Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 3, 317–329

Statistical convergence on probabilistic modular spaces

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Abstract. In this work, we introduce the concepts of statistical convergence and statistical Cauchy sequence on probabilistic modular spaces. After giving some useful characterizations for statistically convergent sequences, we display an example such that our method of convergence works but its classical case does not work. Also we define statistical limit points, statistical cluster points on probabilistic modular spaces. Finally, we give the relations between these notions and limit points of sequences on probabilistic modular spaces.

Mathematics Subject Classification (2010): 40A05, 40A35, 40B05, 42A45.

Keywords: natural density, statistical convergence, modular space, probabilistic modular space.

1. Introduction

The theory of modular spaces was firstly presented by Nakano [13] and subsequently generalized by Musielak and Orlicz [10]. Later many researchers have investigated these spaces in [7, 8]. Also, Menger [9] has introduced the concept of probabilistic metric space which is an interesting and important generalization of the notion of a metric space. The probabilistic generalization of metric space appears when there is an uncertainty about the distance between the points and we know only the probabilities of possible values this distance can take. According to this work instead of associating a number−the distance $d(x, y)$ −to every pair (x, y), one should associate a distribution function $N_{x,y}$ and for any positive real number t, interpret $N_{x,y}(t)$ as the probability that the distance from x to y is less than t. An important family of probabilistic metric spaces are probabilistic normed spaces. For more details, the reader is referred to [1, 15]. After Menger's work, Fallahi and Nourouzi [3] have introduced probabilistic modular spaces in the probabilistic sense which are more general than probabilistic normed spaces and they investigated some basic properties of these spaces.

The concept of statistical convergence for sequences of real numbers was introduced by Fast [6]. Later on some generalizations and applications of this notion have been investigated by many authors [2, 4, 5, 11, 12]. Karakuş studied the concept of statistical convergence in probabilistic normed spaces [14]. In this paper we study the properties of the sequences which are statistically convergent in a probabilistic modular space. Also we define statistical limit points and statistical cluster points in a probabilistic modular space and prove some interesting results.

We recall some notations and basic definitions used in this paper:

A functional $\rho : X \to [0,+\infty]$ is said to be a *modular* on a real linear space X provided that the following conditions hold:

(i) $\rho(x) = 0$ iff x is the null vector θ,
(ii) $\rho(x) = \rho(-x)$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $x, y \in X$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Then the vector subspace $X_\rho = \{x \in X : \rho(ax) \to 0 \text{ as } a \to 0\}$ of X is called a *modular space.*

If A is a subset of $\mathbb{N}$, the set of natural numbers, then the *natural density* of A denoted by $\delta(A)$, is defined by

$$\delta(A) := \lim_n \frac{1}{n} |\{k \leq n : k \in A\}|$$

whenever the limit exists, where $|A|$ denotes the cardinality of the set A. The natural density may not exist for each set A. But the upper density $\bar{\delta}$ always exists for each set A identified as follows:

$$\bar{\delta}(A) := \limsup_n \frac{1}{n} |\{k \leq n : k \in A\}| .$$

A sequence $x = \{x_k\}$ of numbers is *statistically convergent* to L if

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write $st - \lim x = L$.

Note that every convergent sequence is statistically convergent to the same value. If x is statistically convergent, then x needs not to be convergent. It is also not necessarily bounded. For example, let $x = \{x_k\}$ be defined as

$$x_k := \begin{cases} \sqrt{k}, & \text{if } k \text{ is a square} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $st - \lim x = 1$. But x is neither convergent nor bounded.

Definition 1.1. *A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if it is non-decreasing and left-continuous with* $\inf_{t\in\mathbb{R}} f(t) = 0$, *and* $\sup_{t\in\mathbb{R}} f(t) = 1$.

We will denote the set of all distribution functions by $\mathcal{D}$.

Definition 1.2. *A triangular norm, briefly called t-norm, is a binary operation on* $[0,1]$ *which is continuous, commutative, associative, non-decreasing and has 1 as a neutral element, i.e., it is a continuous mapping* $\wedge : [0,1] \times [0,1] \to [0,1]$ *such that for all* $a, b, c \in [0,1]$:

1. $a \wedge 1 = a$,

2. $a \wedge b = b \wedge a$,
3. $c \wedge d \geq a \wedge b$ *if* $c \geq a$ *and* $d \geq b$,
4. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

Definition 1.3. *A pair* (X, μ) *is called a probabilistic modular space* ($\mathcal{P}-$*modular space*) *if* X *is a real vector space,* μ *is a mapping from* X *into* $\mathcal{D}$ (*for* $x \in X$, *the function* $\mu(x)$ *is denoted by* μ_x, *and* $\mu_x(t)$ *is the value* μ_x *at* $t \in \mathbb{R}$) *satisfying the following conditions:*

1. $\mu_x(0) = 0$,
2. $\mu_x(t) = 1$ *for all* $t > 0$ *iff* $x = 0$,
3. $\mu_{-x}(t) = \mu_x(t)$,
4. $\mu_{\alpha x + \beta y}(s + t) \geq \mu_x(s) \wedge \mu_y(t)$ *for all* $x, y \in X$, *and* $\alpha, \beta, s, t \in \mathbb{R}_0^+$, $\alpha + \beta = 1$.

