_t^2 u$, Q_j is a linear second order symmetric elliptic differential operator in the variable x; h is a C^1 function having certain polynomial growth, H_j and G_j contain nonlinear functional (non-local) dependence on u and u', with some polynomial growth.

There are several papers on semilinear hyperbolic differential equations, see, e.g., [3], [4], [10], [14] and the references there. Semilinear hyperbolic functional equations were studied, e.g. in [5], [6], [7], with certain non-local terms, generally in the form of particular integral operators containing the unknown function. First order quasilinear evolution equations with non-local terms were considered, e.g., in [13] and [15], second

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order quasilinear evolution equations with non-local terms were considered in [11], by using the theory of monotone type operators (see [2], [9], [16]).

This work was motivated by the classical book [9] of J.L. Lions on nonlinear PDEs where a single equation was considered in a particular case (semilinear hyperbolic differential equation). We shall use ideas of the above work.

Semilinear hyperbolic functional equations were considered in a previous work of the author (see [12]).

2. Existence in $(0, T)$

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain with sufficiently smooth boundary, and let $Q_T = (0, T) \times \Omega$. Denote by $W^{1,2}(\Omega)$ the Sobolev space with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^n |D_j u|^2 + |u|^2 \right) dx \right]^{1/2}.$$

Further, let $V_j \subset W^{1,2}(\Omega)$ be closed linear subspaces of $W^{1,2}(\Omega)$, $V_j^\star$ the dual space of V_j, $V = (V_1, ..., V_N)$, $V^\star = (V_1^\star, ..., V_N^\star)$, $H = L^2(\Omega) \times ... \times L^2(\Omega)$, the duality between $V_j^\star$ and V_j (and between $V^\star$ and V) will be denoted by $\langle \cdot, \cdot \rangle$, the scalar product in $L^2(\Omega)$ and H will be denoted by $(\cdot, \cdot)$. Denote by $L^2(0, T; V_j)$ and $L^2(0, T; V)$ the Banach space of measurable functions $u : (0, T) \to V_j$, $u : (0, T) \to V$, respectively, with the norm

$$\|u_j\|_{L^2(0,T;V_j)} = \left[\int_0^T \|u_j(t)\|_{V_j}^2 dt \right]^{1/2}, \quad \|u\|_{L^2(0,T;V)} = \left[\int_0^T \|u(t)\|_V^2 dt \right]^{1/2},$$

respectively.

Similarly, $L^\infty(0, T; V_j)$, $L^\infty(0, T; V)$, $L^\infty(0, T; L^2(\Omega))$, $L^\infty(0, T; H)$ is the set of measurable functions $u_j : (0, T) \to V_j$, $u : (0, T) \to V$, $u_j : (0, T) \to L^2(\Omega)$, $u : (0, T) \to H$, respectively, with the $L^\infty(0, T)$ norm of the functions $t \mapsto \|u_j(t)\|_{V_j}$, $t \mapsto \|u(t)\|_V$, $t \mapsto \|u_j(t)\|_{L^2(\Omega)}$, $t \mapsto \|u(t)\|_H$, respectively.

Now we formulate the assumptions on the functions in (1.1).

(A_1). $Q : V \to V^\star$ is a linear continuous operator defined by

$$\langle Q(u), v \rangle = \sum_{j=1}^N \langle Q_j(u), v_j \rangle = \sum_{j=1}^N \left[\sum_{k=1}^N \langle Q_{jk}(u_k), v_j \rangle \right],$$

$$u = (u_1, ..., u_N), \quad v = (v_1, ..., v_N),$$

where $Q_{jk} : W^{1,2}(\Omega) \to [W^{1,2}(\Omega)]^\star$ are continuous linear operators satisfying

$$\langle Q_{jk}(u_k), v_j \rangle = \langle Q_{jk}(v_j), u_k \rangle, \quad Q_{jk} = Q_{kj}, \text{ thus } \langle Q(u), v \rangle = \langle Q(v), u \rangle$$

for all $u, v \in V$ and

$$\langle Q(u), u \rangle \geq c_0 \|u\|_V^2 \text{ with some constant } c_0 > 0.$$

(A_2). $\varphi : \Omega \to \mathbb{R}$ is a measurable function satisfying

$$c_1 \leq \varphi(x) \leq c_2 \text{ for a.a. } x \in \Omega$$

with some positive constants c_1, c_2.

(A_3). $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function satisfying

$$h(\eta) \geq 0, \quad |D_j h(\eta)| \leq \text{const}|\eta|^\lambda \text{ for } |\eta| > 1 \text{ where}$$
$$1 < \lambda \leq \lambda_0 = \frac{n}{n-2} \text{ if } n \geq 3, \quad 1 < \lambda < \infty \text{ if } n = 2.$$

(A_3'). $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function satisfying with some positive constants c_3, c_4

$$h(\eta) \geq c_3|\eta|^{\lambda+1}, \quad |D_j h(\eta)| \leq c_4|\eta|^\lambda \text{ for } |\eta| > 1, \quad n \geq 3 \text{ where } \lambda > \lambda_0 = \frac{n}{n-2},$$
$$|D_j h(\eta)| \leq c_4|\eta|^\lambda \quad \text{ for } |\eta| > 1, \quad n = 2 \text{ where } 1 < \lambda < \infty.$$

(A_4). $H_j : Q_T \times [L^2(Q_T)]^N \to \mathbb{R}$ are functions for which $(t, x) \mapsto H_j(t, x; u)$ is measurable for all fixed $u \in H$, H_j has the Volterra property, i.e. for all $t \in [0, T]$, $H_j(t, x; u)$ depends only on the restriction of u to $(0, t)$; the following inequality holds for all $t \in [0, T]$ and $u \in H$:

$$\int_\Omega |H_j(t, x; u)|^2 dx \leq c^\star \left[\int_0^t \int_\Omega h(u(\tau)) dx d\tau + \int_\Omega h(u) dx\right].$$

Finally, $(u^{(k)}) \to u$ in $[L^2(Q_T)]^N$ and $(u^{(k)}) \to u$ a.e. in Q_T imply

$$H_j(t, x; u^{(k)}) \to H_j(t, x; u) \text{ for a.a. } (t, x) \in Q_T.$$

(A_5). $G_j : Q_T \times [L^2(Q_T)]^N \times L^\infty(0, T; H) \to \mathbb{R}$ is a function satisfying: $(t, x) \mapsto G_j(t, x; u, w)$ is measurable for all fixed $u \in [L^2(Q_T)]^N$, $w \in L^\infty(0, T; H)$, G_j has the Volterra property: for all $t \in [0, T]$, $G_j(t, x; u, w)$ depends only on the restriction of u, w to $(0, t)$ and

$$G_j(t, x; u, u') = \varphi_j(t, x; u) u_j'(t) + \psi_j(t, x; u, u')$$

where

$$\varphi_j \geq 0, \quad |\varphi_j(t, x; u)| \leq \text{const} \tag{2.1}$$

if (A_3) is satisfied.

