Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 3, 385–391

On the geometry of conformal Hamiltonian of the time-dependent coupled harmonic oscillators

Hengameh Raeisi-Dehkordi and Mircea Neagu

Abstract. In this paper we construct the distinguished (d-) geometry (in the sense of d-connections, d-torsions, d-curvatures, momentum geometrical gravitational and electromagnetic theories) for the conformal Hamiltonian of the time-dependent coupled oscillators on the dual 1-jet space $J^{1*}\left(\mathbb{R},\mathbb{R}^2\right)$.

Mathematics Subject Classification (2010): 70S05, 53C07, 53C80.

Keywords: Conformal Hamiltonian of time-dependent coupled harmonic oscillators, Cartan canonical connection, d-torsions, d-curvatures, geometrical Einstein-like equations.

We dedicate this paper to the memory of Professor Gheorghe Atanasiu (1939-2014).

1. Introduction

The model of time-dependent coupled oscillators is used to investigate the dynamics of charged particle motion in the presence of time-varying magnetic fields. At the same time, the model of coupled harmonic oscillator has also been widely used to study the quantum effects in mesoscopic coupled electric circuits. For more details, please see [2].

If m_i $(i=1,2)$, ω_i $(i=1,2)$, and $k(t)$ are the time-dependent mass, frequency, and the coupling parameter, respectively, then the conformal Hamiltonian of the time-dependent coupled harmonic oscillators is given on the dual 1-jet space $J^{1*}\left(\mathbb{R},\mathbb{R}^2\right)$ by [2]

$$H(t,x,p)=h_{11}(t)e^{\sigma(x)}\left[\frac{\left(p_1^1\right)^2}{m_1(t)}+\frac{\left(p_2^1\right)^2}{m_2(t)}\right]+\mathcal{F}(t,x)= \tag{1.1}$$

$$=h_{11}(t)e^{\sigma(x)}\frac{\delta^{ij}}{m_i(t)}p_i^1p_j^1+\mathcal{F}(t,x)=$$

$$=h_{11}(t)g^{ij}(t,x)p_i^1p_j^1+\mathcal{F}(t,x),$$

where $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is a smooth conformal function on $\mathbb{R}^2$, h_{11} is a Riemannian metric on $\mathbb{R}$, $(t, x, p) = (t, x^1, x^2, p_1^1, p_2^1)$ are the coordinates of the space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$, and

$$\mathcal{F}(t,x) = \frac{m_1(t)\omega_1^2(t)x_1^2}{2} + \frac{m_2(t)\omega_2^2(t)x_2^2}{2} + \frac{k(t)\left(x_2 - x_1\right)^2}{2}.$$

The jet coordinates (t, x, p) transform by the rules:

$$\tilde{t} = \tilde{t}(t),\ \ \tilde{x}^i = \tilde{x}^i(x^j),\ \ \tilde{p}_i^1 = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{d\tilde{t}}{dt}p_j^1, \tag{1.2}$$

where $i, j = 1, 2$, rank $(\partial\tilde{x}^i/\partial x^j) = 2$ and $d\tilde{t}/dt \neq 0$.

2. The canonical nonlinear connection

The *fundamental metrical d-tensor* induced by the conformal Hamiltonian metric (1.1) is defined by (see [1] and [3])

$$g^{ij}(t,x) = \frac{h^{11}}{2}\frac{\partial^2 H}{\partial p_i^1 p_j^1} = \frac{\delta^{ij}}{m_i(t)}e^{\sigma(x)} \Longrightarrow g_{jk}(t,x) = \delta_{jk}m_k(t)e^{-\sigma(x)}. \tag{2.1}$$

