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On a class of dichotomous evolution operators with strongly continuous families of projections

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Abstract. The aim of this paper is to present a concept of nonuniform exponential dichotomy through a certain class of strongly continuous evolution operators defined with the aid of a particular family of projections acting on the state space. This class easily emphasizes the fact that, in the case of uniform exponential dichotomy, the uniform exponential growth is essential in order to prove the boundedness of the dichotomic family of projections. The main result of the paper is the extension of the boundedness result in the nonuniform setting.

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1. Introduction

The exponential dichotomy property for linear dynamical systems has gained prominence since the appearance of two fundamental monographs of J. L. Massera and J. J. Schäffer [12], J. L. Daleckii and M. G. Krein [10]. These were followed by the important books of C. Chicone and Y. Latushkin [9] and L. Barreira and C. Valls [5].

Concerning the stability, unstability and dichotomy properties, it is worth to note that their study had an impressive development and several results were obtained, which characterizes these properties, connect them and study their preservations under small perturbations, which were successfully materialized in [2], [14], [17], [11], [16], [22], [19] and the references therein.

The study of concepts of nonuniform exponential dichotomies materialized in a large number of interesting research papers, from where we point out: [6], [7], [8], [13], [18], [21], [17].

In this paper we present a particular family of projections on the Banach space $l^\infty(\mathbb{N}^*, \mathbb{R})$, which satisfies a vast variety of properties, useful in constructing counterexamples (see for example [3] in discrete time). Attached to this family of projections, we give a particular type of evolution operator which will serve as an example to the importance of the growth property assumed in order to prove the existence of a constant upper-bound for the dichotomic family of projections.

In the final part of this paper we took a step forward in this direction: under the hypotheses of exponential growth and *nonuniform* exponential dichotomy, the family of projections is, by conclusion, exponentially bounded.

Several results in the uniform setting were obtained in this sense, and we point out the works [15], [20], [23], [1].

A first approach in the nonuniform case (under the assumption of nonuniform exponential growth and *uniform* asymptotic behavior) was successfully accomplished in [2] in discrete-time, from where the particular cases of exponential and polynomial upper-bounds of the projections were obtained. By using different methods and a stronger concept than the exponential dichotomy (the notion of admissibility), an exponential upper-bound - in terms of an auxiliary norm constructed on the state space - of the family of projections was obtained in [4].

2. Preliminaries

Let X be a real or complex Banach space, and $\mathcal{B}(X)$ the algebra of bounded linear operators acting on X. We denote by $\|\cdot\|$ the norm on X and on $\mathcal{B}(X)$, and let Δ be the set of all pairs of real nonnegative numbers (t,s) satisfying $t \geq s$.

Definition 2.1. *A map $U : \Delta \to \mathcal{B}(X)$ is called an* **evolution operator** *on X if the following conditions hold:*

(e_1) $U(t,t) = I$, for all $t \geq 0$ (I denoting the identity operator on X).
(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$, for all $(t,s),(s,t_0) \in \Delta$.
Moreover, if
(e_3) for all $t \geq 0$ and for all $x \in X$ the maps $[0,t] \ni \tau \mapsto U(t,\tau)x \in X$ and $[t,\infty) \ni \tau \mapsto U(\tau,t)x \in X$ are continuous
then we say that $U : \Delta \to \mathcal{B}(X)$ is a **strongly continuous evolution operator**.

Definition 2.2. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator. We say that $U : \Delta \to \mathcal{B}(X)$ has an* **exponential growth** *if there exist $M, \omega > 0$, $\varepsilon \geq 0$ such that*

$$\|U(t,s)x\| \leq M e^{\varepsilon s} e^{\omega(t-s)} \|x\|, \quad \forall (t,s) \in \Delta, \forall x \in X.$$

In the particular case in which $\varepsilon = 0$, we say that U has a **uniform exponential growth**.