We say (X, μ) is $\beta-$*homogeneous*, where $\beta \in (0, 1]$ if,

$$\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|^\beta}\right),$$

for every $x \in X$, $t > 0$, and $\alpha \in \mathbb{R} \setminus \{0\}$.

Example 1.4. Suppose that X is a real vector space and ρ is a modular on X. Define

$$\mu_x(t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t+\rho(x)}, & t > 0, \end{cases}$$

for all $x \in X$. Then (X, μ) is a $\mathcal{P}-$modular space.

We recall that the concept of convergence and Cauchy sequence in a probabilistic modular space are studied in [3].

Definition 1.5. *Let* (X, μ) *be a* $\mathcal{P}-$*modular space.*

- *A sequence* $\{x_k\}$ *in* X *is said to be* $\mu-$*convergent to a point* $L \in X$ *and denoted by* $x_k \xrightarrow{\mu} L$ *or* $\mu - \lim x = L$, *if for every* $\varepsilon > 0$ *and* $\lambda \in (0, 1)$, *there exists a positive integer* k_0 *such that* $\mu_{x_k - L}(\varepsilon) > 1 - \lambda$, *for all* $k \geq k_0$.
- *A sequence* $\{x_k\}$ *in* X *is called a* $\mu-$*Cauchy sequence if for every* $\varepsilon > 0$ *and* $\lambda \in (0, 1)$, *there exists a positive integer* k_0 *such that* $\mu_{x_k - x_l}(\varepsilon) > 1 - \lambda$, *for all* $k, l \geq k_0$.
- *A subset* F *of* X *is said to be* $\mu-$*bounded if for every* $\lambda \in (0, 1)$, *there exists* $t > 0$ *such that* $\mu_x(t) > 1 - \lambda$ *for all* $x \in F$.
- *For* $x \in X$, $\varepsilon > 0$ *and* $0 < \lambda < 1$, *the ball centered at* x *with radius* λ *is defined by*

$$B(x, \lambda, \varepsilon) = \{y \in X : \mu_{x-y}(\varepsilon) > 1 - \lambda\}.$$

Remark 1.6. Let (X, μ) be a $\mathcal{P}-$modular space, and $\mu_x(t) = \frac{t}{t+\rho(x)}$, where $x \in X$ and $t \geq 0$. Then it can be easily seen that $x_n \xrightarrow{\rho} x$ if and only if $x_n \xrightarrow{\mu} x$.

2. Statistical convergence on $\mathcal{P}-$modular spaces

In this work we deal with the statistical convergence on probabilistic modular spaces. Now, we may obtain our main results.

Definition 2.1. *Let (X, μ) be a $\mathcal{P}-$modular space.*

- *We say that a sequence $x = \{x_k\}$ is statistically convergent to $L \in X$ with respect to the probabilistic modular μ (or briefly $st_\mu-$convergent) provided that for every $\varepsilon > 0$ and $\lambda \in (0,1)$*

$$\delta\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}\right) = 0, \tag{2.1}$$

or equivalently,

$$\lim_n \frac{1}{n} \left|\{k \leq n : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}\right| = 0$$

and denoted by $st_\mu - \lim x = L$.

- *We say that a sequence $x = \{x_k\}$ is a statistical Cauchy sequence with respect to the probabilistic modular μ (or briefly $st_\mu-$Cauchy) provided that for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there is a positive integer $N = N(\varepsilon)$ such that*

$$\delta\left(\{k \in \mathbb{N} : \mu_{x_k - x_N}(\varepsilon) \leq 1 - \lambda\}\right) = 0,$$

or equivalently,

$$\lim_n \frac{1}{n} \left|\{k \leq n : \mu_{x_k - x_N}(\varepsilon) \leq 1 - \lambda\}\right| = 0.$$

By using (2.1) and well-known density properties, we have the following lemma.

Lemma 2.2. *Let (X, μ) be a $\mathcal{P}-$modular space.Then, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, the following statements are equivalent:*

(*i*) $st_\mu - \lim x = L$,
(*ii*) $\delta\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}\right) = 0$,
(*iii*) $\delta\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1 - \lambda\}\right) = 1$,
(*iv*) $st - \lim \mu_{x_k - L}(\varepsilon) = 1$.