(A_5') If (A_3') is satisfied, we assume instead of the second inequality in (2.1)

$$\int_\Omega |\varphi_j(t, x; u)|^2 dx \leq \text{const} \left[\int_{Q_t} |u|^{2\mu} d\tau dx + \int_\Omega |u|^{2\mu} dx\right] \tag{2.2}$$

where $\mu \leq \frac{n+1}{n-1} \frac{\lambda-1}{\lambda+1}$.

Further, on ψ_j we assume

$$\int_\Omega |\psi_j(t, x; u, u')|^2 dx \leq c_1 + c_2 \int_{Q_t} |u'|^2 dx d\tau$$

with some constants c_1, c_2.

Further, if $(u^{(\nu)}) \to u$ in $[L^2(Q_T)]^N$ then

$$\varphi_j(t, x; u^{(\nu)}) \to \varphi_j(t, x; u) \text{ for a.a. } (t, x) \in Q_T$$

and if

$$(u^{(\nu)}) \to u \text{ in } [L^2(Q_T)]^N \text{ and a.e. in } Q_T, \quad (w^{(\nu)}) \to w$$

weakly in $L^\infty(0,T;H)$ in the sense that for all fixed $g_1 \in L^1(0,T;H)$

$$\int_0^T \langle g_1(t), w^{(\nu)}(t)\rangle dt \to \int_0^T \langle g_1(t), w(t)\rangle dt,$$

then for a.a. $(t,x) \in Q_T$

$$\psi_j(t,x;u^{(\nu)},w^{(\nu)}) \to \psi_j(t,x;u,w).$$

Theorem 2.1. *Assume (A_1), (A_2), (A_3), (A_4), (A_5). Then for all $F \in L^2(0,T;H)$, $u^{(0)} \in V$, $u^{(1)} \in H$ there exists $u \in L^\infty(0,T;V)$ such that*

$$u' \in L^\infty(0,T;H), \quad u'' \in L^2(0,T;V^\star),$$

u satisfies the system (1.1) in the sense: for a.a $t \in [0,T]$, all $v \in V$

$$\langle u_j''(t), v_j\rangle + \langle Q_j(u(t)), v_j\rangle + \int_\Omega \varphi(x) D_j h(u(t)) v_j dx + \int_\Omega H_j(t,x;u) v_j dx + \tag{2.3}$$
$$\int_\Omega G_j(t,x;u,u') v_j dx = (F_j(t), v_j) \quad j = 1,...,N$$

and the initial condition (1.2) is fulfilled.

If (A_1), (A_2), (A_3'), (A_4), (A_5) are satisfied then for all $F \in L^2(0,T;H)$, $u^{(0)} \in V \cap [L^{\lambda+1}(\Omega)]^N$, $u^{(1)} \in H$ there exists $u \in L^\infty(0,T;V \cap [L^{\lambda+1}(\Omega)]^N)$ such that

$$u' \in L^\infty(0,T;H),$$
$$u'' \in L^2(0,T;V^\star) + L^\infty(0,T;[L^{\frac{\lambda+1}{\lambda}}(\Omega)]^N) \subset L^2\left(0,T;[V \cap (L^{\lambda+1}(\Omega))^N]^\star\right)$$

and u satisfies (1.1) in the sense: for a.a $t \in [0,T]$, all $v_j \in V_j \cap L^{\lambda+1}(\Omega)$ (2.3) holds, further, the initial condition (1.2) is fulfilled.

Proof. We apply Galerkin's method. Let $w_1^{(j)}, w_2^{(j)}, \dots$ be a linearly independent system in V_j if (A_3) is satisfied and in $V_j \cap L^{\lambda+1}(\Omega)$ if (A_3') is satisfied such that the linear combinations are dense in V_j and $V_j \cap L^{\lambda+1}(\Omega)$, respectively. We want to find the m-th approximation of u in the form

$$u_j^{(m)}(t) = \sum_{l=1}^m g_{lm}^{(j)}(t) w_l^{(j)} \quad (j = 1,2,\dots,N) \tag{2.4}$$

where $g_{lm}^{(j)} \in W^{2,2}(0,T)$ if (A_3) holds and $g_{lm}^{(j)} \in W^{2,2}(0,T) \cap L^\infty(0,T)$ if (A_3') holds such that

$$\langle (u_j^{(m)})''(t), w_k^{(j)}\rangle + \langle Q(u^{(m)}(t)), w_k^{(j)}\rangle + \int_\Omega \varphi(x) D_j h(u^{(m)}(t)) w_k^{(j)} dx \tag{2.5}$$
$$+ \int_\Omega H_j(t,x;u^{(m)}) w_k^{(j)} dx + \int_\Omega G_j(t,x;u^{(m)},(u^{(m)})') w_k^{(j)} dx = \langle F_j(t), w_k^{(j)}\rangle,$$
$$k = 1,\dots,m, \quad j = 1,\dots,N$$

$$u_j^{(m)}(0) = u_{j0}^{(m)}, \quad (u_j^{(m)})'(0) = u_{j1}^{(m)} \tag{2.6}$$

where $u_{j0}^{(m)}$, $u_{j1}^{(m)}$ $(j = 1,2,\dots,N)$ are linear combinations of $w_1^{(j)}, w_2^{(j)} \dots, w_m^{(j)}$ satisfying

$$(u_{j0}^{(m)}) \to u_j^{(0)} \text{ in } V_j \text{ and } V_j \cap L^{\lambda+1}(\Omega), \text{ respectively, as } m \to \infty \text{ and} \tag{2.7}$$

$$(u_{j1}^{(m)}) \to u_j^{(1)} \text{ in } H \text{ as } m \to \infty. \tag{2.8}$$

It is not difficult to show that all the conditions of the existence theorem for a system of functional differential equations with Carathéodory conditions are satisfied.

Thus, by using the Volterra property of G and H, we obtain that there exists a solution of (2.5), (2.6) in a neighbourhood of 0 (see [8]). Further, the maximal solution of (2.5), (2.6) is defined in $[0,T]$. Indeed, multiplying (2.5) by $[g_{lm}^{(j)}]'(t)$ and taking the sum with respect to j, and k we obtain

$$\begin{aligned} &\langle (u^{(m)})''(t), (u^{(m)})'(t)\rangle + \langle Q(u^{(m)}(t)), (u^{(m)})'(t)\rangle \\ &+ \int_\Omega \varphi(x) \frac{d}{dt}[h(u^{(m)}(t))]dx \\ &+ \int_\Omega (H(t,x;u^{(m)}), (u^{(m)})'(t))dx + \int_\Omega (G(t,x;u^{(m)},(u^{(m)})'), (u^{(m)})'(t))dx \\ &= \langle F(t), (u^{(m)})'(t)\rangle. \end{aligned} \tag{2.9}$$