Remark 2.3. If an evolution operator $U : \Delta \to \mathcal{B}(X)$ has a uniform exponential growth then it obviously has a nonuniform exponential growth.

Example 2.4. Let $f : \mathbb{R}_+ \to (0,\infty)$ be a continuous function. For $(t,s) \in \Delta$ we define $U(t,s) : X \to X$ by

$$U(t,s)x = \frac{f(t)}{f(s)} \cdot x, \quad \forall x \in X.$$

We have that U is a strongly continuous evolution operator.

1. If $f(t) = e^t$, for al $t \geq 0$, it is easy to see that $U : \Delta \to \mathcal{B}(X)$ has a uniform exponential growth.
2. If $f(t) = t \cdot \cos t$, for all $t \geq 0$, one can see that $U : \Delta \to \mathcal{B}(X)$ has an exponential growth, which is not uniform.

Below, we will see that Definition 2.2 is not redundant.

Example 2.5. Let $X = \mathbb{R}$ and $A : \mathbb{R}_+ \to \mathbb{R}_+$, $A(t) = e^t$. Consider the Cauchy problem

$$\begin{cases} \dot{x}(t) &= A(t)x(t), \quad t > 0 \\ x(0) &= e \end{cases}$$

The above stated problem has the solution $x(t) = e^{e^t}$, the corresponding evolution operator being $U : \Delta \to \mathcal{B}(\mathbb{R})$,

$$U(t,s)x = e^{e^t - e^s} \cdot x, \quad \forall (t,s,x) \in \Delta \times \mathbb{R}.$$

Assuming that there exist $M \geq 1$, $\varepsilon \geq 0$ and $\omega > 0$ such that

$$\|U(t,s)\| \leq M e^{\varepsilon s} e^{\omega(t-s)}, \quad \forall (t,s) \in \Delta,$$

choosing in the above inequality $s = 0$, we obtain the contradiction

$$e^{e^t} \leq M e^{\omega t}, \quad \forall t \geq 0.$$

The example from above shows us that even in the particular context of evolution operators arising from Cauchy problems, the exponential growth is not assured.

Definition 2.6. *A map $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is called a* **family of projections** *on X if*

$$P(t)^2 = P(t), \quad \text{for every } t \geq 0.$$

In addition,

(i) if there are $M \geq 1$ and $\gamma \geq 0$ such that

$$\|P(t)\| \leq M e^{\gamma t}, \quad \text{for all } t \geq 0$$

then we say that the family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is **exponentially bounded**. *In the particular case when $\gamma = 0$, P is called* **bounded**;

(ii) if for all $t \geq 0$ and for all $x \in X$, the map

$$\mathbb{R}_+ \ni t \mapsto P(t)x \in X$$

is continuous then we say that the family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is **strongly continuous**.

Remark 2.7. If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is a family of projections on X then

$$Q : \mathbb{R}_+ \to \mathcal{B}(X) \text{ defined by } Q(t) = I - P(t)$$

is also a family of projections on X, which is called the **complementary family of projections** of P.

Definition 2.8. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator and $P : \mathbb{R}_+ \to \mathcal{B}(X)$ a family of projections. We say that $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is:*

(i) **invariant** *for the evolution operator $U : \Delta \to \mathcal{B}(X)$ if for all $(t,s) \in \Delta$*

$$U(t,s)P(s) = P(t)U(t,s);$$

(ii) **strongly invariant** *for the evolution operator* $U : \Delta \to \mathcal{B}(X)$ *if it is invariant for* U *and for all* $(t, s) \in \Delta$ *the restriction*

$$U(t, s)_| : KerP(s) \to KerP(t)$$

is an isomorphism.

In what follows, if P is invariant for U, then we say that (U, P) is a **dichotomy pair**.

3. Nonuniform exponential dichotomies

Let (U, P) be a dichotomy pair.

