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# Multiple symmetric solutions for some hemivariational inequalities

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**Abstract.** In the present paper we prove some multiplicity results for hemivariational inequalities defined on the unit ball or on the whole space. By variational methods, we demonstrate that the solutions of these inequalities are invariant by spherical cap symmetrization, the main tools being the symmetric version of Ekeland's variational principle proved by M. Squassina [11] and a nonsmooth version of the symmetric minimax principle due to J. Van Schaftingen [13].

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## 1. Introduction and main results

In this paper we are treating two different problems, which will be detailed below.

### 1.1. The first problem

Consider the following semi-linear elliptic differential inclusion problem, coupled with the homogeneous Dirichlet boundary condition:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \lambda \partial F(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}^1_\lambda$}$$

where $\lambda$ is a positive parameter, $1 < p < N$, $\Omega \subset \mathbb{R}^N$ is the unit ball, $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator, and $\partial F(x, s)$ stands for the generalized gradient of the locally Lipschitz function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ at the point $s \in \mathbb{R}$ with respect to the second variable (see for details Section 2). Here and in the sequel $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

Such problems arise mostly in mathematical physics, where solutions of elliptic problems correspond to certain equilibrium state of the physical system. This is the reason why problems of this type were intensively studied by several authors in the last years.

In the study of PDE-s are often used different symmetrization techniques. We can find many papers where the solutions are for e.g. radially symmetric functions (see Squassina [12]), axially symmetric functions (Kristály, Mezei in [7]) or has some symmetry properties with respect to certain group actions (Farkas, Mezei in [5]). Recently was applied the spherical cap and Schwarz symmetrization for such problems. Van Schaftingen in [13] and Squassina in [11] developed an abstract framework for the symmetrizations. Using their results, Filipucci, Pucci, Varga in [9] obtained existence results of some eigenvalue problems and Farkas, Varga in [6] proved multiplicity results for a model quasi-linear elliptic system in case of $C^1$ functionals.

The purpose of our paper is to extend these results for locally Lipschitz functions. We ensure the existence of multiple spherical cap symmetric solutions for the problem $(\mathcal{P}^1_\lambda)$, where the natural functional space is the Sobolev space $W_0^{1,p}(\Omega)$, endowed with its standard norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p + \int_\Omega |u(x)|^p\right)^{1/p}.$$

In order to obtain our result, we need the following assumptions on the function $F$:

$(\mathbf{F_1})$ $\lim\limits_{|s|\to 0} \dfrac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{|s|^{p-1}} = 0;$

$(\mathbf{F_2})$ $\lim\limits_{|s|\to +\infty} \dfrac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{|s|^{p-1}} = 0;$

$(\mathbf{F_3})$ There exists an $u_0 \in W_0^{1,p}(\Omega)$, $u_0 \neq 0$ such that

$$\int\limits_\Omega F(x, u_0(x))dx > 0.$$

$(\mathbf{F_4})$ $F(x,s) = F(y,s)$ for a.e. $x, y \in \Omega$, with $|x| = |y|$ and all $s \in \mathbb{R}$;

$(\mathbf{F_5})$ $F(x,s) \le F(x,-s)$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^-$.

The first main result of the paper is the following:

**Theorem 1.1.** *Assume that* $1 < p < N$. *Let* $\Omega \subset \mathbb{R}^N$ *be the unit ball and* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a locally Lipschitz function with* $F(x,0) = 0$, *satisfying* $(\mathbf{F_1})$-$(\mathbf{F_5})$. *Then,*

(a) *there exists a* $\lambda_F$ *such that, for every* $0 < \lambda \le \lambda_F$ *the problem* $(\mathcal{P}^1_\lambda)$ *has only the trivial solution;*

(b) *there exists a* $\lambda_1$ *such that, for every* $\lambda > \lambda_1$ *the problem* $(\mathcal{P}^1_\lambda)$ *has at least two weak solutions in* $W_0^{1,p}(\Omega)$, *invariants by spherical cap symmetrization (for details, see Section 2).*

**Remark 1.1.** *Choosing* $p = 3$, *the function* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$F(x,s) = \begin{cases} |x|(s^4 - s^2), & \textit{if } |s| \le 1 \\ |x| \ln s^2, & \textit{if } |s| > 1. \end{cases} \tag{1.1}$$

*fulfills the hypotheses* $(\mathbf{F_1})$-$(\mathbf{F_5})$.

### 1.2. The second problem

Let $\Omega = \mathbb{R}^N$. Consider a real, separable, reflexive Banach space $(X, \|\cdot\|_X)$ and its topological dual $(X^\star, \|\cdot\|_{X^\star})$. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function. In addition, let $p$ be such that $2 \leq p < N$, while $p^\star = \dfrac{Np}{N-p}$ denotes the Sobolev critical exponent.

Our second problem is formulated as follows:

Find $u \in X$ such that

$$\langle Au, v\rangle + \int_{\mathbb{R}^N} F_y^0(x; u(x); -v(x))dx \geq 0,\ \forall v \in X, \tag{$\mathcal{P}_\lambda^2$}$$

where $F_y^0$ denotes the generalized directional derivative of $F$ in the second variable.

In order to derive our second existence result, we need to impose the following hypotheses:

**(CT)** Suppose that for $r \in [p, p^\star]$, the inclusion $X \hookrightarrow L^r(\mathbb{R}^N)$ is continuous with the embedding constant $C_r$.

(**CP**) Assume that for $r \in (p, p^\star)$, the embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact.

Notice that $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $X^\star$ and $X$ and $||\cdot||_r$ is the norm of $L^r(\mathbb{R}^N)$.