Integrating the above equality over $(0,t)$ we find (see, e.g., [16], [12])

$$\begin{aligned} &\frac{1}{2}\|(u^{(m)})'(t)\|_H^2 + \frac{1}{2}\langle Q(u^{(m)}(t)), u^{(m)}(t)\rangle + \int_\Omega \varphi(x) h(u^{(m)}(t))dx \\ &+ \int_0^t \left[\int_\Omega (H(\tau,x;u^{(m)}), (u^{(m)})')dx\right] d\tau + \int_0^t \left[\int_\Omega (G(\tau,x;u^{(m)},(u^{(m)})'), (u^{(m)})')dx\right] d\tau \\ &= \int_0^t \left[\langle F(\tau), (u^{(m)})'(\tau)\rangle\right] d\tau. \end{aligned} \tag{2.10}$$

Hence, by using Young's inequality, Sobolev's imbedding theorem and the assumptions of our theorem, we obtain

$$\begin{aligned} &\|(u^{(m)})'(t)\|_H^2 + \int_\Omega h(u^{(m)}(t))dx + \|u^{(m)}(t)\|_V^2 \\ &\le \text{const}\left\{1 + \int_0^t \left[\|(u^{(m)})'(\tau)\|_H^2 + \int_\Omega h(u^{(m)}(\tau))dx\right] d\tau\right\} \end{aligned}$$

where the constant is not depending on t and m. Thus by Gronwall's lemma

$$\|(u^{(m)})'(t)\|_H^2 + \int_\Omega h(u^{(m)}(t))dx \le \text{const} \tag{2.11}$$

and thus

$$\|u^{(m)}(t)\|_V^2 \le \text{const} \tag{2.12}$$

Further, the estimates (2.11), (2.12) hold for all $t \in [0,T]$ and all m and in the case $\lambda > \lambda_0$, $n \ge 3$

$$\|u^{(m)}(t)\|_{V \cap [L^{\lambda+1}(\Omega)]^N} \le \text{const}. \tag{2.13}$$

By (2.11), (2.12), if (A_3) is satisfied, there exist a subsequence of $(u^{(m)})$, again denoted by $(u^{(m)})$ and $u \in L^\infty(0,T;V)$ such that

$$(u^{(m)}) \to u \text{ weakly in } L^\infty(0,T;V), \tag{2.14}$$

$$(u^{(m)})' \to u' \text{ weakly in } L^\infty(0,T;H) \tag{2.15}$$

in the following sense: for any fixed $g \in L^1(0,T;V^\star)$ and $g_1 \in L^1(0,T;H)$

$$\int_0^T \langle g(t), u^{(m)}(t)\rangle dt \to \int_0^T \langle g(t), u(t)\rangle dt,$$

$$\int_0^T (g_1(t), (u^{(m)})'(t))dt \to \int_0^T (g_1(t), u'(t))dt.$$

Similarly, in the case $\lambda > \lambda_0$, $n \geq 3$, (when (A_3') holds) there exist subsequence of $(u^{(m)})$ and $u \in L^\infty(0,T; V \cap [L^{\lambda+1}(\Omega)]^N)$ such that

$$(u^{(m)}) \to u \text{ weakly in } L^\infty(0,T; V \cap [L^{\lambda+1}(\Omega)]^N), \tag{2.16}$$

which means: for any fixed $g \in L^1(0,T;(V \cap L^{\lambda+1}(\Omega))^\star)$

$$\int_0^T \langle g(t), u^{(m)}(t)\rangle dt \to \int_0^T \langle g(t), u(t)\rangle dt.$$

Since the imbedding $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact, by (2.14) – (2.16) we have for a subsequence

$$(u^{(m)}) \to u \text{ in } L^2(0,T;H) = [L^2(Q_T)]^N \text{ and a.e. in } Q_T. \tag{2.17}$$

(see, e.g., [9]). Finally, we show that the limit function u is a solution of problem (1.1), (1.2).

As $Q : V \to V^\star$ is a linear and continuous operator, by (2.14) for all $v \in V$ and $v \in V \cap \left[L^{\lambda+1}(\Omega)\right]^N$, respectively we have

$$\langle (Q(u^{(m)}m)(t)), v\rangle \to \langle (Q(u(t)), v\rangle \text{ weakly in } L^\infty(0,T) \tag{2.18}$$

and by (2.15)

$$\langle (u^{(m)})''(t), v\rangle = \frac{d}{dt}\langle (u^{(m)})'(t), v\rangle \to \langle u''(t), v\rangle \tag{2.19}$$

with respect to the weak convergence of the space of distributions $D'(0,T)$.

Further, by (2.17) and the continuity of $D_j h$

$$\varphi(x) D_j h(u_m(t)) \to \varphi(x) D_j h(u(t)) \text{ for a.e. } (t,x) \in Q_T. \tag{2.20}$$

Now we show that for any fixed

$$v \in L^2(0,T;V), \quad v \in L^2(0,T;V) \cap L^1(0,T;(L^{\lambda+1}(\Omega))^N),$$

respectively, the sequence of functions

$$\varphi(x) D_j h(u^{(m)}(t)) v \quad j = 1, \dots, N \tag{2.21}$$

is equiintegrable in Q_T. Indeed, if (A_3) is satisfied then by Sobolev's imbedding theorem and (2.12) for all $t \in [0,T]$

$$\|\varphi(x) D_j h(u^{(m)}(t))\|^2_{L^2(\Omega)} \leq \text{const}\|D_j h(u^{(m)}(t))\|^2_{L^2(\Omega)}$$

$$\leq \text{const}\left[1 + \int_\Omega |u^{(m)}(t)|^{2\lambda_0} dx\right] \leq \text{const}\left[1 + \|u_m(t)\|_V^{2\lambda_0}\right] \leq \text{const},$$

because $2\lambda_0 = \frac{2n}{n-2}$ and $W^{1,2}(\Omega)$ is continuously imbedded into $L^{\frac{2n}{n-2}}(\Omega)$, thus Cauchy–Schwarz inequality implies that the sequence of functions (2.21) is equiintegrable in Q_T.

If (A_3') is satisfied then for all $t \in [0,T]$

$$\int_\Omega |\varphi(x) D_j h(u^{(m)}(t))|^{\frac{\lambda+1}{\lambda}} dx \le \text{const} \int_\Omega [h(u^{(m)}(t)) + 1] dx \le \text{const}$$

thus Hölder's inequality implies that the sequence (2.21) is equiintegrable in Q_T. Consequently, by (2.20) and Vitali's theorem we obtain that for any fixed

$$v \in L^2(0,T;V), \quad v \in L^2(0,T;V) \cap L^1(0,T;L^{\lambda+1}(\Omega)),$$

respectively

$$\lim_{m\to\infty} \int_{Q_T} \varphi(x) D_j h(u^{(m)}(t)) v_j dt dx = \int_{Q_T} \varphi(x) D_j h(u(t)) v_j dt dx \tag{2.22}$$

and

$$\varphi(x) D_j h(u(t)) \in L^2(0,T;V^\star), \quad \varphi(x) D_j h(u(t)) \in L^\infty(0,T;L^{\frac{\lambda+1}{\lambda}}(\Omega)) \tag{2.23}$$

if (A_3), (A_3') holds, respectively.