If we use the notations

$$\varkappa_{11}^1 = \frac{h^{11}}{2}\frac{dh^{11}}{dt},$$

$$\Gamma_{ij}^k \stackrel{def}{=} \frac{g^{kl}}{2}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) = \frac{1}{2}\left(-\delta_i^k\sigma_j - \delta_j^k\sigma_i + \delta_{ij}\frac{m_j(t)}{m_k(t)}\sigma_k\right), \tag{2.2}$$

where $h^{11} = 1/h_{11} > 0$ and $\sigma_i = \partial\sigma/\partial x^i$, then we have the following geometrical result:

Proposition 2.1. *For the conformal Hamiltonian metric (*1.1*) the canonical nonlinear connection on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$ has the following components:*

$$N = \left(\underset{1}{N}{}_{(i)1}^{(1)} = \varkappa_{11}^1 p_i^1,\ \underset{2}{N}{}_{(i)j}^{(1)} = -\Gamma_{ij}^k p_k^1\right). \tag{2.3}$$

In other words, we have

$$\underset{2}{N}{}_{(i)j}^{(1)} = \frac{1}{2}\left(\sigma_i p_j^1 + \sigma_j p_i^1 - \delta_{ij}\frac{m_j(t)}{m_k(t)}\sigma_k p_k^1\right).$$

Proof. The canonical nonlinear connection produced by H on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$ has the following components:

$$\underset{1}{N}{}_{(i)1}^{(1)} = \varkappa_{11}^1 p_i^1$$

and

$$\underset{2}{N}{}_{(i)j}^{(1)} = \frac{h^{11}}{4}\left[\frac{\partial g_{ij}}{\partial x^k}\frac{\partial H}{\partial p_k^1} - \frac{\partial g_{ij}}{\partial p_k^1}\frac{\partial H}{\partial x^k} + g_{ik}\frac{\partial^2 H}{\partial x^j \partial p_k^1} + g_{jk}\frac{\partial^2 H}{\partial x^i \partial p_k^1}\right].$$

So, by direct computations, we obtain (2.3). □

3. N-linear Cartan canonical connection. d-Torsions and d-curvatures

The nonlinear connection N produces the following *dual adapted bases* of d-vector fields and d-covector fields:

$$\left\{\frac{\delta}{\delta t},\frac{\delta}{\delta x^i},\frac{\partial}{\partial p_i^1}\right\}\subset\mathcal{X}\left(J^{1*}\left(\mathbb{R},\mathbb{R}^2\right)\right),\quad\left\{dt,dx^i,\delta p_i^1\right\}\subset\mathcal{X}^*\left(J^{1*}\left(\mathbb{R},\mathbb{R}^2\right)\right),\tag{3.1}$$

where

$$\frac{\delta}{\delta t}=\frac{\partial}{\partial t}-\underset{1}{N}{}^{(1)}_{(r)1}\frac{\partial}{\partial p_r^1}=\frac{\partial}{\partial t}-\kappa^1_{11}p^1_r\frac{\partial}{\partial p_r^1},\tag{3.2}$$

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-\underset{2}{N}{}^{(1)}_{(r)i}\frac{\partial}{\partial p_r^1}=\frac{\partial}{\partial x^i}+\Gamma^s_{ri}p^1_s\frac{\partial}{\partial p_r^1},\tag{3.3}$$

$$\delta p_i^1=dp_i^1+\underset{1}{N}{}^{(1)}_{(i)1}dt+\underset{2}{N}{}^{(1)}_{(i)j}dx^j.$$

The naturalness of the geometrical adapted bases (3.1) is coming from the fact that, via a transformation of coordinates (1.2), their elements transform as the classical tensors. Therefore, the description of all subsequent geometrical objects on the dual 1-jet space $J^{1*}(\mathbb{R},\mathbb{R}^2)$ (e.g., the Cartan canonical N-linear connection, its torsion and curvature, etc.) will be done in local adapted components. As a result, by direct computations, we obtain the following geometrical result:

Proposition 3.1. *The Cartan canonical N-linear connection produced by the conformal Hamiltonian metric (*1.1*) has the following adapted local components:*

$$C\Gamma(N)=\left(\kappa^1_{11},\ A^i_{j1}=\delta^i_j\frac{m'_j(t)}{2m_i(t)},\ H^i_{jk}=\Gamma^i_{jk},\ C^{j(k)}_{i(1)}=0\right).\tag{3.4}$$