Definition 3.1. *We say that* (U, P) *is* **exponentially dichotomic** *(e.d) if there exist constants* $N \geq 1, \beta > 0,\ \alpha \geq 0$ *such that for all* $(t, s, x) \in \Delta \times X$ *the following hold:*

(ed_1*)* $\|U(t, s)P(s)x\| \leq Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$*;*
(ed_2*)* $Ne^{\alpha s}\|U(t, s)Q(s)x\| \geq e^{\beta(t-s)}\|Q(s)x\|$.

If $\alpha = 0$ *then we say that* (U, P) *is* **uniformly exponentially dichotomic** *(u.e.d).*

Remark 3.2. If a dichotomy pair (U, P) is (u.e.d) then it is also (e.d). The converse is not generally true, as we can see in the below example.

Example 3.3. Let $X = \mathbb{R}^2$, $f : \mathbb{R}_+ \to \mathbb{R}$,

$$f(t) = \frac{t}{1 + \{t\}}, \quad t \geq 0$$

where by $\{t\}$ we denoted the fractional part of the real number t. For the above defined function we have the following estimation:

$$f(t) - f(s) \geq \frac{1}{2}(t - s) - \frac{s}{2}, \quad \forall (t, s) \in \Delta.$$

We define $U : \Delta \to \mathcal{B}(\mathbb{R}^2)$ by

$$U(t, s)(x_1, x_2) = \left(e^{f(s)-f(t)}, e^{f(t)-f(s)}\right), \quad (t, s, x_1, x_2) \in \Delta \times \mathbb{R}^2.$$

Defining $P : \mathbb{R}_+ \to \mathcal{B}(X)$, by $P(t)(x_1, x_2) = (x_1, 0)$ for $t \geq 0$ and $(x_1, x_2) \in \mathbb{R}^2$, we have that (U, P) is a dichotomy pair and a straightforward estimation shows us that for all $(t, s) \in \Delta$, and for all $x = (x_1, x_2) \in \mathbb{R}^2$

$$\|U(t, s)P(s)x\| \leq e^{\frac{1}{2}s}e^{-\frac{1}{2}(t-s)}\|P(s)x\|$$
$$e^{\frac{1}{2}s}\|U(t, s)Q(s)x\| \geq e^{\frac{1}{2}(t-s)}\|Q(s)x\|.$$

Hence conditions (ed_1) and (ed_1) follow from above, from where (U, P) is e.d, but the dichotomy cannot be uniform since, by assuming the contrary, for $n \in \mathbb{N}$ setting $t_n = n + \frac{3}{2}$ and $s_n = n + 1$, with N, β given by Definition 3.1, we would obtain the contradiction

$$e^{\frac{n}{3}} \leq Ne^{-\frac{\beta}{2}}, \quad \forall n \in \mathbb{N}.$$

Remark 3.4. In [15] it is proven that if $U : \Delta \to \mathcal{B}(X)$ has uniform exponential growth then the uniform exponential dichotomy of $U : \Delta \to \mathcal{B}(X)$ implies that

$$\sup_{t \geq 0} \|P(t)\| < +\infty.$$

We will show that the uniform exponential growth of the dichotomic evolution operator $U : \Delta \to \mathcal{B}(X)$ is essential for the conclusion in the preceding remark to hold.

In what follows, we will present a family of projections which is strongly continuous and, by choosing an appropriate evolution operator, it will give a dichotomy pair with interesting properties.

Example 3.5. Let $X = l^\infty(\mathbb{N}^*, \mathbb{R})$ the Banach space of bounded real-valued sequences, endowed wit the sup-norm

$$\|x\|_\infty = \sup_{n \geq 1} |x_n|, \quad \text{for } x = (x_n)_{n \geq 1} \in l^\infty(\mathbb{N}^*, \mathbb{R}).$$

The norm on $\mathcal{B}(X)$ will be denoted as usual by $\| \cdot \|$.