Further, by (2.17) and (A_4)

$$H_j(t,x;u^{(m)}) \to H_j(t,x;u) \text{ a.e. in } Q_T \tag{2.24}$$

and by (2.11)

$$\int_{Q_T} |H_j(t,x;u_m)|^2 dx dt \le \text{const} \int_{Q_T} h(u_m(t)) dx dt \le \text{const},$$

hence, by Cauchy–Schwarz inequality, for any fixed $v \in L^2(0,T;V)$, the sequence of functions $H_j(t,x;u^{(m)}) v_j$ is equiintegrable in Q_T ($j = 1,\dots,N$), thus by (2.24) and Vitali's theorem

$$\lim_{m\to\infty} \int_{Q_T} H_j(t,x;u^{(m)}) v_j dt dx = \int_{Q_T} H_j(t,x;u) v_j dt dx \tag{2.25}$$

and

$$H(t,x;u) \in L^2(0,T;V^\star).$$

Similarly, (2.15) – (2.17) and (A_5) imply

$$\psi_j(t,x;u^{(m)},(u^{(m)})') \to \psi_j(t,x;u,u') \text{ a.e. in } Q_T \tag{2.26}$$

and for arbitrary $v \in L^2(0,T;V)$ the sequence of functions $\psi_j(t,x;u^{(m)},(u^{(m)})')v_j$ is equintegrable in Q_T by Cauchy – Schwarz inequality, because by (2.11)

$$\int_{Q_T} |\psi_j(t,x;u^{(m)},(u^{(m)})')|^2 dt dx \le \text{const} \left[1 + \int_{Q_T} |(u^{(m)})'|^2 dx\right] dt \le \text{const}.$$

Consequently, Vitali's theorem implies that for $j = 1,\dots,N$

$$\lim_{m\to\infty} \int_{Q_T} \psi_j(t,x;u^{(m)},(u^{(m)})') v_j dt dx = \int_{Q_T} \psi_j(t,x;u,u') v dt dx \tag{2.27}$$

and

$$\psi_j(t,x;u,u') \in L^2(0,T;V^\star).$$

Further, by using Vitali's theorem, we show that for arbitrary fixed $v \in L^2(0,T;V)$

$$\varphi_j(t,x;u^{(m)}) v_j \to \varphi_j(t,x;u) v_j \text{ in } L^2(Q_T), \quad j = 1,\dots,N. \tag{2.28}$$

Indeed, by (A_5) and (2.17)

$$\varphi_j(t,x;u^{(m)}) \to \varphi_j(t,x;u) \text{ for a.e. } (t,x) \in Q_T, \quad j=1,\dots,N. \tag{2.29}$$

Further, by (A_5) $|\varphi_j(t,x;u^{(m)}))|^2$ is bounded and so for fixed $v \in L^2(0,T;V)$ the sequence

$$\int_{Q_T} |\varphi_j(t,x;u^{(m)})v_j - \varphi_j(t,x;u)v_j|^2 dtdx \le \text{const}|v_j|^2$$

is equiintegrable which implies with (2.29) by Vitali's theorem (2.28). Consequently, by (2.15) we obtain

$$\lim \int_{Q_T} \varphi_j(t,x;u^{(m)})(u^{(m)})'(t)v_j dtdx = \int_{Q_T} \varphi_j(t,x;u)u'(t)v_j dtdx, \quad j=1,\dots,N \tag{2.30}$$

and $\varphi(t,x;u)u' \in L^2(0,T;V^\star)$.

If (A_5') (and (A_3')) is satisfied, then for a fixed $v \in L^2(0,T;V) \cap [L^{\lambda+1}(Q_T)]^N$ we also have

$$\varphi_j(t,x;u^{(m)})v_j \to \varphi_j(t,x;u)v_j \text{ in } L^2(Q_T), \quad j=1,\dots,N. \tag{2.31}$$

Indeed, by (2.11), (2.12) $(u^{(m)})$ is bounded in $W^{1,2}(Q_T)$, hence it is bonded in $L^{\frac{2(n+1)}{n-1}}(Q_T)$. Thus Hölder's inequality implies for any measurable $M \subset Q_T$

$$\int_M |\varphi_j(t,x;u^{(m)})v_j - \varphi_j(t,x;u)v_j|^2 dtdx \tag{2.32}$$

$$\le \text{const} \left\{ \int_{Q_T} [|u^{(m)}|^{2\mu} + |u^{(m)}|^{2\mu}]^{q_1} dtdx \right\}^{1/q_1} \cdot \left\{ \int_M |v_j|^{2p_1} \right\}^{1/p_1}$$

$$\le \text{const} \left\{ \int_M |v_j|^{2p_1} \right\}^{1/p_1}$$

where

$$2p_1 = \lambda + 1, \quad \frac{1}{p_1} + \frac{1}{q_1},$$

thus

$$2\mu q_1 = 2\mu \frac{p_1}{p_1 - 1} = 2\mu \frac{\lambda+1}{\lambda-1} \le \frac{2(n+1)}{n-1}$$

since

$$\mu \le \frac{n+1}{n-1} \cdot \frac{\lambda-1}{\lambda+1},$$

hence (2.29), (2.32) and Vitali's theorem imply (2.31). Consequently, by (2.15) we obtain (2.30) (when (A_5') holds).

Now let

$$v = (v_1,\dots,v_N) \in V \text{ and } \chi_j \in C_0^\infty(0,T) \quad (j=1,\dots,N)$$

be arbitrary functions. Further, let $z_j^M = \sum_{l=1}^M b_{lj} w_l^{(j)}$, $b_{lj} \in \mathbb{R}$ be sequences of functions such that

$$(z_j^M) \to v_j \text{ in } V_j \text{ and } V_j \cap L^{\lambda+1}(\Omega), \quad j=1,\dots,N, \tag{2.33}$$

respectively, as $M \to \infty$. Further, by (2.5) we have for all $m \geq M$

$$\int_0^T \langle -(u_j^{(m)})'(t), z_j^M \rangle \chi_j'(t)dt + \int_0^T \langle Q(u^{(m)}(t)), z_j^M \rangle \chi_j(t)dt \tag{2.34}$$
$$+ \int_0^T \int_\Omega \varphi(x) D_j h(u^{(m)}(t)) z_j^M \chi_j(t)dtdx + \int_0^T \int_\Omega H_j(t,x;u^{(m)}) z_j^M \chi_j(t)dtdx$$
$$+ \int_0^T \int_\Omega G_j(t,x;u^{(m)},(u^{(m)})') z_j^M \chi_j(t)dtdx$$
$$= \int_0^T \langle F_j(t), z_j^M \rangle \chi_j(t)dt.$$

