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Bilateral inequalities for harmonic, geometric and Hölder means

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Abstract. For $0 < a < b$, the harmonic, geometric and Hölder means satisfy $H < G < Q$. They are special cases $(p = -1, 0, 2)$ of power means M_p. We consider the following problem: find all $\alpha, \beta \in \mathbb{R}$ for which the bilateral inequalities

$$\alpha H(a,b) + (1-\alpha)Q(a,b) < G(a,b) < \beta H(a,b) + (1-\beta)Q(a,b)$$

hold $\forall\ 0 < a < b$. Then we replace in the bilateral inequalities the mean Q by M_p, $p > 0$ and address the same problem.

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1. Introduction

We consider bivariate means $m : \mathbb{R}_+^2 \to \mathbb{R}$ which are symmetric $(m(b,a) = m(a,b)$ for all $a, b > 0)$ and homogeneous $(m(\lambda a, \lambda b) = \lambda m(a,b)$ for all $a, b, \lambda > 0)$.

For two means m_1 and m_2 we write $m_1 \le m_2$ if and only if $m_1(a,b) \le m_2(a,b)$ for every $a, b > 0$, and $m_1 < m_2$ if and only if $m_1(a,b) < m_2(a,b)$ for all $a, b > 0$ with $a \ne b$.

Since we are dealing with strict inequalities, we may and shall assume in the following that $0 < a < b$.

We consider the bivariate means

$$A(a,b) = \frac{a+b}{2};\ \ G(a,b) = \sqrt{ab};\ \ H(a,b) = \frac{2ab}{a+b};\ \ Q(a,b) = \left(\frac{a^2+b^2}{2}\right)^{1/2};\ \ (1.1)$$

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & \text{for } p \ne 0 \\ \sqrt{ab}, & \text{for } p = 0, \end{cases} \qquad (1.2)$$

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which are known as the *arithmetic*, *geometric*, *harmonic*, *Hölder* and *power means*, respectively. Properties and comparison of standard means can be found in [3].

The means from (1.1) are comparable:

$$\min < H < G < A < Q < \max,$$

where min and max are the trivial means given by $(a,b) \mapsto \min(a,b)$ and $(a,b) \mapsto \max(a,b)$. The power means are monotonic in p, and $M_{-1} = H$, $M_0 = G$, $M_1 = A$, and $M_2 = Q$.

Recently, many bilateral inequalities between means have been proved ([1], [2], [4], [5], [6]). We mention one of them, which was the starting point for this paper, and refers to the means G, A and Q.

Theorem 1.1. [2] *The double inequality*

$$\alpha G(a,b) + (1-\alpha)Q(a,b) < A(a,b) < \beta G(a,b) + (1-\beta)Q(a,b), \ \forall\ 0 < a < b$$

holds if and only if $\alpha \geq 1/2$ *and* $\beta \leq 1 - \sqrt{2}/2$.

In what follows we shall prove a similar result for the means H, G and Q. Afterwards we consider the more general case of the means H, G and M_p, $p > 0$. We show that for $p = 5/2$ the auxiliary function f is still monotone and we formulate an open problem.

2. Main result

For $0 < a < b$, the geometric, harmonic and Hölder means satisfy $H < G < Q$. We shall find all the values of α and β in order that the geometric mean to be strictly between the combination of H and Q with parameters α, respectively β. Due to the homogeneity of all the means considered here, we may denote $t = b/a$, $t > 1$, and write in the following $m(t)$ instead of $m(1,t) = (1/a)\, m(a,b)$. For any three means $m_1 < m_2 < m_3$, the double inequality

$$\alpha m_1(t) + (1-\alpha)m_3(t) < m_2(t) < \beta m_1(t) + (1-\beta)m_3(t) \tag{2.1}$$

is equivalent to

$$\beta < f(t) < \alpha, \tag{2.2}$$

where

$$f(t) = \frac{m_3(t) - m_2(t)}{m_3(t) - m_1(t)}. \tag{2.3}$$

We shall prove the following result.

Theorem 2.1. *The double inequality*

$$\alpha H(t) + (1-\alpha)Q(t) < G(t) < \beta H(t) + (1-\beta)Q(t), \ \forall t > 1$$

holds if and only if $\alpha \geq 1$ *and* $\beta \leq 2/3$. *The function*

$$f_1(t) = \frac{Q(t) - G(t)}{Q(t) - H(t)}$$

is strictly increasing on $(1, \infty)$.