For every $t \in \mathbb{R}_+$ we define $P(t) : l^\infty(\mathbb{N}^*, \mathbb{R}) \to l^\infty(\mathbb{N}^*, \mathbb{R})$, for $x = (x_1, x_2, \ldots, x_n, \ldots) \in l^\infty(\mathbb{N}^*, \mathbb{R})$, by

$$P(t)x = \big(x_1 + (e^t - 1)x_2 \ , \ 0 \ , \ x_3 + (e^t - 1)x_4 \ , \ 0 \ , \ \ldots\big)\,.$$

We denote by $Q(t) = I - P(t)$, for all $t \in \mathbb{R}_+$.

The properties of the family of operators $P : \mathbb{R}_+ \to \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ are pointed out by the following result.

Proposition 3.6. *For all $t, s \in \mathbb{R}_+$ and for all $x = (x_n)_{n \geq 1} \in l^\infty(\mathbb{N}^*, \mathbb{R})$, the following assertions hold:*

(i) $P(t)$ *is correctly defined,* $P(t) \in \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ *and* $\|P(t)\| = e^t$*;*

(ii) $P(t)$ *is a projection on* $l^\infty(\mathbb{N}^*, \mathbb{R})$*;*

(iii) $Q(t)x = ((1 - e^t)\, x_2 \ , \ x_2 \ , \ (1 - e^t)\, x_4 \ , \ x_4 \ , \ \ldots)$ *and* $\|Q(t)\| = \max\{1, e^t - 1\}$*;*

(iv)

$$\begin{aligned} Range P(t) &= Range P(s) \\ &= \{(y_n)_{n \geq 1} \in l^\infty(\mathbb{N}^*, \mathbb{R}) \ : \ y_{2n} = 0, \forall n \in \mathbb{N}^*\} =: \mathcal{H}; \end{aligned}$$

(v)

$$\begin{aligned} Range Q(t) = \big\{(x_n) \in l^\infty(\mathbb{N}^*, \mathbb{R}) \ : \ x_{2n-1} + \big(e^t - 1\big)\, x_{2n} = 0, \forall n \in \mathbb{N}^*\big\} \\ =: \mathcal{K}(t); \end{aligned}$$

(vi) the decomposition $l^\infty(\mathbb{N}^*, \mathbb{R}) = \mathcal{H} \oplus \mathcal{K}(t)$ *holds;*

(vii) $P(t)P(s) = P(s)$*;*

(viii) $Q(t)Q(s) = Q(t)$*;*

(ix) $Q(t)P(s) = 0$*.*

Proof. Let $t, s \in \mathbb{R}_+$ and $x = (x_n)_{n \geq 1} \in l^\infty(\mathbb{N}^*, \mathbb{R})$.

(i) Obviously $P(t)$ is a linear operator, and from

$$\|P(t)x\|_\infty = \sup_{n\geq 1} \left|x_{2n-1} + \left(e^t - 1\right)x_{2n}\right| \leq \left(1 + \left|e^t - 1\right|\right)\|x\|_\infty = e^t\|x\|_\infty,$$

we have that $P(t)$ is correctly defined, and $P(t) \in \mathcal{B}(X)$ with $\|P(t)\| \leq e^t$. Choosing $x = (1,1,1,\ldots) \in l^\infty(\mathbb{N}^*, \mathbb{R})$ we have that $\|x\|_\infty = 1$ and from

$$P(t)x = \left(1 + \left(e^t - 1\right)\cdot 1\ ,\ 0\ ,\ 1 + \left(e^t - 1\right)\cdot 1\ ,\ 0\ ,\ \ldots\right) = \left(e^t\ ,\ 0\ ,\ e^t\ ,\ 0\ ,\ \ldots\right),$$

we have that

$$\|P(t)x\|_\infty = e^t = e^t\|x\|_\infty,$$

from which it follows that $\|P(t)\| = e^t$.