Proof. The adapted components of Cartan canonical connection are given by the formulas

$$A^i_{j1}=\frac{g^{il}}{2}\frac{\delta g_{lj}}{\delta t}=\frac{g^{il}}{2}\frac{dg_{lj}}{dt}=\frac{\delta^i_j}{2m_i(t)}\frac{dm_j}{dt}=\delta^i_j\frac{m'_j(t)}{2m_i(t)},$$

$$H^i_{jk}=\frac{g^{ir}}{2}\left(\frac{\delta g_{jr}}{\delta x^k}+\frac{\delta g_{kr}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^r}\right)=\Gamma^i_{jk},$$

$$C^{j(k)}_{i(1)}=-\frac{g_{ir}}{2}\left(\frac{\partial g^{jr}}{\partial p^c_k}+\frac{\partial g^{kr}}{\partial p^c_j}-\frac{\partial g^{jk}}{\partial p^c_r}\right)=0.$$

Using the derivative operators (3.2) and (3.3), the direct calculations lead us to the desired results. □

Proposition 3.2. *The Cartan canonical connection of the conformal Hamiltonian metric (*1.1*) has four effective d-torsions:*

$$T^r_{1j}=-A^r_{j1},\ P^{(1)\ (j)}_{(r)1(1)}=A^j_{r1},$$

$$R^{(1)}_{(r)1j}=\frac{\partial\Gamma^s_{rj}}{\partial t}p^1_s,$$

$$R^{(1)}_{(r)ij}=-\mathfrak{R}^k_{rij}p^1_k,$$

where

$$\mathfrak{R}^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{ij}\Gamma^l_{sk} - \Gamma^s_{ik}\Gamma^l_{sj}.$$

Proof. The Cartan canonical connection on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$ generally has six effective local d-tensors of torsion. For our particular Cartan canonical connection (3.4) these reduce only to four (the others are zero) [1]:

$$T^r_{1j} = -A^r_{j1}, \ P^{r(j)}_{i(1)} = C^{r(j)}_{i(1)} = 0,$$

$$P^{(1)\ (j)}_{(r)1(1)} = \frac{\partial \underset{1}{N}{}^{(1)}_{(r)1}}{\partial p^1_j} + A^j_{r1} - \delta^j_r \chi^1_{11} = A^j_{r1},$$

$$P^{(1)\ (j)}_{(r)i(1)} = \frac{\partial \underset{2}{N}{}^{(1)}_{(r)i}}{\partial p^1_j} + \Gamma^j_{ri} = 0,$$

$$R^{(1)}_{(r)1j} = \frac{\delta \underset{1}{N}{}^{(1)}_{(r)1}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta t} = -\frac{\partial \underset{2}{N}{}^{(1)}_{(r)j}}{\partial t} = \frac{\partial \Gamma^s_{rj}}{\partial t} p^1_s,$$

$$R^{(1)}_{(r)ij} = \frac{\delta \underset{2}{N}{}^{(1)}_{(r)i}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta x^i} = -\mathfrak{R}^k_{rij} p^1_k. \qquad \square$$

Proposition 3.3. *The Cartan canonical connection of the conformal Hamiltonian metric (*1.1*) has two effective d-curvatures:*

$$R^l_{i1k} = \frac{\partial A^l_{i1}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial t} + A^r_{i1}\Gamma^l_{rk} - \Gamma^r_{ik}A^l_{r1},$$

$$R^l_{ijk} = \mathfrak{R}^l_{ijk}.$$

Proof. A Cartan canonical connection on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R})$ generally has five local d-tensors of curvature. For our particular Cartan canonical connection (3.4) these reduce only to two (the others are zero). So, we have [1]:

$$R^l_{i1k} = \frac{\delta A^l_{i1}}{\delta x^k} - \frac{\delta \Gamma^l_{ik}}{\delta t} + A^r_{i1}\Gamma^l_{rk} - \Gamma^r_{ik}A^l_{r1} = \frac{\partial A^l_{i1}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial t} + A^r_{i1}\Gamma^l_{rk} - \Gamma^r_{ik}A^l_{r1},$$

$$R^l_{ijk} = \frac{\delta \Gamma^l_{ij}}{\delta x^k} - \frac{\delta \Gamma^l_{ik}}{\delta x^j} + \Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj} = \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj} = \mathfrak{R}^l_{ijk},$$

$$P^{l\ (k)}_{i1(1)} = \frac{\partial A^l_{i1}}{\partial p^1_k} = 0, \quad P^{l\ (k)}_{ij(1)} = \frac{\partial \Gamma^l_{ij}}{\partial p^1_k} = 0, \quad S^{l(j)(k)}_{i(1)(1)} = 0. \qquad \square$$

4. From the conformal Hamiltonian of the time-dependent coupled oscillators to field-like geometrical models

4.1. Momentum gravitational-like geometrical model

The conformal Hamiltonian metric (1.1) produces on the momentum phase space $J^{1*}(\mathbb{R},\mathbb{R}^2)$ the adapted metrical d-tensor (*momentum gravitational potential*)

$$\mathbb{G}=h_{11}dt\otimes dt+g_{ij}dx^i\otimes dx^j+h_{11}g^{ij}\delta p_i^1\otimes\delta p_j^1,$$

where g_{jk} and g^{ij} are given by (2.1). We postulate that the momentum gravitational potential $\mathbb{G}$ is governed by the *geometrical Einstein equations*

$$\mathrm{Ric}(C\Gamma(N))-\frac{\mathrm{Sc}(C\Gamma(N))}{2}\mathbb{G}=\mathcal{K}\mathbb{T},\tag{4.1}$$

where:

• $\mathrm{Ric}(C\Gamma(N))$ is the Ricci d-tensor associated with the Cartan canonical linear connection (3.4);

• $\mathrm{Sc}(C\Gamma(N))$ is the *scalar curvature*;

• $\mathcal{K}$ is the Einstein constant and $\mathbb{T}$ in an intrinsic momentum *stress-energy* d-tensor of matter.

Therefore, using the adapted basis of vector fields, we can locally describe the global geometrical Einstein equations (4.1). Consequently, some direct computations lead to:

Proposition 4.1. *The Ricci tensor of the Cartan canonical connection of the conformal Hamiltonian metric (*1.1*) has the following two effective Ricci d-tensors:*

$$\begin{aligned}
&R_{11}:=\mathrm{K}_{11}=0, && R_{1i}=R^1_{1i1}=0,\\
&R_{i1}=R^r_{i1r}, && R_{ij}=R^r_{ijr}=\mathfrak{R}^r_{ijr}:=\mathfrak{R}_{ij},\\
&R_{i(1)}^{\ \ (j)}:=-P_{i(1)}^{\ \ (j)}=-P^{r\ \ (j)}_{ir(1)}=0, && R^{(i)}_{(1)1}:=-P^{(i)}_{(1)1}=-P^{i\ \ (r)}_{r1(1)}=0,
\end{aligned}$$

$$\begin{aligned}
&R_{1(1)}^{\ \ (j)}=-P^{1(j)}_{1(1)1}=0,\\
&R^{(i)(j)}_{(1)(1)}:=-S^{(i)(j)}_{(1)(1)}=-S^{i(j)(r)}_{r(1)(1)}=0,\\
&R^{(i)}_{(1)j}:=-P^{(i)}_{(1)j}=-P^{i\ \ (r)}_{rj(1)}=0.
\end{aligned}$$