Let $A : X \to X^\star$ be a potential operator with the potential $a : X \to \mathbb{R}$, that is, $a$ is Gâteaux differentiable and for every $u, v \in X$ we have

$$\lim_{t\to 0} \frac{a(u+tv) - a(u)}{t} = \langle A(u), v\rangle.$$

For a potential we always assume that $a(0) = 0$. In addition, we suppose that $A : X \to X^\star$ satisfies the following properties:

($\mathbf{A_1}$) $A$ is hemicontinuous, i.e. $A$ is continuous on line segments in $X$ and $X^\star$ equipped with the weak topology.

($\mathbf{A_2}$) $A$ is homogeneous of degree $p-1$, i.e. for every $u \in X$ and $t > 0$ we have $A(tu) = t^{p-1}A(u)$.

($\mathbf{A_3}$) $A : X \to X^\star$ is a strongly monotone operator, i.e. there exists a continuous function $\tau : [0, \infty) \to [0, \infty)$ which is strictly positive on $(0, \infty)$, $\tau(0) = 0$, $\lim\limits_{t\to\infty} \tau(t) = \infty$ and

$$\langle A(u) - A(v), u - v\rangle \geq \tau(||u - v||_X)||u - v||_X,$$

for all $u, v \in X$.

($\mathbf{A_4}$) $a(u) \geq c\|u\|_X^p$, for all $u \in X$, where $c$ is a positive constant.

($\mathbf{A_5}$) $a(u^H) \leq a(u)$, for all $u \in X$, where $u^H$ denotes the polarization of $u$ (for details, see Section 2.).

**Remark 1.2.** *By conditions* ($\mathbf{A_1}$) *and* ($\mathbf{A_2}$), *we have* $a(u) = \frac{1}{p}\langle A(u), u\rangle$.

Furthermore, we suppose that the following additional condition holds: there exists $c > 0$ and $r \in (p, p^\star)$ such that

($\mathbf{F_1'}$) $|\xi| \leq c(|s|^{p-1} + |s|^{r-1}), \forall s \in \mathbb{R}, \xi \in F(x, s)$ and a.e. $x \in \mathbb{R}^N$.

Moreover, instead of ($\mathbf{F_2}$), we assume that:

($\mathbf{F_2'}$) there exists $q \in (0,p)$, $\nu \in (p, p^\star)$, $\alpha \in L^{\frac{\nu}{\nu-q}}(\mathbb{R}^N)$, $\beta \in L^1(\mathbb{R}^N)$ such that

$$F(z,s) \leq \alpha(z)|s|^q + \beta(z)$$

for all $s \in \mathbb{R}$ and a.e. $z \in \mathbb{R}^N$.

**Remark 1.3.** *When $\Omega$ is the unit ball, by conditions ($\mathbf{F_1}$) and ($\mathbf{F_2}$), we can deduce the assumption ($\mathbf{F_1'}$). But in the case of this second problem when we assume that $\Omega = \mathbb{R}^N$, we really need the condition ($\mathbf{F_1'}$).*

Now we can state our second main result:

**Theorem 1.2.** *Assume that $2 \leq p < N$ and let $\Omega = \mathbb{R}^N$. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function, $A : X \to X^\star$ be a potential operator such that the conditions ($\mathbf{A_1}$) $-$ ($\mathbf{A_5}$) and ($\mathbf{CT}$), ($\mathbf{CP}$), ($\mathbf{F_1}$), ($\mathbf{F_1'}$), ($\mathbf{F_2'}$), ($\mathbf{F_4}$), ($\mathbf{F_5}$) are fulfilled. Then, there exists $\lambda_2 > 0$ such that for every $\lambda > \lambda_2$ the problem ($\mathcal{P}_\lambda^2$) has two nontrivial solutions, which are invariants by spherical cap symmetrization.*

The energy functional related to the problem ($\mathcal{P}_\lambda^2$) is defined as follows:

$$\mathscr{A}_\lambda(u) = a(u) - \lambda\tilde{\mathcal{F}}(u),$$

where $\tilde{\mathcal{F}} : X \to \mathbb{R}$ is a function defined by $\tilde{\mathcal{F}}(u) = \displaystyle\int_{\mathbb{R}^N} F(x, u(x))dx$.

**Remark 1.4.** *We observe that, using Proposition 5.1.2. from Cs. Varga and A. Kristály [8], due to condition ($\mathbf{F_1'}$), we have that*

$$\tilde{\mathcal{F}}^0(u;v) \leq \int_{\mathbb{R}^N} F_y^0(x, u(x); v(x))dx. \tag{1.2}$$

*Therefore, it follows that the critical points of the energy functional $\mathscr{A}_\lambda$ are the (weak) solutions of the problem ($\mathcal{P}_\lambda^2$).*

## 2. Preliminaries and abstract framework

In this section we give a brief overview on some preparatory results used in the sequel.

### 2.1. Locally Lipschitz functions

In the following, we recall some basic definitions and properties from the theory developed by F. Clarke [4].

Let $E$ be a Banach space, $E^*$ be its topological dual space, $V$ be an open subset of $E$ and $f : V \to \mathbb{R}$ be a functional.

**Definition 2.1.** *The functional $f : V \to \mathbb{R}$ is called locally Lipschitz if every point $v \in V$ possesses a neighborhood $\mathcal{V}$ such that*

$$|f(z) - f(w)| \leq K_v \|z - w\|_E, \quad \forall w, z \in \mathcal{V},$$

*for a constant $K_v > 0$ which depends on $\mathcal{V}$.*

**Definition 2.2.** *The generalized derivative of a locally Lipschitz functional $f : V \to \mathbb{R}$ at the point $v \in V$ along the direction $w \in E$ is denoted by $f^0(v;w)$, i.e.*

$$f^0(v;w) = \limsup_{\substack{z \to v \\ t \searrow 0}} \frac{f(z+tw) - f(z)}{t}.$$

We recall here some useful properties of the generalized directional derivative for locally Lipschitz functions (see F. Clarke [4]).