(ii) Let $y = P(t)x$. Then we have that for all $n \in \mathbb{N}^*$, $y_{2n-1} = x_{2n-1} + (e^t - 1)\,x_{2n}$ and $y_{2n} = 0$. It follows from here that

$$\begin{aligned} P(t)^2x = P(t)y &= \Big(y_1 + \left(e^t - 1\right)y_2\ ,\ 0\ ,\ y_3 + \left(e^t - 1\right)y_4\ ,\ 0\ ,\ \ldots\Big) \\ &= (y_1, 0, y_3, 0, \ldots) = P(t)x. \end{aligned}$$

(iii) The expression defining $Q(t)$ follows from a straightforward computation. Let $y = Q(t)x$. It follows that for $n \in \mathbb{N}^*$, $y_{2n-1} = (1 - e^t)\,x_{2n}$ and $y_{2n} = x_{2n}$. This implies that

$$\begin{aligned} \|Q(t)x\|_\infty = \sup_{n\geq 1}|y_n| &= \max\left\{\sup_{n\geq 1}|y_{2n-1}|, \sup_{n\geq 1}|y_{2n}|\right\} \\ &= \max\left\{\left(e^t - 1\right)\sup_{n\geq 1}|x_{2n}|, \sup_{n\geq 1}|x_{2n}|\right\} \\ &= \max\left\{1, e^t - 1\right\}\cdot\sup_{n\geq 1}|x_{2n}| \leq \max\left\{1, e^t - 1\right\}\cdot\|x\|_\infty. \end{aligned}$$

Choosing $x_0 = (0,1,0,1,\ldots) \in l^\infty(\mathbb{N}^*, \mathbb{R})$ having $\|x_0\|_\infty = 1$, from

$$Q(t)x_0 = \left(1 - e^t\ ,\ 1\ ,\ 1 - e^t\ ,\ 1\ ,\ \ldots\right)$$

we obtain that $\|Q(t)x_0\|_\infty = \max\{1, e^t - 1\}\,\|x_0\|_\infty$, from which the validity of the assertion follows.

(iv) Let $y \in RangeP(t)$. Then there exists $z \in l^\infty(\mathbb{N}^*, \mathbb{R})$ with $y = P(t)z$, from which we deduce that $y_{2n} = 0$, for all $n \in \mathbb{N}^*$.
Conversely, let $y \in l^\infty(\mathbb{N}^*, \mathbb{R})$ be a bounded sequence having $y_{2n} = 0$, for all $n \in \mathbb{N}^*$. A straightforward calculation shows us that $P(t)y = y$, so $y \in RangeP(t)$.

(v) From the equivalence $P(t)x = 0 \Leftrightarrow x_{2n-1} + (e^t - 1)\,x_{2n} = 0$, for all $n \in \mathbb{N}^*$, it follows that the assertion is true.

(vi) The decomposition takes place, provided by the fact that $P(t) \in \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$.

(vii) From $P(s)x \in \mathcal{H} = RangeP(t)$, $P(t)$ acting as the identity operator on its range, we deduce that $P(t)P(s)x = P(s)x$.

(viii) The desired relation follows from

$$\begin{aligned} Q(t)Q(s)x &= (I - P(t))(I - P(s))x = x - P(s)x - P(t)x + P(t)P(s)x \\ &= x - P(s)x - P(t)x + P(s)x = x - P(t)x = Q(t)x. \end{aligned}$$

(ix) From $P(s)x \in \mathcal{H} = RangeP(t)$, we have that $Q(t)P(s)x = 0$. □

Another property of the above defined family of projections is given by:

Proposition 3.7. *The family of projections* $P : \mathbb{R}_+ \to \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ *defined in Example* 3.5 *is strongly continuous.*

Proof. Let $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$ and $\tau_0 \in \mathbb{R}_+$. A simple computation gives us that

$$P(\tau)x - P(\tau_0)x = ((e^\tau - e^{\tau_0})x_2 \ , \ 0 \ , \ (e^\tau - e^{\tau_0})x_4 \ , \ 0 \ , \ \ldots),$$

hence

$$\|P(\tau)x - P(\tau_0)x\|_\infty = |e^\tau - e^{\tau_0}| \sup_{n\geq 1} |x_{2n}| \leq |e^\tau - e^{\tau_0}| \, \|x\|_\infty \xrightarrow[\tau\to\tau_0]{} 0.$$