Theorem 2.3. *Let (X, μ) be a $\mathcal{P}-$modular space. If a sequence $x = \{x_k\}$ is $st_\mu-$convergent, then the $st_\mu-$limit is unique.*

Proof. Assume that $st_\mu - \lim x = L_1$, $st_\mu - \lim x = L_2$ and $\lambda \in (0,1)$. Choose $\eta \in (0,1)$ such that $(1-\eta) \wedge (1-\eta) > 1 - \lambda$. Then, for any $\varepsilon > 0$, define the following sets:

$$\begin{aligned} T_{\mu,1}(\eta, \varepsilon) &: = \{k \in \mathbb{N} : \mu_{x_k - L_1}(\varepsilon) \leq 1 - \eta\}, \\ T_{\mu,2}(\eta, \varepsilon) &: = \{k \in \mathbb{N} : \mu_{x_k - L_2}(\varepsilon) \leq 1 - \eta\}. \end{aligned}$$

Since $st_\mu - \lim x = L_1$, $\delta\left(T_{\mu,1}(\eta, \varepsilon)\right) = 0$ for all $\varepsilon > 0$. Also because of $st_\mu - \lim x = L_2$, we get $\delta\left(T_{\mu,2}(\eta, \varepsilon)\right) = 0$ for all $\varepsilon > 0$. Let $T_\mu(\eta, \varepsilon) = T_{\mu,1}(\eta, \varepsilon) \cap T_{\mu,2}(\eta, \varepsilon)$.

Then it can be easily seen that $\delta\left(T_{\mu}\left(\eta,\varepsilon\right)\right)=0$ which implies $\delta\left(\mathbb{N}/T_{\mu}\left(\eta,\varepsilon\right)\right)=1$. If $k\in\mathbb{N}/T_{\mu}\left(\eta,\varepsilon\right)$, then we get

$$\begin{aligned}\mu_{\frac{1}{2}(L_1-L_2)}\left(\varepsilon\right) &= \mu_{\frac{1}{2}(x_k-L_1)+\frac{1}{2}(L_2-x_k)}\left(\varepsilon\right)\\ &\geq \mu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right)\wedge\mu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right)\\ &> (1-\eta)\wedge(1-\eta).\end{aligned}$$

Because of $(1-\eta)\wedge(1-\eta)>1-\lambda$, it follows that

$$\mu_{L_1-L_2}\left(\varepsilon\right)>1-\lambda. \tag{2.2}$$

Since $\lambda>0$ is arbitrary, by (2.2) we have $\mu_{L_1-L_2}\left(\varepsilon\right)=1$ for all $\varepsilon>0$. This implies that $L_1=L_2$. □

Theorem 2.4. *Let (X,μ) be a $\beta-$homogeneous $\mathcal{P}-$modular space. If $\{x_k\}$ is a $st_\mu-$Cauchy sequence, possessing a subsequence which is $st_\mu-$convergent, then $\{x_k\}$ is $st_\mu-$convergent to the same limit.*

Proof. For a given $\lambda>0$ choose $\eta\in(0,1)$ such that $(1-\eta)\wedge(1-\eta)>1-\lambda$. Because of $\{x_k\}$ is a $st_\mu-$*Cauchy* sequence , there exists $t_1\in\mathbb{N}$, a positive integer $N=N(\varepsilon)$ and subset A_1 of density 1 such that $\mu_{x_k-x_N}\left(\frac{\varepsilon}{2^{\beta+1}}\right)>1-\eta$ holds for all $\varepsilon>0$, $k\in A_1$and $k\geq t_1$. Let $\{x_{k_i}\}$ be a subsequence of $\{x_k\}$ which is $st_\mu-$*convergent to* $L\in X$, then there exists $t_2\in\mathbb{N}$ and subset A_2 of density 1 such that $\mu_{x_{k_i}-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)>1-\eta$ holds for all $\varepsilon>0$, $k_i\in A_2$ and $k_i\geq t_2$. Take $t_0=\max\{t_1,t_2\}$ and $A=A_1\cap A_2$, then $\delta(A)=1$ and for all $\varepsilon>0$, $k\in A$ and $k\geq t_0$ we have

$$\begin{aligned}\mu_{x_k-L}\left(\varepsilon\right) &\geq \mu_{2\left(x_k-x_{k_i}\right)}\left(\frac{\varepsilon}{2}\right)\wedge\mu_{2\left(x_{k_i}-L\right)}\left(\frac{\varepsilon}{2}\right)\\ &= \mu_{x_k-x_{k_i}}\left(\frac{\varepsilon}{2^{\beta+1}}\right)\wedge\mu_{x_{k_i}-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)\\ &> (1-\eta)\wedge(1-\eta)>1-\lambda.\end{aligned}$$

That is $st_\mu-\lim x=L$. □

Theorem 2.5. *Let (X,μ) be a $\beta-$homogeneous $\mathcal{P}-$modular space. Then every $st_\mu-$convergent sequence is also a $st_\mu-$Cauchy sequence.*

Proof. Suppose that $\{x_k\}$ is a $st_\mu-$*convergent to* $L\in X$. Let $\lambda\in(0,1)$ and choose $\eta\in(0,1)$ such that $(1-\eta)\wedge(1-\eta)>1-\lambda$. There exists $k_0\in\mathbb{N}$ and subset A of density 1 such that $\mu_{x_k-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)>1-\eta$ holds for all $\varepsilon>0$, $k\in A$ and $k\geq k_0$. If $N=N(\varepsilon)$ is a positive integer,

$$\begin{aligned}\mu_{x_k-x_N}\left(\varepsilon\right) &\geq \mu_{2(x_N-L)}\left(\frac{\varepsilon}{2}\right)\wedge\mu_{2(x_k-L)}\left(\frac{\varepsilon}{2}\right)\\ &= \mu_{x_N-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)\wedge\mu_{x_k-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)\\ &> (1-\eta)\wedge(1-\eta)>1-\lambda,\end{aligned}$$

for every $\varepsilon>0$, $k\in A$ and $k\geq k_0$. That is $\{x_k\}$ is a $st_\mu-$*Cauchy* sequence. □