**Definition 2.3.** *Let $E$ be a Banach space. A locally Lipschitz functional $h : E \to \mathbb{R}$ is said to satisfy the* non-smooth Palais-Smale condition at level $c \in \mathbb{R}$ (*for brevity we shall use the notation $(PS)_c$-condition*) *if any sequence $\{u_n\} \subset E$ which satisfies*

(i) *$h(u_n) \to c$;*
(ii) *there exists $\{\varepsilon_n\} \subset \mathbb{R}$, $\varepsilon_n \downarrow 0$ such that $h^0(u_n; v - u_n) + \varepsilon_n \|v - u_n\|_e \geq 0$, for all $v \in E$ and all $n \in \mathbb{N}$*

*admits a convergent subsequence. If this is true for every $c \in \mathbb{R}$, we say that $h$ satisfies the non-smooth $(PS)$-condition.*

**Remark 2.1.** *If we use the notation $\lambda_h(u) = \inf\limits_{w \in \partial h(u)} \|w\|_{E^\star}$ (see K.-C. Chang* [3]*) and we replace the condition $(ii)$ from the above definition with the following one:*
(ii)' $\lambda_h(u_n) \to 0$,
*we obtain an equivalent definition with the Definition* 2.3.

**Definition 2.4.** *The generalized gradient of $f : V :\to \mathbb{R}$ at the point $v \in V$ is a subset of $E^*$, defined by*

$$\partial f(v) = \{y^* \in E^* : \langle y^*, w\rangle \leq f^0(v;w), \quad \text{for each } w \in E\}. \tag{2.1}$$

**Remark 2.2.** *Using the Hahn-Banach theorem (see, for example H. Brezis* [2]*), it is easy to see that the set $\partial f(v)$ is nonempty for every $v \in E$.*

The next result will be crucial in the proofs of our main result.

**Theorem 2.1.** (Lebourg's Mean Value Theorem, F. Clarke [4]) *Let $U$ be an open subset of a Banach space $E$, let $x, y$ be two points of $U$ such that the line segment $[x,y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$ is contained in $U$ and let $f : U \to \mathbb{R}$ be a locally Lipschitz function. Then there exists $u \in [x,\ y]\backslash\{x,\ y\}$ such that*

$$f(y) - f(x) = \langle z,\ y - x\rangle,$$

*for some $z \in \partial f(u)$.*

### 2.2. Abstract framework of symmetrization

Now we recall the definition of spherical cap symmetrization and polarization.

**Definition 2.5** (Spherical cap symmetrization)**.** *Let $P \in \partial B(0,1) \cap \mathbb{R}^N$. The spherical cap symmetrization of the set $A$ with respect to $P$ is the unique set $A^*$ such that $A^* \cap \{0\} = A \cap \{0\}$ and for any $r \geq 0$,*

$$A^* \cap \partial B(0,r) = B_g(rP,\rho) \cap \partial B(0,r) \text{ for some } \rho \geq 0,$$
$$\mathcal{H}^{N-1}(A^* \cap \partial B(0,r)) = \mathcal{H}^{N-1}(A \cap \partial B(0,r)),$$

*where $\mathcal{H}^{N-1}$ is the outer Hausdorff $(N-1)$-dimensional measure and $B_g(rP,\rho)$ denotes the geodeisc ball on the sphere $\partial B(0,r)$ of center $rP$ and radius $\rho$. By definition $B_g(rP,0)=\emptyset$.*

**Definition 2.6.** *The spherical cap symmetrization of a function $f:\Omega\to\overline{\mathbb{R}}$ is the unique function $u^*:\Omega^*\to\overline{\mathbb{R}}$ such that, for all $c\in\mathbb{R}$,*

$$\{u^*>c\}=\{u>c\}^*.$$

**Definition 2.7** (Polarization)**.** *A subset $H$ of $\mathbb{R}^N$ is called a polarizer if it is a closed affine half-space of $\mathbb{R}^N$, namely the set of points $x$ which satisfy $\alpha\cdot x\le\beta$ for some $\alpha\in\mathbb{R}^N$ and $\beta\in\mathbb{R}$ with $|\alpha|=1$. Given $x$ in $\mathbb{R}^N$ and a polarizer $H$ the reflection of $x$ with respect to the boundary of $H$ is denoted by $x_H$. The polarization of a function $u:\mathbb{R}^N\to\mathbb{R}^+$ by a polarizer $H$ is the function $u^H:\mathbb{R}^N\to\mathbb{R}^+$ defined by*

$$u^H(x)=\begin{cases}\max\{u(x),u(x_H)\}, & \text{if } x\in H\\ \min\{u(x),u(x_H)\}, & \text{if } x\in\mathbb{R}^N\setminus H.\end{cases}\tag{2.2}$$

*The polarization $C^H\subset\mathbb{R}^N$ of a set $C\subset\mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{C^H}=(\chi_C)^H$, where $\chi$ denotes the characteristic function. The polarization $u^H$ of a positive function $u$ defined on $C\subset\mathbb{R}^N$ is the restriction to $C^H$ of the polarization of the extension $\tilde{u}:\mathbb{R}^N\to\mathbb{R}^+$ of $u$ by zero outside $C$. The polarization of a function which may change sign is defined by $u^H:=|u|^H$, for any given polarizer $H$.*

Following J. Van Schaftingen [13], consider the abstract framework below:

Let $X$, $V$ and $W$ be three real Banach spaces, with $X\subset V\subset W$ and let $S\subset X$. For the clarity, we present some crucial abstract symmetrization and polarization results of J. Van Schaftingen [13] and of M. Squassina [11]. Let us first introduce the following main assumption.