Corollary 4.2. *The scalar curvature of the Cartan canonical connection of the conformal Hamiltonian metric (*1.1*) is given by the following formula:*

$$\mathrm{Sc}(C\Gamma(N))=g^{ij}R_{ij}=g^{ij}\mathfrak{R}_{ij}:=\mathfrak{R}.$$

Corollary 4.3. *The geometrical momentum Einstein-like equations produced by the conformal Hamiltonian metric* (1.1) *are locally described by*

$$\begin{cases} -\dfrac{\mathfrak{R}}{2}h_{11} = \mathcal{K}\mathbb{T}_{11} \\ \mathfrak{R}_{ij} - \dfrac{\mathfrak{R}}{2}g_{ij} = \mathcal{K}\mathbb{T}_{ij} \\ -\dfrac{\mathfrak{R}}{2}h_{11}g^{ij} = \mathcal{K}\mathbb{T}^{(i)(j)}_{(1)(1)} \\ 0 = \mathbb{T}_{1i}, \quad R_{i1} = \mathcal{K}\mathbb{T}_{i1} \\ 0 = \mathbb{T}_{1(1)}^{\;\;(i)}, \quad 0 = \mathbb{T}_{i(1)}^{\;\;(j)} \\ 0 = \mathbb{T}^{(i)}_{(1)1}, \quad 0 = \mathbb{T}^{(i)}_{(1)j}. \end{cases}$$

The *geometrical momentum conservation-like laws* produced by the conformal Hamiltonian metric (1.1) are postulated by the following formulas (for more details, please see [1]):

$$\begin{cases} \left[\dfrac{\mathfrak{R}}{2}\right]_{/1} = R^r_{1|r} \\ \left[\mathfrak{R}^r_j - \dfrac{\mathfrak{R}}{2}\delta^r_j\right]_{|r} = 0 \\ \left[\dfrac{\mathfrak{R}}{2}\delta^j_r\right]|^{(r)}_{(1)} = 0, \end{cases}$$

where "$_{/1}$", "$_{|r}$" and "$|^{(r)}_{(1)}$" are the local covariant derivatives induced by the Cartan canonical connection $C\Gamma(N)$, and we have

$$R^i_1 = g^{iq}R_{q1}, \quad \mathfrak{R}^i_j = g^{iq}\mathfrak{R}_{qj}.$$

4.2. Geometrical momentum electromagnetic-like 2-form

On the momentum phase space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$, the distinguished geometrical electromagnetic 2-form is defined by

$$\mathbb{F} = F^{(i)}_{(1)j}\delta p^1_i \wedge dx^j,$$

where

$$F^{(i)}_{(1)j} = \frac{h^{11}}{2}\left[g^{jk}\underset{2}{N}{}^{(1)}_{(k)i} - g^{ik}\underset{2}{N}{}^{(1)}_{(k)i} + (g^{jk}\Gamma^r_{ki} - g^{ik}\Gamma^r_{kj})p^1_r\right].$$

By a direct calculation, the conformal Hamiltonian metric (1.1) produces the null momentum electromagnetic components

$$F^{(i)}_{(1)j} = 0.$$

References

[1] Atanasiu, Gh., Neagu, M., Oană, A., *The Geometry of Jet Multi-Time Lagrange and Hamiltonian Spaces: Applications in Theoretical Physics*, Fair Partners Publishers, Bucharest, 2013.

[2] Macedo, D.X., Guedes, I., *Time-dependent coupled harmonic oscillators*, Journal of Mathematical Physics **53**(2012), 10 pages.

[3] Miron, R., Hrimiuc, D., Shimada, H., Sabău, S.V., *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

Hengameh Raeisi-Dehkordi
Instituto de Matemática e Estatística,
Universidade de São Paulo
Rua do Matão 1010, 05508 090 São Paulo, Brazil
e-mail: `hengameh@ime.usp.br`

Mircea Neagu
University Transilvania of Braşov
Department of Mathematics and Informatics
50 Iuliu Maniu Blvd., 500091 Braşov, Romania
e-mail: `mircea.neagu@unitbv.ro`