Theorem 2.6. *Let (X, μ) be a $\mathcal{P}-$modular space. If $x = \{x_k\}$ is a $\mu-$convergent sequence, then x is also a $st_\mu-$convergent sequence.*

Proof. Since x is $\mu-$*convergent*, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there is a number $k_0 \in \mathbb{N}$ such that $\mu_{x_k - L} > 1 - \lambda$ for all $k \geq k_0$. So the set $\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}$ has at most finitely many terms. Because every finite subset of the natural numbers has density zero, we can easily see that $\delta(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}) = 0$, which completes the proof. □

Example 2.7. Define $\rho : \mathbb{R} \to \mathbb{R}$ by

$$\rho(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$$

for all $x \in \mathbb{R}$. Then $(\mathbb{R}, \rho)$ is a modular space. Let $a \wedge b = ab$ and $\mu_x(t) = \frac{t}{t + \rho(x)}$, where $x \in X$ and $t \geq 0$. Observe that $(\mathbb{R}, \mu)$ is a $\beta-$*homogeneous* $\mathcal{P}-$modular space. Now we define the sequence $x = \{x_k\}$ whose terms are given by

$$x_k := \begin{cases} 1, & \text{if } k = m^2 \ \ (m \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Then for *any* $\varepsilon > 0$ and for *every* $\lambda \in (0,1)$, let

$$T_\mu(\lambda, \varepsilon) := \{k \leq n : \mu_{x_k}(\varepsilon) \leq 1 - \lambda\}.$$

Since

$$\begin{aligned} T_\mu(\lambda, \varepsilon) &= \left\{k \leq n : \frac{\varepsilon}{\varepsilon + \rho(x_k)} \leq 1 - \lambda\right\} \\ &= \left\{k \leq n : \rho(x_k) \geq \frac{\lambda\varepsilon}{1 - \lambda} > 0\right\} \\ &= \{k \leq n : x_k = 1\} \\ &= \left\{k \leq n : k = m^2 \text{ and } m \in \mathbb{N}\right\}, \end{aligned}$$

we get

$$\frac{1}{n} |T_\mu(\lambda, \varepsilon)| \leq \frac{1}{n} \left|\left\{k \leq n : k = m^2 \text{ and } m \in \mathbb{N}\right\}\right| \leq \frac{\sqrt{n}}{n}$$

that is

$$\lim_n \frac{1}{n} |T_\mu(\lambda, \varepsilon)| = 0.$$

So, we have $st_\mu - \lim x = 0$. But, because $x = \{x_k\}$ given by (2.3) is not convergent in the space $(\mathbb{R}, \rho)$, by Remark 1.6, we also see that x is not convergent with respect to the probabilistic modular μ.

Theorem 2.8. *Let (X, μ) be a $\mathcal{P}-$modular space. Let $st_\mu - \lim x = L$ if and only if there exists an increasing index sequence $T = \{k_n\}_{n \in \mathbb{N}}$ of natural numbers such that $\delta(T) = 1$ and $\mu - \lim_{n \in T} x_n = L$, i.e., $\mu - \lim_n x_{k_n} = L$.*

Proof. Necessity: First suppose that $st_\mu - \lim x = L$. Then, for every $\varepsilon > 0$ and $j \in \mathbb{N}$, let

$$T(j,\varepsilon) := \left\{ n \in \mathbb{N} : \mu_{x_n - L}(\varepsilon) > 1 - \frac{1}{j} \right\}$$

Observe that, for $\varepsilon > 0$ and $j \in \mathbb{N}$,

$$T(j+1,\varepsilon) \subset T(j,\varepsilon). \tag{2.4}$$

Because $st_\mu - \lim x = L$, we can write

$$\delta(T(j,\varepsilon)) = 1, \quad (\varepsilon > 0 \text{ and } j \in \mathbb{N}). \tag{2.5}$$

Let r_1 be an arbitrary number of $T(1,\varepsilon)$. Then, by (2.5), there is a number $r_2 \in T(2,\varepsilon)$, $(r_2 > r_1)$, such that, for all $n \geq r_2$,

$$\frac{1}{n}\left|\left\{ k \leq n : \mu_{x_k - L}(\varepsilon) > 1 - \frac{1}{2} \right\}\right| > \frac{1}{2}.$$