From this it easily follows that

$$\|Q(\tau)x - Q(\tau_0)x\|_\infty = \|x - P(\tau)x - x + P(\tau_0)x\|_\infty = \|P(\tau_0)x - P(\tau)x\|_\infty \xrightarrow[\tau\to\tau_0]{} 0$$

hence its complementary is also strongly continuous. □

For $(t, s) \in \Delta$, we define $U_P(t, s) : l^\infty(\mathbb{N}^*, \mathbb{R}) \to l^\infty(\mathbb{N}^*, \mathbb{R})$ by

$$U_P(t, s)x = e^{s-t}P(s)x + e^{t-s}Q(t)x, \quad \forall x \in l^\infty(\mathbb{N}^*, \mathbb{R}).$$

The following result will point out the basic properties that the above defined two-parameter family of bounded linear operators verifies.

Proposition 3.8. U_P *is a strongly continuous evolution operator on* $l^\infty(\mathbb{N}^*, \mathbb{R})$.

Proof. (e_1) We have that $U_P(t, t)x = e^0P(t)x + e^0Q(t)x = x$, for all $(t, s, x) \in \Delta \times l^\infty(\mathbb{N}^*, \mathbb{R})$.
(e_2) Let $(t, s), (s, t_0) \in \Delta$ and $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$.

$$\begin{aligned}
&U_P(t, s)U_P(s, t_0)x = e^{s-t}P(s)U(s, t_0)x + e^{t-s}Q(t)U(s, t_0)x \\
&= e^{s-t}P(s)\left(e^{t_0-s}P(t_0)x + e^{s-t_0}Q(s)x\right) + e^{t-s}Q(t)\left(e^{t_0-s}P(t_0)x + e^{s-t_0}Q(s)x\right) \\
&= e^{t_0-t}P(s)P(t_0)x + e^{s-t}e^{s-t_0}P(s)Q(s)x + e^{t-s}e^{t_0-s}Q(t)P(t_0)x + e^{t-t_0}Q(t)Q(s)x \\
&= e^{t_0-t}P(t_0)x + e^{t-t_0}Q(t)x = U_P(t, t_0)x.
\end{aligned}$$

(e_3) Let $t \geq 0$ and $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$. The continuity of the map $[0, t] \ni \tau \mapsto U(t, \tau)x \in l^\infty(\mathbb{N}^*, \mathbb{R})$ follows from the below estimations:

$$\begin{aligned}
&\|U_P(t, \tau)x - U_P(t, \tau_0)x\|_\infty \\
&\leq \left\|e^{\tau-t}P(\tau)x + e^{t-\tau}Q(t)x - e^{\tau_0-t}P(\tau_0)x - e^{t-\tau_0}Q(t)x\right\|_\infty \\
&\leq \left\|e^{\tau-t}P(\tau)x - e^{\tau_0-t}P(\tau_0)x\right\|_\infty + \left\|e^{t-\tau}Q(t)x - e^{t-\tau_0}Q(t)x\right\|_\infty \\
&\leq \left\|e^{\tau-t}P(\tau)x - e^{\tau-t}P(\tau_0)x\right\|_\infty + \left\|e^{\tau-t}P(\tau_0)x - e^{\tau_0-t}P(\tau_0)x\right\|_\infty + \\
&+ \left|e^{t-\tau} - e^{t-\tau_0}\right| \|Q(t)x\|_\infty \leq \\
&\leq \|P(\tau)x - P(\tau_0)x\|_\infty + \left|e^{\tau-t} - e^{\tau_0-t}\right| \|P(\tau_0)x\|_\infty + e^t \left|e^{-\tau} - e^{-\tau_0}\right| \|Q(t)x\|_\infty.
\end{aligned}$$