By (2.15), (2.18), (2.22), (2.25), (2.27), (2.30) we obtain from (2.34) as $m \to \infty$

$$-\int_0^T \langle u_j'(t), z_j^M \rangle \chi_j'(t)dt + \int_0^T \langle Q_j(u(t)), z_j^M \rangle \chi_j(t)dt \tag{2.35}$$
$$+ \int_0^T \int_\Omega \varphi(x) D_j h(u(t)) z_j^M \chi_j(t)dtdx$$
$$+ \int_0^T \int_\Omega H_j(t,x;u) z_j^M \chi_j(t)dtdx + \int_0^T \int_\Omega G_j(t,x;u,u') z_j^M \chi_j(t)dtdx$$
$$= \int_0^T \langle F_j(t), z_j^M \rangle \chi(t)dt.$$

From equality (2.35) and (2.33) we obtain as $M \to \infty$

$$-\int_0^T \langle u_j'(t), v_j \rangle \chi_j'(t)dt + \int_0^T \langle Q_j(u(t)), v_j \rangle \chi_j(t)dt \tag{2.36}$$
$$+ \int_0^T \int_\Omega \varphi(x) D_j h(u(t)) v_j \chi_j(t)dtdx$$
$$+ \int_0^T \int_\Omega H_j(t,x;u) v_j \chi_j(t)dtdx + \int_0^T \int_\Omega G_j(t,x;u,u') v_j \chi_j(t)dtdx$$
$$= \int_0^T \langle F_j(t), v_j \rangle \chi_j(t)dt.$$

Since $v_j \in V_j$ and $\chi_j \in C_0^\infty(0,T)$ are arbitrary functions, (2.36) means that

$$u_j'' \in L^2(0,T;V_j^\star) \text{ and } u_j'' \in L^2(0,T;(V \cap L^{\lambda+1}(\Omega))^\star), \tag{2.37}$$

respectively (see, e.g. [16]) and for a.a. $t \in [0,T]$

$$u_j'' + Q_j(u(t)) + \varphi(x) D_j h(u(t)) + H_j(t,x;u) + G_j(t,x;u,u') = F_j, \quad j = 1,\ldots,N, \tag{2.38}$$

i.e. we proved (1.1).

Now we show that the initial condition (1.2) holds. Since $u \in L^\infty(0,T;V)$, $u' \in L^\infty(0,T;H)$, we have $u \in C([0,T];H)$ and for arbitrary $\chi_j \in C^\infty[0,T]$ with the properties $\chi_j(0) = 1$, $\chi_j(T) = 0$, all j, k

$$\int_0^T \langle u_j'(t), w_k^{(j)} \rangle \chi_j(t)dt = -(u_j(0), w_k^{(j)})_{L^2(\Omega)} - \int_0^T \langle u_j(t), w_k^{(j)} \rangle \chi_j'(t)dt,$$

$$\int_0^T \langle (u_j^{(m)})'(t), w_k^{(j)} \rangle \chi_j(t)dt = -(u_j^{(m)}(0), w_k^{(j)})_{L^2(\Omega)} - \int_0^T \langle u_j^{(m)}(t), w_k^{(j)} \rangle \chi_j'(t)dt.$$

Hence by (2.6), (2.7), (2.8), (2.14), (2.15), we obtain as $m \to \infty$

$$(u^{(0)}, w_k^{(j)})_{L^2(\Omega)} = \lim_{m\to\infty} (u_{j0}^{(m)}, w_k^{(j)})_{L^2(\Omega)}$$

$$= \lim_{m\to\infty} (u_j^{(m)}(0), w_k^{(j)})_{L^2(\Omega)} = (u_j(0), w_k^{(j)})_{L^2(\Omega)}$$

for all j and k which implies $u(0) = u^{(0)}$.

Similarly can be shown that $u'(0) = u^{(1)}$.

3. Examples

Let the operator Q be defined by

$$\langle Q_{jk}(u_k), v_j \rangle = \int_\Omega \left[\sum_{i,l=1}^n a_{il}^{jk}(x)(D_l u_k)(D_i v_j) + d^{jk}(x) u_k v_j \right] dx$$

where $a_{il}^{jk}, d^{jk} \in L^\infty(\Omega)$, $a_{il}^{jk} = a_{li}^{jk}$, $\sum_{i,l=1}^n a_{il}^{jj}(x)\xi_i\xi_l \geq c_1|\xi|^2$, $d^{ii}(x) \geq c_0$ with some positive constants c_0, c_1; further, $a_{il}^{jk} = a_{il}^{kj}$ and for some $\tilde{c}_0 < c_1$

$$\|a_{il}^{jk}\|_{L^\infty(\Omega)} < \frac{\tilde{c}_0}{n-1}, \quad \|d^{jk}\|_{L^\infty(\Omega)} < \frac{\tilde{c}_0}{n-1} \text{ for } j \neq k.$$

Then assumption (A_1) is satisfied.

If h is a C^1 function such that $h(\eta) = |\eta|^{\lambda+1}$ if $|\eta| > 1$ then (A_3), (A_3'), respectively, are satisfied.

Further, let $\tilde{h}_j : \mathbb{R}^N \to \mathbb{R}$ be continuous functions satisfying

$$|\tilde{h}_j(\eta)| \leq \text{const } |\eta|^{\frac{\lambda+1}{2}} \text{ for } |\eta| > 1, \quad j = 1, \ldots, N$$

with some positive constant. It is not difficult to show that operators H_j defined by one of the formulas

$$H_j(t,x;u) = \chi_j(t,x)\tilde{h}_j\left(\int_{Q_t} u_1(\tau,\xi)d\tau d\xi, \ldots, \int_{Q_t} u_N(\tau,\xi), d\tau d\xi \right),$$

$$H_j(t,x;u) = \chi_j(t,x)\tilde{h}_j\left(\int_0^t u_1(\tau,x)d\tau, \ldots, \int_0^t u_N(\tau,x)d\tau \right),$$

$$H_j(t,x;u) = \chi_j(t,x)\tilde{h}_j\left(\int_\Omega u_1(t,\xi)d\xi, \ldots, \int_\Omega u_N(t,\xi)d\xi \right),$$

$$H_j(t,x;u) = \chi_j(t,x)\tilde{h}_j(u_1(\tau_1(t),x), \ldots, u_N(\tau_k(t),x)) \text{ where}$$

$$\tau_k \in C^1, \quad 0 \leq \tau_k(t) \leq t, \quad \tau_k'(t) \geq c_1 > 0, \quad k = 1, \ldots, N$$

satisfy (A_4) if $\chi_j \in L^\infty(Q_T)$.

The operators φ_j, ψ_j may have forms, similar to the above forms of H_j with bounded continuous functions $\tilde{h}_j$. Then (A_5) is fulfilled.