Proof. The function f_1 is given by

$$f_1(t) = \frac{((2t^2+2)^{1/2} - 2t^{1/2})(t+1)}{(2t^2+2)^{1/2}t + (2t^2+2)^{1/2} - 4t}. \tag{2.4}$$

We substitute $t = s^2$, $s > 1$ and get

$$f_1(s^2) = \frac{((2s^4+2)^{1/2} - 2s)(s^2+1)}{(2s^4+2)^{1/2}s^2 + (2s^4+2)^{1/2} - 4s^2}.$$

The numerator of the derivative of this expression is

$$\begin{aligned}&4\left(s^8 - 4s^7 + 2s^6 + 2(2s^4+2)^{1/2}s^4 - 2(2s^4+2)^{1/2}s^2 - 2s^2 + 4s - 1\right)\\ &= 4(s^2-1)(s^6 - 4s^5 + 3s^4 - 4s^3 + 3s^2 - 4s + 1 + 2(2s^4+2)^{1/2}s^2)\end{aligned}$$

and the denominator is obviously positive. We shall prove that

$$g_1(s) = s^6 - 4s^5 + 3s^4 - 4s^3 + 3s^2 - 4s + 1 + 2(2s^4+2)^{1/2}s^2$$

is positive for $s > 1$, hence f_1 is strictly increasing. We write $g_1(s) = 0$ as

$$s^6 - 4s^5 + 3s^4 - 4s^3 + 3s^2 - 4s + 1 = -2(2s^4+2)^{1/2}s^2, \tag{2.5}$$

square both sides and get

$$(s^8 - 4s^7 - 4s^5 + 6s^4 - 4s^3 - 4s + 1)(s-1)^4 = 0.$$

Denoting by $h_1(s) = s^8 - 4s^7 - 4s^5 + 6s^4 - 4s^3 - 4s + 1$ we get

$$h_1(s+4) = s^8 + 28s^7 + 336s^6 + 2236s^5 + 8886s^4 + 20956s^3 + 26640s^2 + 12604s - 2831,$$

which has only one change of sign. We apply Descartes' rule of signs for $h_1(s+4)$ and we obtain that the polynomial $h_1(s)$ has a single root greater than 4. We denote by $k_1(s)$ the 6th degree polynomial in the left hand side of (2.5) and get

$$k_1(s+4) = s^6 + 20s^5 + 163s^4 + 684s^3 + 1523s^2 + 1620s + 545. \tag{2.6}$$

Then the polynomial (2.6) is positive on $s > 4$, hence $g_1(s) = 0$ has no solutions on $s > 1$. It follows that f_1 is strictly increasing on $(1, \infty)$. Since $\lim_{t\to 1} f_1(t) = 2/3$ and $\lim_{t\to\infty} f_1(t) = 1$, the theorem is proved. □

We try to see if a similar result can be obtained by taking instead of $M_2 = Q$ another power mean. For $p = 5/2$ we can prove

Theorem 2.2. *The double inequality*

$$\alpha H(t) + (1-\alpha)M_{5/2}(t) < G(t) < \beta H(t) + (1-\beta)M_{5/2}(t), \ \forall t > 1$$

holds if and only if $\alpha \geq 1$ *and* $\beta \leq 5/7$. *The function*

$$f_2(t) = \frac{M_{5/2}(t) - G(t)}{M_{5/2}(t) - H(t)}$$

is strictly increasing on $(1, \infty)$.

Proof. We have

$$f_2(t) = \frac{(\frac{1}{2}t^{5/2} + \frac{1}{2})^{2/5} - t^{1/2}}{(\frac{1}{2}t^{5/2} + \frac{1}{2})^{2/5} - \frac{2t}{t+1}}. \tag{2.7}$$

By substituting $t = s^2$, $s > 1$ we get

$$f_2(s^2) = \frac{((16s^5 + 16)^{2/5} - 4s)(s^2 + 1)}{(s^2 + 1)(16s^5 + 16)^{2/5} - 8s^2}.$$