To prove that the map $[t,\infty) \ni \tau \mapsto U(\tau,t)x \in l^\infty(\mathbb{N}^*,\mathbb{R})$ is continuous, we will proceed as above:

$$\begin{aligned}
&\|U_P(\tau,t)x - U_P(\tau_0,t)x\|_\infty \\
&= \left\|e^{t-\tau}P(t)x + e^{\tau-t}Q(\tau)x - e^{t-\tau_0}P(t)x - e^{\tau_0-t}Q(\tau_0)x\right\|_\infty \\
&\le \left|e^{t-\tau} - e^{t-\tau_0}\right| \|P(t)x\|_\infty + e^{\tau-t}\|Q(\tau)x - Q(\tau_0)x\|_\infty + \left|e^{\tau-t} - e^{\tau_0-t}\right| \|Q(\tau_0)x\|_\infty,
\end{aligned}$$

the right-hand side tending to zero as $\tau \to \tau_0$, provided by the fact that the map $\tau \mapsto e^{\tau-t}$ is bounded on the interval $[t,\tau_0+1]$. □

Regarding the growth of the above defined evolution operator, we state the following two results.

Proposition 3.9. *The evolution operator $U_P : \Delta \to \mathcal{B}(l^\infty(\mathbb{N}^*,\mathbb{R}))$ has an exponential growth.*

Proof. Let $(t,s) \in \Delta$ and $x \in l^\infty(\mathbb{N}^*,\mathbb{R})$. Setting $M = \omega = 2$ and $\varepsilon = 1$, and having in mind that

$$\begin{aligned}
\|U_P(t,s)x\|_\infty &= \left\|e^{s-t}P(s)x + e^{t-s}Q(t)x\right\|_\infty \le e^{t-s}\left(\|P(s)\| + \|Q(t)\|\right)\|x\|_\infty = \\
&= e^{t-s}\left(e^s + \max\{1, e^t - 1\}\right)\|x\|_\infty \le 2e^t e^{t-s}\|x\|_\infty = \\
&= 2e^s e^{2(t-s)}\|x\|_\infty,
\end{aligned}$$

we obtain the desired conclusion. □

Proposition 3.10. *The evolution operator U_P does not admit a uniform exponential growth.*

Proof. Assume by a contradiction that there exist $M,\omega > 0$ such that

$$\|U_P(t,s)x\|_\infty \le Me^{\omega(t-s)}\|x\|_\infty, \quad \forall (t,s,x) \in \Delta \times l^\infty(\mathbb{N}^*,\mathbb{R}).$$

Let, in the above inequality, $t \ge 3$, $s = t-1$ and $x = (0,1,0,1,\ldots) \in l^\infty(\mathbb{N}^*,\mathbb{R})$, having $\|x\|_\infty = 1$. This implies that

$$\|U_P(t,t-1)x\|_\infty \le Me^\omega. \tag{3.1}$$

We have that

$$\frac{1}{e}P(t-1)x = \left(e^{t-2} - \frac{1}{e}, 0, e^{t-2} - \frac{1}{e}, 0, \ldots\right) \tag{3.2}$$

$$eQ(t)x = \left(e - e^{t+1}, e, e - e^{t+1}, e, \ldots\right). \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$U_P(t,t-1)x = \left(e^{t-2} - e^{t+1} + e - \frac{1}{e}, e, e^{t-2} - e^{t+1} + e - \frac{1}{e}, e, \ldots\right),$$

from which we deduce that

$$\|U_P(t,t-1)x\|_\infty = \max\left\{\left|e^{t-2} - e^{t+1} + e - \frac{1}{e}\right|, e\right\}. \tag{3.4}$$