Remark. One can show uniqueness and continuous dependence of the solution of (1.1), (1.2) if the following additional conditions are satisfied:

$$G_j(t,x;u,u') = \tilde{\varphi}_j(x)u'_j(t)$$

where $\tilde{\varphi}_j$ is measurable and $0 \le \tilde{\varphi}_j(x) \le \text{const}$, h is twice continuously differentiable and

$$|D_i D_k h(\eta)| \le \text{const}|\eta|^{\lambda-1} \text{ for } |\eta| > 1.$$

Further $H_j(t,x;u)$ satisfy some Lipschitz condition with respect to u.

4. Solutions in $(0,\infty)$

Now we formulate and prove existence of solutions for $t \in (0,\infty)$. Denote by $L^p_{loc}(0,\infty;V)$ the set of functions $u:(0,\infty)\to V$ such that for each fixed finite $T>0$, their restrictions to $(0,T)$ satisfy $u|_{(0,T)} \in L^p(0,T;V)$ and let $Q_\infty = (0,\infty)\times\Omega$, $L^\alpha_{loc}(Q_\infty)$ the set of functions $u: Q_\infty \to \mathbb{R}^N$ such that $u_j|_{Q_T} \in L^\alpha(Q_T)$ $(j=1,\dots,N)$ for any finite T.

Now we formulate assumptions on H_j and G_j.

(B_4) The functions $H_j : Q_\infty \times \left[L^2_{loc}(Q_\infty)\right]^N \to \mathbb{R}$ are such that for all fixed $u \in \left[L^2_{loc}(Q_\infty)\right]^N$ the functions $(t,x)\mapsto H_j(t,x;u)$ are measurable, H_j have the Volterra property (see (A_4)) and for each fixed finite $T>0$, the restrictions of H_j to $Q_T \times \left[L^2(Q_T)\right]^N$ satisfy (A_4).

Remark. Since H_j has the Volterra property, this restriction H_j^T is well defined by the formula

$$H_j^T(t,x;\tilde{u}) = H_j(t,x;u), \quad (t,x)\in Q_T, \quad \tilde{u}\in[L^2(Q_T)]^N$$

where $u \in [L^2_{loc}(Q_\infty)]^N$ may be any function satisfying $u(t,x)=\tilde{u}(t,x)$ for $(t,x)\in Q_T$.

(B_5) The operators

$$G_j : Q_\infty \times [L^2_{loc}(Q_\infty)]^N \times L^\infty_{loc}(0,\infty;H) \to \mathbb{R}$$

are such that for all fixed $u \in L^2_{loc}(0,\infty;V)$, $w \in L^\infty_{loc}(0,\infty;H)$ the functions $(t,x)\mapsto G_j(t,x;u,w)$ are measurable, G_j have the Volterra property and for each fixed finite $T>0$, the restrictions G_j^T of G_j to $Q_T\times[L^2(Q_T)]^N\times L^\infty(0,T;H)$ satisfy (A_5).

(B'_5) It is the same as (B_5) but G_j^T satisfy (A'_5).

Theorem 4.1. *Assume (A_1) – (A_3), (B_4), (B_5). Then for all $F \in L^2_{loc}(0,\infty;H)$, $u^{(0)}\in V$, $u^{(1)}\in H$ there exists*

$$u \in L^\infty_{loc}(0,\infty;V) \text{ such that } u' \in L^\infty_{loc}(0,\infty;H), \quad u'' \in L^2_{loc}(0,\infty;V^\star),$$

u satisfies (1.1) for a.a. $t\in(0,\infty)$ (in the sense, formulated in Theorem 2.1) and the initial condition (1.2).

If (A_1), $A_2)$, (A'_3), (B_4), (B_5) are fulfilled then for all $F \in L^2_{loc}(0,\infty;H)$, $u^{(0)} \in V\cap[L^{\lambda+1}(\Omega)]^N$, $u^{(1)}\in H$ there exists

$$u \in L^\infty_{loc}(0,\infty;V\cap[L^{\lambda+1}(\Omega)]^N) \text{ such that } u' \in L^\infty_{loc}(0,\infty;H),$$

$$u'' \in L^2_{loc}(0,\infty;V^\star) + L^\infty_{loc}(0,\infty;[L^{\frac{\lambda+1}{\lambda}}(\Omega)]^N) \subset L^2_{loc}(0,\infty;[V\cap(L^{\lambda+1}(\Omega))^N]^\star),$$

u satisfies (1.1) for a.a. $t\in(0,\infty)$ (in the sense, formulated in Theorem 2.1) and the initial condition (1.2).

Assume that the following additional conditions are satisfied: there exist T_0 and a function $\gamma\in L^2(T_0,\infty)$ such that for $t>T_0$

$$|G(t,x;u,u')| \le \gamma(t), |H(t,x;u)| \le \gamma(t) \text{ and } \|F(t)\|_{V^\star} \le \gamma(t). \tag{4.1}$$

Then for the above solution u we have

$$u\in L^\infty(0,\infty;V), \quad u\in L^\infty(0,\infty;V\cap[L^{\lambda+1}(\Omega)]^N), \text{ respectively and} \tag{4.2}$$
$$u'\in L^\infty(0,\infty;H).$$

Further, assume that there exists a positive constant $\tilde c$ such that

$$\varphi_j(t,x;u) \ge \tilde c, \quad (t,x)\in Q_\infty, \quad j=1,\dots,N \tag{4.3}$$

and there exist $F_\infty\in H$, $u_\infty\in V$ such that

$$Q(u_\infty)=F_\infty, \quad F-F_\infty\in L^2(0,\infty;H), \tag{4.4}$$

$$|H_j(t,x;u)|\le\beta(t,x), \quad |\psi_j(t,x;u,u')|\le\beta(t,x), \quad |\varphi_j(t,x;u)|\le const \tag{4.5}$$

with some $\beta\in L^2(0,\infty;L^2(\Omega))$. Then for the above solution we have

$$u\in L^\infty(0,\infty;V), \quad u\in L^\infty(0,\infty;v\cap[L^{\lambda+1}(\Omega)]^N), \tag{4.6}$$

$$\|u'(t)\|_H \le const\ e^{-\tilde c t}, \quad t\in(0,\infty) \tag{4.7}$$

and there exists $w^{(0)}\in H$ such that

$$u(T)\to w^{(0)} \text{ in } H \text{ as } T\to\infty, \quad \|u(T)-w^{(0)}\|_H \le const\ e^{-\tilde c T}. \tag{4.8}$$

Finally, $w^{(0)}\in V$ and

$$Q(w^{(0)})+\varphi Dh(w^{(0)})=F_\infty. \tag{4.9}$$

Proof. Similarly to the proof of Theorem 2.1, we apply Galerkin's method and we want to find the m-th approximation of solution $u=(u_1,\dots,u_N)$ for $t\in(0,\infty)$ in the form (see (2.4))

$$u_j^{(m)}(t)=\sum_{l=1}^m g_{lm}^{(j)}(t)w_l^{(j)}, \quad j=1,\dots,N$$

where $g_{lm}^{(j)}\in W^{2,2}_{loc}(0,\infty)$ if (A_3) is satisfied and $g_{lm}^{(j)}\in W^{2,2}_{loc}(0,\infty)\cap L^\infty_{loc}(0,\infty)$ if (A_3') is satisfied. Here $W^{2,2}_{loc}(0,\infty)$ and $L^\infty_{loc}(0,\infty)$ denote the set of functions $g:(0,\infty)\to\mathbb{R}$ such that for all T the restriction of g to $(0,T)$ belongs to $W^{2,2}(0,T)$, $L^\infty(0,T)$, respectively.