**Definition 2.8.** *Let $\mathcal{H}_\star$ be a pathconnected topological space and denote by $h:S\times\mathcal{H}_\star\to S$, $(u,H)\mapsto u^H$, the polarization map. Let $\star:S\to V, u\mapsto u^\star$, be any symmetrization map. Assume that the following properties hold.*

1) *The embeddings $X\hookrightarrow V$ and $V\hookrightarrow W$ are continuous;*
2) *$h$ is continuous;*
3) *$(u^\star)^H=(u^H)^\star=u^\star$ and $(u^H)^H=u^H$ for all $u\in S$ and $H\in\mathcal{H}_\star$;*
4) *for all $u\in S$ there exists a sequence $(H)_m\subset\mathcal{H}_\star$ such that $u^{H_1\dots H_m}\to u^\star$ in $V$;*
5) *$\|u^H-v^H\|_V\le\|u-v\|_V$ for all $u$, $v\in S$ and $H\in\mathcal{H}_\star$.*

Since there exists a map $\Theta:(X,\|\cdot\|_V)\to(S,\|\cdot\|_V)$ which is Lipschitz continuous, with Lipschitz constant $C_\Theta>0$, and such that $\Theta|_S=Id|_S$, both maps $h:S\times\mathcal{H}_\star\to S$ and $\star:S\to V$ can be extended to $h:X\times\mathcal{H}_\star\to S$ and $\star:X\to V$ by setting $u=(\Theta(u))^H$ and $u^\star=(\Theta(u))^\star$ for every $u\in X$ and $H\in\mathcal{H}_\star$.

The previous properties, in particular 4) and 5), and the definition of $\Theta$ easily yield that

$$\|u^H-v^H\|_V\le C_\Theta\|u-v\|_V,\qquad \|u^\star-v^\star\|_V\le C_\Theta\|u-v\|_V\tag{2.3}$$

for all $u$, $v\in X$ and for all $H\in\mathcal{H}_\star$.

Some known examples of spherical cap symmetrization with Dirichlet boundary and of Schwarz symmetrization are given by J. Van Schaftingen in [13].

### 2.3. Variational framework

We recall three results which will play an essential role in what follows.

**Proposition 2.1.** (*Proposition 3.3. of R. Filippucci et al.* [9]) *Let $G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, satisfying property* ($\mathbf{F_4}$), *that is $G(x,u) = G(y,u)$ for a.e. $x$, $y \in \mathbb{R}^N$, with $|x| = |y|$, and all $u \in \mathbb{R}$. Then, for all $H \in \mathcal{H}_\star$*

$$\int_{\mathbb{R}^N} G(x,u(x))dx = \int_{\mathbb{R}^N} G(x,u^H(x))dx \tag{2.4}$$

*along any $u : \mathbb{R}^N \to \mathbb{R}_0^+$, with $G(\cdot,u(\cdot)) \in L^1(\mathbb{R}^N)$.*

**Remark 2.3.** *The statement of the above proposition remains valid if we choose $\Omega = \Omega^H \subset \mathbb{R}$ instead of the whole space $\mathbb{R}^N$ (see J. Van Schaftingen* [13, Proposition 2.19]*).*

In the paper of Cs. Varga and V. Varga [14] a quantitative deformation lemma is proved for locally Lipschitz functions. J. Van Schaftingen in [13], proves a symmetric version of this variational principle (see Theorem 3.5) for $C^1$ functionals. Using the mentioned results with slight modifications, we can prove the following symmetric variational principle for locally Lipschitz functionals.

**Theorem 2.2.** *Let $(X, V, \star, \mathcal{H}_\star, S)$ satisfy the assumptions of Definition* 2.8. *Denote by $\kappa > 0$ any constant with the property $\|u\|_V \le \kappa\|u\|_X$ for all $u \in X$. Let $e \in X \setminus \{0\}$ be fixed and*

$$\Gamma = \{\gamma : C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

*Consider also the locally Lipschitz functional $\Phi : X \to \mathbb{R}$, which satisfies:*

1) $\infty > c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t)) > a := \max\{\Phi(0), \Phi(e)\}$,
2) $\Phi(u^H) \le \Phi(u)$, *for all $u \in S$ and $H \in \mathcal{H}_\star$.*

*Then for every $0 < \varepsilon < \frac{c-a}{2}$, $\delta > 0$ and $\gamma \in \Gamma$, with the properties*

i) $\sup_{t\in[0,1]} \Phi(\gamma(t)) \le c + \varepsilon$;
ii) $\gamma([0,1]) \subset S$;
iii) $\{\gamma(0), \gamma(1)\}^{H_0} = \{\gamma(0), \gamma(1)\}$ *for some $H_0 \in \mathcal{H}_\star$,*

*there exists $u_\varepsilon \in X$ such that*

a) $c - 2\varepsilon \le \Phi(u_\varepsilon) \le c + 2\varepsilon$;
b) $\|u_\varepsilon - u_\varepsilon^\star\|_V \le 2(2\kappa + 1)\delta$;
c) $\lambda_\Phi(u) \le 8\varepsilon/\delta$.

## 3. Proof of Theorem 1.1

**Definition 3.1.** *We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution to problem $(\mathcal{P}_\lambda^1)$ if there exists $\xi_F \in \partial F(x, u(x))$ for a.e. $x \in \Omega$ such that for all $v \in W_0^{1,p}(\Omega)$ we have*

$$\int_\Omega (|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv)dx = \lambda \int_\Omega \xi_F v(x)dx. \tag{3.1}$$

We consider the functionals $I, \mathcal{F} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)dx, \qquad \mathcal{F}(u) = \int_\Omega F(x, u(x))dx.$$

Now, we can define the energy functional associated to the problem $(\mathcal{P}_\lambda^1)$ by

$$\mathscr{E}_\lambda(u) = I(u) - \lambda\mathcal{F}(u).$$

**Remark 3.1.** *If $\Omega$ is bounded, using* [10, Theorem 1.3]*, we have*

$$\partial\mathcal{F}(u) \subset \int_\Omega \partial F(x, u(x))dx.$$

*Hence, the critical points of the energy functional $\mathscr{E}_\lambda$ are exactly the (weak) solutions of the problem $(\mathcal{P}_\lambda^1)$. So, instead of seeking for the solutions of the problem $(\mathcal{P}_\lambda^1)$, it is enough to look for the critical points of the energy functional $\mathscr{E}_\lambda$.*

Before proving our main result, we prove that the functional $\mathscr{E}_\lambda$ is coercive and it satisfies the non-smooth Palais-Smale condition on $W_0^{1,p}(\Omega)$.