Further, by (2.5), there is a number $r_3 \in T(3,\varepsilon)$, $(r_3 > r_2)$, such that, for all $n \geq r_3$,

$$\frac{1}{n}\left|\left\{ k \leq n : \mu_{x_k - L}(\varepsilon) > 1 - \frac{1}{3} \right\}\right| > \frac{2}{3},$$

and so on. Hence, by induction we can construct an increasing index sequence $\{r_j\}_{j\in\mathbb{N}}$ of natural numbers such that $r_j \in T(j,\varepsilon)$ and such that the following statement holds for all $n \geq r_j$ $(j \in \mathbb{N})$:

$$\frac{1}{n}\left|\left\{ k \leq n : \mu_{x_k - L}(\varepsilon) > 1 - \frac{1}{j} \right\}\right| > \frac{j-1}{j}. \tag{2.6}$$

Now we set the increasing index sequence T as follows:

$$T := \{n \in \mathbb{N} : 1 < n < r_1\} \cup \left\{ \bigcup_{j\in\mathbb{N}} \{n \in T(j,\varepsilon) : r_j \leq n < r_{j+1}\} \right\}. \tag{2.7}$$

Then by (2.4), (2.6) and (2.7) we conclude, for all n, $(r_j \leq n < r_{j+1})$, that

$$\frac{1}{n}\left|\{k \leq n : k \in T\}\right| \geq \frac{1}{n}\left|\left\{ k \leq n : \mu_{x_k - L}(\varepsilon) > 1 - \frac{1}{j} \right\}\right| > 1 - \frac{1}{j}.$$

Therefore it follows that $\delta(T) = 1$. Now choose a number $j \in \mathbb{N}$ and let $\varepsilon > 0$ such that $\frac{1}{j} < \varepsilon$. Suppose that $n \geq v_j$ and $n \in T$. Then, from the definition of T, there exists a number $m \geq j$ such that $v_m \leq n < v_{m+1}$ and $n \in T(j,\varepsilon)$. Hence, we get, for every $\varepsilon > 0$,

$$\mu_{x_n - L}(\varepsilon) > 1 - \frac{1}{j} > 1 - \varepsilon$$

for all $n \geq v_j$ and $n \in T$, which implies

$$\mu - \lim_{n\in T} x_n = L.$$

This completes the proof of necessity.

Sufficiency: Assume that there exists an increasing index sequence $T = \{k_n\}_{n\in\mathbb{N}}$ such that $\delta(T) = 1$ and $\mu - \lim_{n\in T} x_n = L$. Now, for any $\varepsilon > 0$ and $\lambda \in (0,1)$, there is a number n_0 such that for each $n \geq n_0$ the inequality $\mu_{x_n - L}(\varepsilon) > 1-\lambda$ holds. Now define $S(\lambda,\varepsilon) := \{n \in \mathbb{N} : \mu_{x_n-L}(\varepsilon) \leq 1-\lambda\}$. Then we have

$$S(\lambda,\varepsilon) \subset \mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\}.$$

Since $\delta(T) = 1$, we get $\delta(\mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\}) = 0$, which yields that $\delta(S(\lambda,\varepsilon)) = 0$.

Hence, we get $st_\mu - \lim x = L$. □

By a similar technique as in the above theorem one can get the following result at once.

Theorem 2.9. *Let (X,μ) be a $\mathcal{P}-$modular space and $x = \{x_k\}$ is a sequence in X. The following statements are equivalent:*

- (a) *x is a $st_\mu-$Cauchy sequence.*
- (b) *There exists an increasing index sequence $T = \{k_n\}$ of natural numbers such that $\delta(T) = 1$ and the subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ is a $\mu-$Cauchy sequence.*

Remark 2.10. If $st_\mu - \lim_n x_n = L$, then there exists a sequence $y = \{y_n\}$ such that $\mu - \lim_n y_n = L$ and $\delta(\{n \in \mathbb{N} : x_n = y_n\}) = 1$.

Now, we show that statistical convergence on a $\mathcal{P}-$modular space has some properties similar to the properties of the usual convergence on $\mathbb{R}$.

Lemma 2.11. *Let (X,μ) be a $\beta-$homogeneous $\mathcal{P}-$modular space.*

1. *If $st_\mu - \lim x = L_1$ and $st_\mu - \lim y = L_2$,*
   then $st_\mu - \lim(x+y) = L_1 + L_2$.
2. *If $st_\mu - \lim x = L$ and $\alpha \in \mathbb{R}$, then $st_\mu - \lim \alpha x = \alpha L$.*
3. *If $st_\mu - \lim x = L_1$ and $st_\mu - \lim y = L_2$,*
   then $st_\mu - \lim(x-y) = L_1 - L_2$.