According to the arguments in the proof of Theorem 2.1, there exists a solution of (2.5), (2.6) in a neighbourhood of $t=0$. Further, we obtain estimates (2.11), (2.12) and (2.13), respectively, for $t\in[0,T]$ with sufficiently small T where on the right hand side are finite constants (depending on T). Consequently, the maximal solutions of (2.5), (2.6) are defined in $(0,\infty)$ and the estimates (2.11), (2.12), (2.13) hold for all finite $T > 0$ (if $t \in [0,T]$), the constants on the right hand sides are depending only on T.

Let $(T_k)_{k\in\mathbb{N}}$ be a monotone increasing sequence, converging to $+\infty$. According to the arguments in the proof of Theorem 2.1, there is a subsequence $(u^{(m1)})$ of $(u^{(m)})$ for which (2.14), (2.15) and (2.16) hold, respectively, with $T = T_1$. Further, there is a subsequence $(u^{(m2)})$ of $(u^{(m1)})$ for which (2.14), (2.15) and (2.16) hold, respectively, with $T = T_2$, etc. By a diagonal process we obtain a sequence $(u^{(mm)})_{m\in\mathbb{N}}$ such that (2.14), (2.15), (2.16) hold for every fixed $T > 0$; further,

$$u \in L^\infty_{loc}(0,\infty;V), \quad u' \in L^\infty_{loc}(0,\infty;H), \quad u'' \in L^2_{loc}(0,\infty;V^\star) \text{ and}$$
$$u \in L^\infty_{loc}(0,\infty;V \cap [L^{\lambda+1}(\Omega)]^N), \quad u' \in L^\infty_{loc}(0,\infty;H),$$
$$u'' \in L^2_{loc}(0,\infty;V^\star) + L^\infty_{loc}(0,\infty;[L^{\frac{\lambda+1}{\lambda}}(\Omega)]^N),$$

respectively and (1.1) holds for $t \in (0,\infty)$.

Now we consider the case when (4.1) holds. Then by (2.10) we obtain for all $t \geq T_1 \geq T_0$

$$\frac{1}{2}\|(u^{(m)})'(t)\|^2_H + \frac{1}{2}\langle(Q(u^{(m)})(t), u^{(m)}(t)\rangle + c_1\int_\Omega h(u^{(m)}(t))dx$$
$$\leq \int_0^{T_1}\int_\Omega |\langle G(\tau,x;u^{(m)},(u^{(m)})'),(u^{(m)})'(\tau)\rangle| d\tau + \int_0^{T_1}\int_\Omega |\langle H(\tau,x;u^{(m)}),(u^{(m)})'(\tau)\rangle| d\tau$$
$$+\int_0^{T_1}\int_\Omega |\langle F(\tau),(u^{(m)})'(\tau)\rangle| d\tau + 3\lambda(\Omega)\left[\int_{T_1}^\infty |\gamma(\tau)|d\tau\right] \sup_{\tau\in[0,t]} \|(u^{(m)})'(\tau)\|_H.$$

Choosing sufficiently large $T_1 > 0$, since $\lim_{T_1\to\infty}\int_{T_1}^\infty |\gamma(\tau)|d\tau = 0$, we find

$$\frac{1}{4}\|(u^{(m)})'(t)\|^2_H + \frac{1}{2}\langle Q(u^{(m)}(t)), u^{(m)}(t)\rangle + c_1\int_\Omega h(u^{(m)}(t)dx \leq \text{const}$$

for all $t > 0$, m which implies (4.2).

Finally, consider the case when (4.3) – (4.5) are satisfied, too. Denoting $u^{(mm)}$ by $u^{(m)}$, for simplicity, by (2.9), $Qu_\infty = F_\infty$ we obtain for $w_m = u_m - u_\infty$ (since $(w^{(m)})' = (u^{(m)})'$):

$$\langle(w^{(m)})''(t),(w^{(m)})'(t)\rangle + \langle(Qw^{(m)})(t),(w^{(m)})'(t)\rangle + \int_\Omega \varphi(x)\frac{d}{dt}[h(u^{(m)}(t))]dx \quad (4.10)$$
$$+\int_\Omega (H(t,x;u^{(m)}),(w^{(m)})'(t))dx + \int_\Omega (G(t,x;u^{(m)},(u^{(m)})'),(w^{(m)})'(t)dx$$
$$= \langle F(t) - F_\infty,(w^{(m)})'(t)\rangle.$$

Integrating over $[0,t]$ we find (similarly to (2.10))

$$\frac{1}{2}\|(w^{(m)})'(t)\|^2_H + \frac{1}{2}\langle Q(w^{(m)}(t)), w^{(m)}(t)\rangle + c_1\int_\Omega h(u^{(m)}(t))dx \quad (4.11)$$
$$+\tilde{c}\int_0^t\left[\int_\Omega |(w^{(m)})'(\tau)|^2 dx\right]d\tau$$
$$\leq \varepsilon\int_0^t\left[\int_\Omega |(w^{(m)})'(\tau)|^2 dx\right]d\tau + C(\varepsilon)\int_0^t \|F(\tau) - F_\infty\|^2_H d\tau$$

$$+\frac{1}{2}\|(u^{(m)})'(0)\|_H^2+\frac{1}{2}\langle (Qu^{(m)})(0),u^{(m)}(0)\rangle + c_2\int_\Omega h(u^{(m)}(0))dx$$
$$+\varepsilon\int_0^t\left[\int_\Omega |(w^{(m)})'(\tau)|dx\right]d\tau + C(\varepsilon)\|\beta\|_{L^2(0,\infty;H)}.$$

Choosing $\varepsilon=\tilde{c}/4$ we obtain

$$\int_0^t\left[\int_\Omega |(w^{(m)})'(\tau)|^2dx\right]d\tau \le \text{const.} \tag{4.12}$$

Further, from (4.11), (4.12) we obtain

$$\|(u^{(m)})'(t)\|_H^2+\tilde{c}\int_0^t \|(u^{(m)})'(\tau)\|_H^2 d\tau \le c^\star$$

with some positive constant $c^\star$ not depending on m and t. Thus by Gronwall's lemma we find

$$\|(u^{(m)})'(t)\|_H^2=\|(w^{(m)})'(t)\|_H^2\le c^\star \mathrm{e}^{-\tilde{c}t},\quad t>0$$

which implies (4.7) as $m\to\infty$ (since $(u^{(m)})'\to u'$ weakly in $L^\infty(0,T;H)$). Further, by (A_1) one obtains from (4.11) that for all $t>0$, m

$$\|w^{(m)}(t)\|_V\le \text{const},\quad \|w^{(m)}(t)\|_{V\cap[L^{\lambda+1}(\Omega)]^N}\le \text{const},$$

respectively, which implies (4.6).