**Lemma 3.1.** *The functional $\mathscr{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ is coercive for every $\lambda \geq 0$, that is, $\mathscr{E}_\lambda(u) \to \infty$ as $\|u\| \to \infty$, for all $u \in W_0^{1,p}(\Omega)$.*

*Proof.* Let us fix a $\lambda \geq 0$. In particular, from $(\mathbf{F_1})$, there exists a $\delta_1 > 0$ such that

$$|\xi| \leq \frac{1}{2}\cdot\frac{1}{p}c_p^{-p}\frac{1}{1+\lambda}|s|^{p-1}, |s| < \delta_1, \tag{3.2}$$

where $c_p$ is the best Sobolev constant in the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)(q \in [1, p^*)])$.

Due to $(\mathbf{F_2})$, it follows that for every $\varepsilon > 0$ there exists $\delta_2 = \delta_2(\varepsilon) > 0$, such that

$$\max\{|\xi| : \xi \in \partial F(x, s)\} \leq \varepsilon|s|^{p-1}, |s| > \delta_2.$$

Moreover, if $\varepsilon = \frac{1}{2}\cdot\frac{1}{p}c_p^{-p}\cdot\frac{1}{1+\lambda}$, then for every $\xi \in \partial F(x, s)$ one, has

$$|\xi| \leq \frac{1}{2}\cdot\frac{1}{p}c_p^{-p}\frac{1}{1+\lambda}|s|^{p-1}, |s| > \delta_2. \tag{3.3}$$

Since the set-valued mapping $\partial F$ is upper-semicontinuous, then there exists $C_F = \sup\{\partial F(x, [\delta_2, \delta_1])\}$, thus

$$|\xi| \leq \frac{1}{p}c_p^{-p}\frac{1}{1+\lambda}|s|^{p-1} + C_F, \quad \text{for all } s \in \mathbb{R}. \tag{3.4}$$

Now we can use Lebourg's mean value theorem (see Theorem 2.1), obtaining that:

$$|F(x, s)| = |F(x, s) - F(x, 0)| \leq |\xi_\theta s| \text{ for some } \xi_\theta \in \partial F(x, \theta s), \theta \in (0, 1).$$

Combining this inequality with the relation (3.4), we get

$$|F(x, s)| \leq \frac{1}{p}c_p^{-p}\frac{1}{1+\lambda}|s|^p + C_F|s|.$$

Moreover,

$$\mathscr{E}_\lambda(u) \geq \frac{1}{p}\|u\|^p - \frac{1}{p}\frac{\lambda}{1+\lambda}\left(\frac{\|u\|_p^p}{c_p^p}\right) - \lambda C_F\|u\|_1.$$

Therefore,

$$\begin{aligned}\mathscr{E}_\lambda(u) &\geq \frac{1}{p}\|u\|^p - \frac{1}{p}\frac{\lambda}{1+\lambda}\|u\|^p - \lambda\cdot C_1\|u\| \\ &= \frac{1}{p}\left(1-\frac{\lambda}{1+\lambda}\right)\|u\|^p - \lambda C_1\|u\| \to \infty\end{aligned}$$

as $\|u\| \to \infty$,where $C_1$ is a constant, which concludes our proof. □

**Lemma 3.2.** *For every $\lambda > 0$, $\mathscr{E}_\lambda$ satisfies the non-smooth Palais-Smale condition.*

*Proof.* Let $\lambda > 0$ be fixed. We consider a Palais-Smale sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ for $\mathscr{E}_\lambda$, i.e., for some $\varepsilon_n \to 0^+$, we have

$$\mathscr{E}_\lambda^o(u_n; u-u_n) \geq -\varepsilon_n\|u-u_n\| \tag{3.5}$$

and $\{\mathscr{E}_\lambda(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$. Since $\mathscr{E}_\lambda$ is coercive, the sequence $\{u_n\}$ is bounded. Therefore taking a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \to u$ strongly in $L^p$ (note that $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, see H. Brezis [2]). One clearly has,

$$\langle I'(u_n), u-u_n\rangle = \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n\right)(\nabla u - \nabla u_n) + \int_\Omega |u_n|^{p-2}u_n(u-u_n),$$

and

$$\langle I'(u), u_n-u\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u) + \int_\Omega |u|^{p-2}u(u_n-u).$$

Adding these two relations and from the fact that $|v-w|^p \leq (|v|^{p-2}v - |w|^{p-2}w)(v-w)$, one can conclude that

$$\begin{gathered}\langle I'(u_n), u-u_n\rangle + \langle I'(u), u_n-u\rangle = \\ \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u - \nabla u_n) + \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u-u_n) \\ \leq \int_\Omega (-|\nabla u_n - \nabla u|^p - |u_n-u|^p) = -\|u_n-u\|^p.\end{gathered} \tag{3.6}$$

On the other hand, by the relations

$$\mathscr{E}_\lambda^o(u_n; u-u_n) = \langle I'(u_n); u-u_n\rangle + \lambda\mathcal{F}^o(u_n; u_n-u),$$

$$\mathscr{E}_\lambda^o(u; u_n-u) = \langle I'(u); u_n-u\rangle + \lambda\mathcal{F}^o(u; u-u_n),$$

and the inequalities (3.5) and (3.6), we have

$$\begin{gathered}\|u_n-u\|^p \leq \varepsilon_n\|u-u_n\| - \mathscr{E}_\lambda^o(u; u_n-u) + \\ +\lambda(\mathcal{F}^o(u_n; u_n-u) + \mathcal{F}^o(u; u-u_n))\end{gathered} \tag{3.7}$$