Proof. (1) Let $st_\mu - \lim x = L_1$, $st_\mu - \lim y = L_2$ and $\lambda \in (0,1)$. There exists $\eta \in (0,1)$ such that $(1-\eta)\wedge(1-\eta) > 1-\lambda$. There exists $k_1 \in \mathbb{N}$ and subset A_1 of density 1 such that $\mu_{x_k-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right) > 1-\eta$ holds for all $\varepsilon > 0$, $k \in A_1$ and $k \geq k_1$. Also, there exists $k_2 \in \mathbb{N}$ and subset A_2 of density 1 such that $\mu_{y_k-L_2}\left(\frac{\varepsilon}{2^{\beta+1}}\right) > 1-\eta$ holds for all $\varepsilon > 0$, $k \in A_2$ and $k \geq k_2$. Take $k_0 = \max\{k_1,k_2\}$ and $A = A_1 \cap A_2$, then $\delta(A) = 1$ and for every $\varepsilon > 0$, $k \in A$ and $k \geq k_0$ we have

$$\begin{aligned}
\mu_{(x_k-L_1)+(y_k-L_2)}(\varepsilon) &\geq \mu_{2(x_k-L_1)}\left(\frac{\varepsilon}{2}\right) \wedge \mu_{2(y_k-L_2)}\left(\frac{\varepsilon}{2}\right) \\
&= \mu_{x_k-L_1}\left(\frac{\varepsilon}{2^{\beta+1}}\right) \wedge \mu_{y_k-L_2}\left(\frac{\varepsilon}{2^{\beta+1}}\right) \\
&> (1-\eta)\wedge(1-\eta) > 1-\lambda,
\end{aligned}$$

That is $st_\mu - \lim(x+y) = L_1 + L_2$.

(2) Let $st_\mu - \lim x = L$, $\varepsilon > 0$, $\lambda > 0$. We may assume that $\alpha = 0$. In this case

$$\mu_{0x_k-0L}(\varepsilon) = \mu_0(\varepsilon) = 1 > 1-\lambda.$$

So we get $\mu - \lim x_k = 0$. Then from Theorem 2.5 we have $st_\mu - \lim x_k = 0$.

Now we consider the case of $\alpha \in \mathbb{R}\,(\alpha \neq 0)$. Since $st_\mu - \lim x = L$, if we define the set

$$T_\mu(\lambda,\varepsilon) := \{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}$$

then we can say $\delta(T_\mu(\lambda,\varepsilon)) = 0$ for all $\varepsilon > 0$ which implies $\delta(\mathbb{N}/T_\mu(\lambda,\varepsilon)) = 1$. If $k \in \mathbb{N}/T_\mu(\lambda,\varepsilon)$, then we get

$$\begin{aligned}
\mu_{\alpha x_k - \alpha L}(\varepsilon) &= \mu_{x_k - L}\left(\frac{\varepsilon}{|\alpha|}\right) \\
&\geq \mu_{x_k - L}(\varepsilon) \wedge \mu_0\left(\frac{\varepsilon}{|\alpha|} - \varepsilon\right) \\
&= \mu_{x_k - L}(\varepsilon) \wedge 1 \\
&= \mu_{x_k - L}(\varepsilon) > 1 - \lambda
\end{aligned}$$

for $\alpha \in \mathbb{R}\,(\alpha \neq 0)$. That is

$$\delta(\{k \in \mathbb{N} : \mu_{\alpha x_k - \alpha L}(\varepsilon) \leq 1 - \lambda\}) = 0.$$

So $st_\mu - \lim \alpha x = \alpha L$.

(3) The proof is clear from (1) and (2). □

Theorem 2.12. *Let (X,μ) be a $\mathcal{P}-$modular space and $S_b^\mu(X)$ the space of bounded statistically convergent sequences on the $\mathcal{P}-$modular space. Then the set $S_b^\mu(X)$ is a closed linear subspace of the set $l_\infty^\mu(X)$.*

Proof. It is clear that $S_b^\mu(X) \subset \overline{S_b^\mu(X)}$. Now we show that $\overline{S_b^\mu(X)} \subset S_b^\mu(X)$. Let $y \in \overline{S_b^\mu(X)}$, then because of $B(y,\lambda,\varepsilon) \cap S_b^\mu(X) \neq \varnothing$, there is an $x \in B(y,\lambda,\varepsilon) \cap S_b^\mu(X)$.

Let $\varepsilon > 0$ and for a given $\eta > 0$ choose $\lambda \in (0,1)$ such that $(1-\lambda) \wedge (1-\lambda) > 1 - \eta$. Since $x \in B(y,\lambda,\varepsilon) \cap S_b^\mu(X)$, there is a set $T \subseteq \mathbb{N}$ with $\delta(T) = 1$ such that

$$\mu_{y_n - x_n}\left(\frac{\varepsilon}{2}\right) > 1 - \lambda \text{ and } \mu_{x_n}\left(\frac{\varepsilon}{2}\right) > 1 - \lambda$$

for all $n \in T$. Then we have

$$\begin{aligned}
\mu_{y_n}(\varepsilon) &= \mu_{y_n - x_n + x_n}(\varepsilon) \\
&\geq \mu_{y_n - x_n}\left(\frac{\varepsilon}{2}\right) \wedge \mu_{x_n}\left(\frac{\varepsilon}{2}\right) \\
&> 1 - \eta,
\end{aligned}$$

for all $n \in T$. Therefore

$$\delta(\{n \in T : \mu_{y_n}(\varepsilon) > 1 - \eta\}) = 1$$

and so $y \in S_b^\mu(X)$. □

3. Statistical limit points and statistical cluster points on $\mathcal{P}-$modular spaces

The concepts of statistical limit points and statistical cluster points of real number sequences were given by Fridy in 1993 [5]. Also, he gives relations between them and the set of ordinary limit points. In this section we study the analogues of these notions on probabilistic modular spaces.