We differentiate the above function and obtain its numerator

$$32(s-1)(2s^8 - 6s^7 - 2s^6 - 2s^5 - 2s^3 - 2s^2 - 6s + 2 + s^2\,(s+1)\,(16s^5 + 16)^{2/5}),$$

the denominator being positive. We denote

$$g_2(s) = 2s^8 - 6s^7 - 2s^6 - 2s^5 - 2s^3 - 2s^2 - 6s + 2 + s^2\,(s+1)\,(16s^5 + 16)^{2/5}$$

and we write $g_2(s) = 0$ as

$$\frac{2(s^8 - 3s^7 - s^6 - s^5 - s^3 - s^2 - 3s + 1)}{s^2\,(s+1)} = -(16s^5 + 16)^{2/5}. \tag{2.8}$$

We apply the 5th power to both sides of (2.8) and get $h_2(s) = 0$, where

$$\begin{aligned} h_2(s) &= s^{30} - 10s^{29} + 25s^{28} + 20s^{27} - 50s^{26} - 196s^{25} - 150s^{24} + 320s^{23} \\ &+1305s^{22} + 2090s^{21} + 2439s^{20} + 2320s^{19} + 2550s^{18} + 3460s^{17} + 4760s^{16} \\ &+5240s^{15} + 4760s^{14} + 3460s^{13} + 2550s^{12} + 2320s^{11} + 2439s^{10} + 2090s^9 \\ &+1305s^8 + 320s^7 - 150s^6 - 196s^5 - 50s^4 + 20s^3 + 25s^2 - 10s + 1. \end{aligned}$$

Using the Sturm sequence, we obtain that $h_2(s)$ has no roots in $(1, \infty)$. It follows that $h_2(s) > 0$ on $(1, \infty)$, and the derivative of $f_2(t)$ is positive on this interval, hence $f_2(t)$ is strictly increasing. Since $\lim_{t\to 1} f_2(t) = 5/7$, $\lim_{t\to\infty} f_2(t) = 1$, the theorem is proved. □

Remark 2.3. We can consider the function

$$f_3(t) = \frac{M_p(t) - G(t)}{M_p(t) - H(t)}$$

for arbitrary $p > 0$. It is easy to check that $\lim_{t\to 1} f_3(t) = p/(p+1)$ and $\lim_{t\to\infty} f_3(t) = 1$. It remains to study the monotonicity of f_3. In the following theorem we prove that, for $p > 5/2$, the function f_3 is not monotone on $(1, \infty)$.

Theorem 2.4. *For $p > 5/2$, the infimum of the function f_3 on $(1, \infty)$ satisfies the inequality*

$$\inf_{t>1} f_3(t) < \frac{p}{p+1}.$$

Proof. Let $p > 5/2$. The function f_3 is given by

$$f_3(t) = \frac{(\frac{1}{2}t^p + \frac{1}{2})^{1/p} - t^{1/2}}{(\frac{1}{2}t^p + \frac{1}{2})^{1/p} - \frac{2t}{t+1}},$$

and after the substitution $t = s^2$, $s > 1$ we get

$$f_3(s^2) = \frac{((\frac{1}{2}s^{2p} + \frac{1}{2})^{1/p} - s)(s^2+1)}{(s^2+1)(\frac{1}{2}s^{2p} + \frac{1}{2})^{1/p} - 2s^2}.$$

The Taylor series for $s_0 = 1$ reads

$$\frac{p}{p+1} - \frac{p(2p-5)}{12(p+1)}(s-1)^2 + \frac{p(2p-5)}{12(p+1)}(s-1)^3 + O((s-1)^4), \text{ for } s \to 1$$

and its derivative will be

$$-\frac{p(2p-5)}{6(p+1)}(s-1) + O((s-1)^2).$$

It follows that the derivative is negative at least for $s > 1$ close to 1, hence f_3 decreases and $\inf_{t>1} f_3(t) < p/(p+1)$. □

Based on the results in theorems 2.1 and 2.2, we formulate the following **Open problem.** Prove that the function f_3 is strictly increasing on $(1, \infty)$ for each $p \in (0, 5/2]$. Then, for each $p \in (0, 5/2]$, the double inequality

$$\alpha H(t) + (1-\alpha)M_p(t) < G(t) < \beta H(t) + (1-\beta)M_p(t), \ \forall t > 1$$

will be true if and only if $\alpha \geq 1$ and $\beta \leq p/(p+1)$.

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