Since the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we clearly have

$$\lim_{n\to\infty} \varepsilon_n \|u - u_n\| = 0. \tag{3.8}$$

Now fix $w^* \in \partial\mathscr{E}_\lambda(u)$. In particular, by the definition (2.1), we have $\langle w^*; u_n - u\rangle \leq \mathscr{E}_\lambda^o(u; u_n - u)$. Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, we obtain

$$\liminf_{n\to\infty} \mathscr{E}_\lambda^o(u; u_n - u) \geq 0. \tag{3.9}$$

Now, for the remaining two terms in the estimation (3.7), we use the fact that

$$\mathcal{F}^o(u; v) \leq \int_\Omega F^o(x, u(x); v(x))dx, \forall u, v \in W_0^{1,p}(\Omega).$$

Therefore,

$$\begin{aligned}
\mathcal{F}^o(u_n; u_n - u) &\leq \int_\Omega F^o(x, u_n(x); u_n(x) - u(x))dx \\
&= \int_\Omega \max\{\xi(u_n(x) - u(x)) : \xi \in \partial F(x, u_n(x))\}dx \\
&\leq \int_\Omega |u_n(x) - u(x)| \cdot \max\{|\xi| : \xi \in \partial F(x, u_n(x))\}dx.
\end{aligned}$$

From the upper semi-continuity property of $\partial F$, one has

$$\sup_{\substack{n\in\mathbb{N}\\ x\in\Omega}} \{|\xi| : \xi \in \partial F(x, u_n(x)\} < \infty.$$

Proceeding in the same way for $\mathcal{F}^o(u; u - u_n)$ and adding the outcomes, we obtain

$$\mathcal{F}^o(u_n; u_n - u) + \mathcal{F}^o(u; u - u_n) \leq K \cdot \int_\Omega |u_n(x) - u(x)| = K||u_n - u||_{L^1}, \tag{3.10}$$

where $K$ is a constant. Since $u_n \to u$ strongly in $L^1(\Omega)$, we have that

$$\limsup_{n\to\infty} (\mathcal{F}^o(u_n; u_n - u) + \mathcal{F}^o(u; u - u_n)) \leq 0. \tag{3.11}$$

Now, combining the inequalities (3.8), (3.9) and (3.11), we obtain

$$\limsup_{n\to\infty} \|u - u_n\|^p \leq 0,$$

which means that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. □

From the symmetric Ekeland's variational principle, given by M. Squassina in [11] (see Theorem 2.8), we can state the following corollary for locally Lipschitz functions.

**Lemma 3.3.** *Let $(X, V, \star, \mathcal{H}_\star, S)$ satisfy the assumptions given in Definition 2.8, with $V = L^p(\Omega)$, $X = W_0^{1,p}(\Omega)$ and with the further property that if $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$, then $u_n^\star \to u^\star$ in $L^p(\Omega)$. Assume that $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a locally Lipschitz functional bounded from below such that*

$$\Phi(u^H) \leq \Phi(u) \quad \textit{for all } u \in S \textit{ and } H \in \mathcal{H}_\star. \tag{3.12}$$

*and for all $u \in W_0^{1,p}(\Omega)$ there exists $\xi \in S$, with $\Phi(\xi) \leq \Phi(u)$.*

*If $\Phi$ satisfies the $(PS)_{\inf \Phi}$ condition, then there exists $v \in W_0^{1,p}(\Omega)$, such that $\Phi(v) = \inf \Phi$ and $v = v^\star$ in $L^p(\Omega)$.*

*Proof.* Put $\inf \Phi = d$. For the minimizing sequence $(u_n)_n$ we consider the following sequence:

$$\varepsilon_n = \begin{cases} \Phi(u_n) - d, & \text{if } \Phi(u_n) - d > 0 \\ \dfrac{1}{n}, & \text{if } \Phi(u_n) - d = 0. \end{cases}$$

Then $\Phi(u_n) \leq d + \varepsilon_n$ and $\varepsilon_n \to 0$ as $n \to \infty$. By [11, Theorem 2.8], there exists a sequence $(v_n)_n \subset W_0^{1,p}(\Omega)$ such that:

a) $\Phi(v_n) \leq \Phi(u_n)$;
b) $\lambda_\Phi(u_n) \to 0$;
c) $\|v_n - v_n^\star\|_p \to 0$;

Since $\Phi$ satisfies the $(PS)_d$ condition, there exists $v \in W_0^{1,p}(\Omega)$ such that $v_n \to v$ in $W_0^{1,p}(\Omega)$. Hence $v_n \to v$ in $L^p(\Omega)$ (because $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$) and so $v_n^\star \to v^\star$ in $L^p(\Omega)$ by assumption. In particular,

$$\|v - v^\star\|_p \leq \|v - v_n\|_p + \|v_n - v_n^\star\|_p + \|v_n^\star - v^\star\|_p \to 0.$$

Therefore $v = v^\star$ in $L^p(\Omega)$, as stated. $\square$

**Lemma 3.4.** *One has,*

$$\mathscr{E}_\lambda(u^H) \leq \mathscr{E}_\lambda(u).$$

*Proof.* One has that $\|\nabla u^H\|_p = \|\nabla u\|_p$, and $\|u^H\|_p \leq \|u\|_p$ (see Van Schaftingen [13]). On the other hand, due to Proposition 2.1, one has

$$\int_\Omega F(x, u(x))dx = \int_\Omega F(x, u^H(x))dx,$$

therefore

$$\mathscr{E}_\lambda(u^H) \leq \mathscr{E}_\lambda(u).$$

$\square$

Now we can prove our main result.