Definition 3.1. *Let (X, μ) be a $\mathcal{P}-$modular space. Then $l \in X$ is called a limit point of the sequence $x = \{x_k\}$ with respect to the probabilistic modular μ (or $\mu-$limit point) if there is a subsequence of x that converges to l with respect to the probabilistic modular μ. The set of all limit points of the sequence x is denoted by $L_\mu(x)$.*

Definition 3.2. *Let (X, μ) be a $\mathcal{P}-$modular space. If $\{x_{k(j)}\}$ is a subsequence of $x = \{x_k\}$ and $K := \{k(j) \in \mathbb{N} : j \in \mathbb{N}\}$ then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$ in this case $\delta(K) = 0$. $\{x\}_K$ is called a thin subsequence or subsequence of density zero. Additionally, $\{x\}_K$ is a nonthin subsequence of x if K does not have density zero.*

Definition 3.3. *Let (X, μ) be a $\mathcal{P}-$modular space. $L \in X$ is called a statistical limit point of the sequence $x = \{x_k\}$ with respect to the probabilistic modular μ (or $st_\mu-$limit point) if there is a nonthin subsequence of x that converges to L with respect to the probabilistic modular μ and we say L is a $st_\mu-$limit point of the sequence $x = \{x_k\}$. The set of all $st_\mu-$limit points of the sequence x is denoted by $\Lambda_\mu(x)$.*

Definition 3.4. *Let (X, μ) be a $\mathcal{P}-$modular space. Then $\gamma \in X$ is called a statistical cluster point of the sequence $x = \{x_k\}$ with respect to the probabilistic modular μ (or $st_\mu-$cluster point) if for all $\varepsilon > 0$ and $\lambda \in (0,1)$*

$$\overline{\delta}\left(\{k \in \mathbb{N} : \mu_{x_k - \gamma}(\varepsilon) > 1 - \lambda\}\right) > 0.$$

In this case we say that γ is a $st_\mu-$cluster point of the sequence $x = \{x_k\}$. The set of all $st_\mu-$cluster points of the sequence x is denoted by $\Gamma_\mu(x)$.

Theorem 3.5. *Let (X, μ) be a $\mathcal{P}-$modular space. For any sequence $x \in X$ it holds $\Lambda_\mu(x) \subset \Gamma_\mu(x)$.*

Proof. Assume $L \in \Lambda_\mu(x)$, then there is a nonthin subsequence $\{x_{k(j)}\}$ of $x = \{x_k\}$ that is $st_\mu-$*convergent* to L, i.e. for all $\varepsilon > 0$ and $\lambda \in (0,1)$

$$\delta\left(\left\{k(j) \in \mathbb{N} : \mu_{x_{k(j)} - L}(\varepsilon) > 1 - \lambda\right\}\right) = d > 0.$$

So

$$\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1 - \lambda\} \supset \left\{k(j) \in \mathbb{N} : \mu_{x_{k(j)} - L}(\varepsilon) > 1 - \lambda\right\},$$

we have

$$\begin{aligned}\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1 - \lambda\} \quad \supset \quad & \{k(j) \in \mathbb{N} : j \in \mathbb{N}\} \\ & \setminus \left\{k(j) \in \mathbb{N} : \mu_{x_{k(j)} - L}(\varepsilon) \leq 1 - \lambda\right\}.\end{aligned}$$

Since $\{x_{k(j)}\}$ is $st_\mu-$*convergent* to L, the set

$$\left\{k(j) \in \mathbb{N} : \mu_{x_{k(j)} - L}(\varepsilon) \leq 1 - \lambda\right\}$$

is finite for every $\varepsilon > 0$. Hence,

$$\begin{aligned}\overline{\delta}\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1-\lambda\}\right) &\geq \overline{\delta}\left(\{k(j) \in \mathbb{N} : j \in \mathbb{N}\}\right) \\ &\quad -\overline{\delta}\left(\left\{k(j) \in \mathbb{N} : \mu_{x_{k(j)} - L}(\varepsilon) \leq 1-\lambda\right\}\right).\end{aligned}$$

Therefore

$$\overline{\delta}\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1-\lambda\}\right) > 0$$

that is $L \in \Gamma_\mu(x)$. □

Theorem 3.6. *Let (X, μ) be a $\mathcal{P}-$modular space. For any sequence $x \in X$ it holds $\Gamma_\mu(x) \subseteq L_\mu(x)$.*

Proof. Suppose $\gamma \in \Gamma_\mu(x)$, then

$$\delta\left(\{k \in \mathbb{N} : \mu_{x_k - \gamma}(\varepsilon) > 1-\lambda\}\right) > 0$$

for any $\varepsilon > 0$ and $\lambda \in (0,1)$. Set $\{x\}_K$ a nonthin subsequence of x such that

$$K := \left\{k(j) \in \mathbb{N} : \mu_{x_{k(j)} - \gamma}(\varepsilon) > 1-\lambda\right\}$$

for every $\varepsilon > 0$ and $\delta(K) \neq 0$. Since there are infinitely many elements in K, $\gamma \in L_\mu(x)$. □