Further, for arbitrary $T_1<T_2$

$$\|u(T_2)-u(T_1)\|_H^2=(u(T_2),u(T_2)-u(T_1))_H-(u(T_1),u(T_2)-u(T_1))_H$$
$$=\int_{T_1}^{T_2}\langle u'(t),u(T_2)-u(T_1)\rangle dt=\int_{T_1}^{T_2}(u'(t),u(T_2)-u(T_1))_H dt$$
$$\le \|u(T_2)-u(T_1)\|_H\int_{T_1}^{T_2}\|u'(t)\|_H dt$$

which implies

$$\|u(T_2)-u(T_1)\|_H\le\int_{T_1}^{T_2}\|u'(t)\|_H dt. \tag{4.13}$$

Hence by (4.7)

$$\|u(T_2)-u(T_1)\|_H\to 0 \text{ as } T_1,T_2\to\infty$$

which implies (4.8) and by (4.10), (4.7) we obtain

$$\|u(T)-w_0\|_H\le\int_T^\infty \|u'(t)\|_H dt\le \text{const } \mathrm{e}^{-\tilde{c}T}.$$

Now we show $w_0\in V$ and (4.9) holds. Since $u\in L^\infty(0,\infty;V)$,

$$(u(T_k))\to w_0^\star \text{ weakly in } V,\quad w_0^\star\in V \tag{4.14}$$

for some sequence (T_k), $\lim(T_k)=+\infty$. Clearly, (4.14) implies

$$(u(T_k))\to w_0^\star \text{ weakly in } H,$$

thus by (4.8) $w_0=w_0^\star\in V$ and (4.14) holds for arbitrary sequence (T_k) converging to $+\infty$.

In order to prove (4.9), consider arbitrary fixed $v \in V$, $v \in V \cap [L^{\lambda+1}(\Omega)]^N$, respectively and

$$\chi_T(t) = \chi(t-T) \text{ where } \chi \in C_0^\infty(\mathbb{R}),\ \mathrm{supp}\chi \subset [0,1], \quad \int_0^1 \chi(t)dt = 1.$$

Multiply (2.3) by $\chi_T(t)$ and integrate with respect to t on $(0,\infty)$ and take the sum with respect to j, then we obtain

$$\int_0^\infty \langle u''(t), v\rangle \chi_T(t)dt + \int_0^\infty \langle Q(u(t)), v\rangle \chi_T(t)dt \tag{4.15}$$

$$+ \int_0^\infty \left[\int_\Omega \varphi(x)((Dh)(u(t)), v)dx\right] \chi_T(t)dt + \int_0^\infty \left[\int_\Omega (H(t,x;u), v)dx\right] \chi_T(t)dt$$

$$+ \int_0^\infty \left[\int_\Omega (G(t,x;u,u'), v)dx\right] \chi_T(t)dt = \int_0^\infty (F(t), v)\chi_T(t)dt.$$

Let (T_k) be an arbitrary sequence converging to $+\infty$ and consider (4.15) with $T = T_k$. For the first term on the left hand side of this equation we have by (4.7) (if $T_k > 1$)

$$\int_0^\infty \langle u''(t), v\rangle \chi_{T_k}(t)dt = -\int_0^\infty \langle u'(t), v\rangle (\chi_{T_k})'(t)dt \to 0 \text{ as } k \to \infty. \tag{4.16}$$

Further, by (A_1), (4.14) and Lebesgue's dominated convergence theorem

$$\int_0^\infty \langle Q(u(t)), v\rangle \chi_{T_k}(t)dt = \int_0^\infty \langle Q(v), u(t)\rangle \chi_{T_k}(t)dt \tag{4.17}$$

$$= \int_0^1 \langle Q(v), u(T_k+\tau)\rangle \chi(\tau)d\tau \to \int_0^1 \langle Q(v), w_0\rangle \chi(\tau)d\tau = \langle Q(v), w_0\rangle$$

$$= \langle Q(w_0), v\rangle \text{ as } k \to \infty.$$

For the third term on the left hand side of (4.15) we have

$$\int_0^\infty \left[\int_\Omega \varphi(x)((Dh)(u(t)), v)dx\right] \chi_{T_k}(t)dt \tag{4.18}$$

$$= \int_0^1 \left[\int_\Omega \varphi(x)((Dh)(u(T_k+\tau)), v)dx\right] \chi(\tau)d\tau$$

$$\to \int_0^1 \left[\int_\Omega \varphi(x)((Dh)(w_0), v)dx\right] \chi(\tau)d\tau = \int_\Omega \varphi(x)((Dh)(w_0), v)dx$$

as $k \to \infty$ since by (4.8)

$$u(T_k+\tau) \to w_0 \text{ in } [L^2((0,1)\times\Omega)]^N \text{ as } k \to \infty$$

and thus for a.a. $(\tau, x) \in (0,1)\times\Omega$ (for a subsequence), consequently

$$(Dh)(u(T_k+\tau, x)) \to (Dh)(w_0(x)) \text{ for a.a. } (\tau,x) \in (0,1)\times\Omega. \tag{4.19}$$

By using Hölder's inequality, (A_3), (A_3'), respectively and Vitali's theorem, we obtain (4.18) from (4.19).

The fourth and fifth terms on the left hand side of (4.15) can be estimated by (4.5) and (4.7) as follows: for sufficiently large k

$$\left|\int_0^\infty \left[\int_\Omega (H(t,x;u),v)dx\right]\chi_{T_k}(t)dt\right| = \left|\int_0^\infty \left[\int_\Omega (H(T_k+\tau,x;u),v)dx\right]\chi(\tau)d\tau\right| \tag{4.20}$$

$$\leq \int_0^\infty \left[\int_\Omega \beta(T_k+\tau,x)|v|dx\right]|\chi(\tau)|d\tau \to 0 \text{ as } k\to\infty,$$

$$\left|\int_0^\infty \left[\int_\Omega (G(t,x;u,u'),v)dx\right]\chi_{T_k}(t)dt\right| \tag{4.21}$$

$$\leq \int_0^1 \left[\int_\Omega \{c_5|u'(T_k+\tau)|+\beta(T_k+\tau,x)\}|v|dx\right]|\chi(\tau)|d\tau \to 0.$$

Finally, for the right hand side of (4.15) we obtain by using (4.4) and the Cauchy – Schwarz inequality

$$\int_0^\infty (F(t),v)\chi_{T_k}(t)dt = \int_0^1 (F(T_k+\tau),v)\chi(\tau)d\tau \to \int_0^1 (F_\infty,v)\chi(\tau)d\tau = (F_\infty,v). \tag{4.22}$$

From (4.15) – (4.18), (4.20) – (4.22) one obtains (4.9).

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