*Proof of Theorem* 1.1: (a) Suppose that $u \in W_0^{1,p}(\Omega)$ is a weak solution of $(\mathcal{P}_\lambda^1)$. Now, if we put $v = u$ as the test function in the relation (3.1), we obtain

$$||u||^p = \int_\Omega (|\nabla u|^p + |u|^p)dx = \lambda \int_\Omega \xi_F u dx \leq \lambda c_F \int_\Omega |u|^p dx \leq \lambda c_F c_p^p ||u||^p,$$

where $c_F = \max\limits_{s>0} \dfrac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{s^{p-1}} > 0$. Therefore, if $\lambda < \frac{1}{c_F c_p^p}$, then $u = 0$.

(b) By Lemma 3.3 there exists the global minimum $v_\lambda = v_\lambda^*$ of the energy functional $\mathscr{E}_\lambda$.

We now turn to establish the existence of the second nontrivial solution of $(\mathcal{P}_\lambda^1)$. From the assumption $(\mathbf{F_3})$, one has

$$\mathscr{E}_\lambda(u_0) = \frac{1}{p}\|u_0\|^p - \lambda \int_\Omega F(x, u_0(x))dx = A - \lambda B,$$

where $A = \frac{1}{p}\|u_0\|^p > 0$, and $B = \int_\Omega F(x, u_0(x))dx > 0$. Consequently, there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, we have that $h(\lambda) = A - \lambda B < 0$, therefore

$$\mathscr{E}_\lambda(u_0) = \frac{1}{p}\|u_0\|^p - \lambda \int_\Omega F(x, u_0(x))dx < 0.$$

In fact, we may choose,

$$\lambda_0 = \frac{1}{p} \inf \left\{ \frac{\|u\|^p}{\mathcal{F}(u)} : u \in W_0^{1,p}(\Omega), \mathcal{F}(u) > 0 \right\}.$$

Now, fix $\lambda > \lambda_0$. From $(\mathbf{F_1})$ it follows that for fixed $\frac{1}{p\lambda c_p^p} > \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$\max\{|\xi| : \xi \in \partial F(x, s)\} \leq \varepsilon |s|^{p-1}, |s| < \delta,$$

therefore for every $\xi \in \partial F(x, s)$, $|s| \leq \delta$ one has,

$$|\xi| \leq \varepsilon \cdot |s|^{p-1}. \tag{3.13}$$

Using the Lebourg's mean value theorem (see Theorem 2.1), we obtain:

$$|F(x, s)| = |F(x, s) - F(x, 0)| \leq |\xi_\theta s| \text{ for some } \xi_\theta \in \partial F(x, \theta s), \theta \in (0, 1),$$

which means that using the (3.13) iequality, we have

$$|F(x, s)| \leq \varepsilon \cdot |s|^p,$$

whenever $|s| \leq \delta$.

Thus, if $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho < \min\left\{\frac{\delta}{c_p}, \|u_0\|\right\}$, then

$$\begin{aligned}
\mathscr{E}_\lambda(u) &= \frac{1}{p}||u||^p - \lambda \mathcal{F}(u) \\
&\geq \frac{1}{p}||u||^p - \varepsilon \lambda c_p^p \|u\|^p \\
&= ||u||^p \left(\frac{1}{p} - \varepsilon \lambda c_p^p\right) \\
&= \rho^p \left(\frac{1}{p} - \varepsilon \lambda c_p^p\right) > 0.
\end{aligned}$$

Since $\mathscr{E}_\lambda$ satisfies the Palais-Smale condition and

$$\inf_{\|u\|=\rho} \mathscr{E}_\lambda(u) > 0 = \mathscr{E}_\lambda(0) > \mathscr{E}_\lambda(u_0),$$

we are in the position to apply the Mountain Pass theorem, which means that $c = \inf\limits_{\gamma \in \Gamma} \sup\limits_{t \in [0,1]} \mathscr{E}_\lambda(\gamma(t))$ is a critical value of $\mathscr{E}_\lambda$, therefore there exists a critical point $u$ such that $\mathscr{E}_\lambda(u) = c$.

From the definition of $c$, we have

$$\sup_{t \in [0,1]} \mathscr{E}_\lambda(\gamma(t)) \leq c + \frac{1}{n^2}.$$

From the above inequality and from the fact that we can choose $\gamma(0) = 0$ and $\gamma(1) = u = u^H$, we can apply Theorem 2.2 for $\varepsilon = \frac{1}{n^2}$, and $\delta = \frac{1}{n}$. Thus, there exists $u_n \in W_0^{1,p}(\Omega)$ such that

(a) $|\mathscr{E}_\lambda(u_n) - c| \leq \frac{2}{n^2}$;
(b) $\|u_n - u_n^*\|_p \leq 2(2\kappa + 1)\frac{1}{n}$;
(c) $\lambda_{\mathscr{E}_\lambda}(u_n) \leq \frac{8}{n}$.

Since $\mathscr{E}_\lambda$ satisfies the Palais-Smale condition, up to a subsequence $u_n$ converges to $u$ in $W_0^{1,p}(\Omega)$, with $\mathscr{E}_\lambda(u) = c$, $\lambda_{\mathscr{E}_\lambda}(u) = 0$ and $u = u^*$. This means that $u$ is a critical point for the energy functional $\mathcal{E}_\lambda$, different from the critical point obtained in (a) and it is invariant by spherical cap symmetrization as well. □

## 4. Proof of Theorem 1.2

Similarly to the previous section, we start this paragraph with the proofs of two properties of the energy functional $\mathscr{A}_\lambda$, namely that $\mathscr{A}_\lambda$ is coercive and it satisfies the Palais-Smale condition for every $\lambda > 0$.