Theorem 3.7. *Let (X, μ) be a $\mathcal{P}-$modular space. For a sequence $x = \{x_k\}$ with $st_\mu - \lim x = L$ it follows that $\Lambda_\mu(x) = \Gamma_\mu(x) = \{L\}$.*

Proof. First we show that $\Lambda_\mu(x) = \{L\}$. Assume that $\Lambda_\mu(x) = \{L, N\}$ $(L \neq N)$. Then we can write that there exist nonthin subsequences $\left\{x_{k(j)}\right\}$ and $\left\{x_{l(i)}\right\}$ of $x = \{x_k\}$ that are $st_\mu-$*convergent* to L and N, respectively. Because of $\left\{x_{l(j)}\right\}$ is $st_\mu-$*convergent* to N for every $\varepsilon > 0$ and $\lambda \in (0,1)$

$$K := \left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) \leq 1-\lambda\right\}$$

is a finite set, so $\delta(K) = 0$. Then we observe that

$$\begin{aligned}\{l(i) \in \mathbb{N} : i \in \mathbb{N}\} &= \left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) > 1-\lambda\right\} \\ &\quad \cup \left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) \leq 1-\lambda\right\}\end{aligned}$$

which implies that

$$\delta\left(\left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) > 1-\lambda\right\}\right) \neq 0. \tag{3.1}$$

Since $st_\mu - \lim x = L$,

$$\delta\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1-\lambda\}\right) = 0 \tag{3.2}$$

for every $\varepsilon > 0$. Hence, we get

$$\delta\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1-\lambda\}\right) \neq 0.$$

For every $L \neq N$

$$\left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) > 1-\lambda\right\} \cap \{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1-\lambda\} = \varnothing.$$

So

$$\left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) > 1-\lambda\right\} \subseteq \{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1-\lambda\}.$$

Therefore

$$\overline{\delta}\left(\left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)}-N}(\varepsilon) > 1-\lambda\right\}\right) \leq \overline{\delta}\left(\left\{k \in \mathbb{N} : \mu_{x_k-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

This contradicts (3.1). Hence $\Lambda_\mu(x) = \{L\}$.

Now suppose that $\Gamma_\mu(x) = \{L, M\}$ $(L \neq M)$. Then

$$\overline{\delta}\left(\left\{k \in \mathbb{N} : \mu_{x_k-M}(\varepsilon) > 1-\lambda\right\}\right) \neq 0. \tag{3.3}$$

Since

$$\left\{k \in \mathbb{N} : \mu_{x_k-L}(\varepsilon) > 1-\lambda\right\} \cap \left\{k \in \mathbb{N} : \mu_{x_k-M}(\varepsilon) > 1-\lambda\right\} = \varnothing$$

for every $L \neq M$, so

$$\left\{k \in \mathbb{N} : \mu_{x_k-L}(\varepsilon) \leq 1-\lambda\right\} \supseteq \left\{k \in \mathbb{N} : \mu_{x_k-M}(\varepsilon) > 1-\lambda\right\}.$$

Hence

$$\overline{\delta}\left(\left\{k \in \mathbb{N} : \mu_{x_k-L}(\varepsilon) \leq 1-\lambda\right\}\right) \geq \overline{\delta}\left(\left\{k \in \mathbb{N} : \mu_{x_k-M}(\varepsilon) > 1-\lambda\right\}\right). \tag{3.4}$$

From (3.3), the right hand-side of (3.4) is greater than zero and from (3.2), the left hand-side of (3.4) equals to zero. This is a contradiction. So $\Gamma_\mu(x) = \{L\}$. □

Theorem 3.8. *Let (X, μ) be a $\mathcal{P}-$modular space. Then the set Γ_μ is closed in X for each sequence $x = \{x_k\}$ of elements of X.*

Proof. Let $y \in \overline{\Gamma_\mu(x)}$. Take $\varepsilon > 0$ and $0 < \lambda < 1$. There exists $\gamma \in \Gamma_\mu(x) \cap B(y, \lambda, \varepsilon)$ such that

$$B(y, \lambda, \varepsilon) = \left\{x \in X : \mu_{y-x}(\varepsilon) > 1-\lambda\right\}.$$

Choose $\zeta > 0$ such that $B(\gamma, \zeta, \varepsilon) \subset B(y, \lambda, \varepsilon)$. We get

$$\left\{k \in \mathbb{N} : \mu_{y-x_k}(\varepsilon) > 1-\lambda\right\} \supset \left\{k \in \mathbb{N} : \mu_{\gamma-x_k}(\varepsilon) > 1-\zeta\right\}$$

so

$$\delta\left(\left\{k \in \mathbb{N} : \mu_{y-x_k}(\varepsilon) > 1-\lambda\right\}\right) \neq 0$$

and $y \in \Gamma_\mu$. □

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