**Lemma 4.1.** *Let the conditions* $(\mathbf{F_2'})$ *and* $(\mathbf{A_4})$ *be satisfied. Then the functional* $\mathscr{A}_\lambda : X \to \mathbb{R}$ *is coercive for each* $\lambda > 0$*, that is,* $\mathscr{A}_\lambda(u) \to \infty$ *as* $\|u\|_X \to \infty$*, for all* $u \in X$*.*

*Proof.* Due to $(\mathbf{F_2'})$, for all $u \in X$ we have:

$$F(x, u(x)) \leq \alpha(x)|u(x)|^q + \beta(x).$$

Hence, by using Hölder's inequality, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} F(x, u(x))dx &\leq \int_{\mathbb{R}^N} \alpha(x)|u(x)|^q dx + \int_{\mathbb{R}^N} \beta(x)dx \\
&\leq \left[\int_{\mathbb{R}^N} \alpha(x)^{\frac{\nu}{\nu-q}}\right]^{\frac{\nu-q}{\nu}} \cdot \left[\int_{\mathbb{R}^N} [|u(x)|^q]^{\frac{\nu}{q}}\right]^{\frac{q}{\nu}} dx + \int_{\mathbb{R}^N} \beta(x)dx \\
&\leq \|\alpha\|_{\frac{\nu}{\nu-q}} \cdot \|u\|_\nu^q + \|\beta\|_1. \qquad (4.1)
\end{aligned}$$

Since $X \hookrightarrow L^\nu(\mathbb{R}^N)$, when $\nu \in [p, p^*]$, one can find a number $C_\nu \geq 0$ such that

$$\|u\|_\nu^q \leq C_\nu^q \|u\|_X^q. \qquad (4.2)$$

Combining the relations (4.1) and (4.2), we obtain that for all $\lambda > 0$, we have

$$-\lambda \int_{\mathbb{R}^N} F(x, u(x))dx \geq -\lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q \|u\|_X^q - \lambda\|\beta\|_1.$$

Therefore, from the definition of the energy functional $\mathscr{A}_\lambda$ and using the condition $(\mathbf{A_4})$, we get

$$\begin{aligned}
\mathscr{A}_\lambda(u) &\geq a(u) - \lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q \|u\|_X^q - \lambda\|\beta\|_1 \\
&\geq c\|u\|^p - \lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q \|u\|_X^q - \lambda\|\beta\|_1.
\end{aligned}$$

Taking into account $(\mathbf{F_2})'$ and the fact that $q \in (0, p)$, it follows that $\mathscr{A}_\lambda(u) \to +\infty$, whenever $\|u\|_X \to +\infty$. This completes the proof. □

**Lemma 4.2.** *If the conditions hold then for every $\lambda > 0$ the functional $\mathscr{A}_\lambda : X \to \mathbb{R}$ satisfies the Palais-Smale condition.*

*Proof.* The proof of this lemma is similar to the proofs of the Lemma 3.2 and of [8, Theorem 5.1.1]. □

**Lemma 4.3.** *Assume that* $(\mathbf{F_4}) - (\mathbf{F_5})$ *and* $(\mathbf{A_5})$ *are satisfied. Then, for all $H \in H_*$, we have*

$$\mathscr{A}_\lambda(u^H) \leq \mathscr{A}_\lambda(u), \forall u \in X.$$

*Proof.* From $(\mathbf{A_5})$, we have that $a(u^H) \leq a(u)$. Therefore, using $(\mathbf{F_4}) - (\mathbf{F_5})$ and taking inspiration from the proof of Lemma 4.6. in M. Squassina [11], we obtain

$$\int_{\mathbb{R}^N} F(x, u(x))dx \leq \int_{\mathbb{R}^N} F(x, u^H(x))dx.$$

Hence, by the definition of $\mathscr{A}_\lambda$, we have that for all $\lambda > 0$:

$$\mathscr{A}_\lambda(u) = \frac{1}{p}a(u) - \lambda \int_{\mathbb{R}^N} F(x, u(x))dx \geq \frac{1}{p}a(u^H) - \lambda \int_{\mathbb{R}^N} F(x, u^H(x))dx = \mathscr{A}_\lambda(u^H).$$

□

*Proof of Theorem* 1.2*:* The proof is similar to the proof of Theorem 1.1 so it is left to the reader.

### 4.1. Particular case

Let $V : \mathbb{R}^N \to \mathbb{R}$ a function such that:

($\mathbf{V1}$) $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$;

($\mathbf{V2}$) For every $M > 0$, we have $\text{meas}(\{x \in \mathbb{R}^N : b(x) \leq M\}) < \infty$;

($\mathbf{V3}$) For $x, y \in \mathbb{R}^N$, if $|x| \leq |y|$ then $V(x) \leq V(y)$.

The space $H = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2 dx < \infty\}$, equipped with the inner product

$$\langle u, v \rangle_H = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx$$

is a Hilbert space. It is known that $H$ is compactly embedded into $L^s(\mathbb{R}^n)$ for $s \in [2, 2^*)$ (see T. Bartsch, Z.-Q. Wang [1]).

A particular case of the problem $(P^2_\lambda)$ can be formulated as follows: Find a positive $u \in H$ such that for each $v \in H$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx + \int_{\mathbb{R}^N} F^0_y(x, u(x) - v(x))dx \geq 0. \qquad (\mathbf{P}'_\lambda).$$

Similarly to the proof of Theorem 1.2, we can prove the next result:

**Lemma 4.4.** *If $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions* $(\mathbf{F1})$, $(\mathbf{F'_2})$, $(\mathbf{F_4})$, $(\mathbf{F_5})$ *and* $(\mathbf{V1}) - (\mathbf{V3})$, *then there exists two nontrivial solutions of the problem* $(\mathbf{P^2_\lambda})$, *which are invariants by the spherical cap symmetrization.*

*Proof.* Theorem 1.2 can be applied since the conditions $(\mathbf{A1}) - (\mathbf{A5})$ are fulfilled. Indeed, the assumptions $(\mathbf{A1}) - (\mathbf{A5})$ follow from the fact that $a(u) = \frac{1}{2}\langle u, u\rangle$. On the other hand, the condition $(\mathbf{V3})$, implies $(\mathbf{A5})$. Then, by Theorem 1.2, it follows that problem $(\mathbf{P}_\lambda^\mathbf{2})$ has two nontrivial solutions, which are invariants by the spherical cap symmetrization. □

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