By Lagrange's mean value theorem applied to the exponential function on the interval $[t-2, t+1] \subset [1, \infty)$, there exists $\xi_t \in (t-2, t+1)$ such that

$$e^{t+1} - e^{t-2} = 3e^{\xi_t} > 3e^{t-2} \geq 3e. \tag{3.5}$$

Hence

$$e^{t-2} - e^{t+1} + e - \frac{1}{e} = -3e^{\xi_t} + e - \frac{1}{e} < -2e - \frac{1}{e} < -2e < -e < 0. \tag{3.6}$$

By (3.4) and (3.6), we have that

$$\|U_P(t, t-1)x\|_\infty = e^{t+1} - e^{t-2} + \frac{1}{e} - e. \tag{3.7}$$

Finally, using (3.7), (3.5) and (3.1), we obtain the contradicting inequality

$$3e^{t-2} + \frac{1}{e} - e \leq e^{t+1} - e^{t-2} + \frac{1}{e} - e \leq Me^{\omega}, \quad \forall t \geq 3.$$

□

Proposition 3.11. *(U_P, P) is a dichotomy pair.*

Proof. Let $(t, s) \in \Delta$ and $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$. The conclusion easily follows from

$$\begin{aligned} U_P(t,s)P(s)x &= e^{s-t}P(s)P(s)x + e^{t-s}Q(t)P(s)x \\ &= e^{s-t}P(s)x; \\ P(t)U_P(t,s)x &= P(t)\left(e^{s-t}P(s)x + e^{t-s}Q(t)x\right) \\ &= e^{s-t}P(t)P(s)x + e^{t-s}P(t)Q(t)x \\ &= e^{s-t}P(s)x. \end{aligned}$$

□

Corollary 3.12. *From the above proposition, we can state that for all $(t, s) \in \Delta$ we have:*

(i) $U_P(t, s)Q(s) = Q(t)U_P(t, s)$;
(ii) $U_P(t, s)\mathcal{H} \subset \mathcal{H}$;
(iii) $U_P(t, s)\mathcal{K}(s) \subset \mathcal{K}(t)$.

Proposition 3.13. *For all $(t, s) \in \Delta$ the restriction $U_P(t, s)_| : \mathcal{K}(s) \to \mathcal{K}(t)$ is an isomorphism.*

Proof. Let $(t, s) \in \Delta$. To prove the injectivity of $U_P(t, s)_|$, let $y \in \mathcal{K}(s)$ satisfying $U_P(t, s)_| y = 0$. Using the definition of $\mathcal{K}(s)$, we have that there exists $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$ with $y = Q(s)x$. It follows that

$$U_P(t, s)_| y = U_P(t, s)Q(s)x = e^{t-s}Q(t)x$$

which implies $Q(t)x = 0$, so $P(t)x = x$. Hence $y = Q(s)x = Q(s)P(t)x = 0$.
To prove the surjectivity of the operator, let $z \in \mathcal{K}(t)$. It follows that there exists $y \in l^\infty(\mathbb{N}^*, \mathbb{R})$ with $z = Q(t)y$. Let $x = e^{s-t}Q(s)y \in \mathcal{K}(s)$. Then

$$U_P(t, s)_| x = e^{t-s}Q(t)x = e^{t-s}e^{s-t}Q(t)Q(s)y = Q(t)y = z.$$

□

For $t \geq 0$ we will refer to $\mathcal{H}$ and $\mathcal{K}(t)$ as to the *stable* and *unstable* subspaces at time t respectively.

Proposition 3.14. *There exist constants $N, \beta > 0$ such that*

$$\|U_P(t,s)P(s)x\|_\infty \le Ne^{-\beta(t-s)}\|P(s)x\|_\infty, \quad \forall (t,s,x) \in \Delta \times l^\infty(\mathbb{N}^*, \mathbb{R}).$$

Proof. Choose $N = \beta = 1$. By Proposition 3.11 we have that for $(t,s,x) \in \Delta \times l^\infty(\mathbb{N}^*, \mathbb{R})$,

$$\|U_P(t,s)P(s)x\|_\infty = e^{s-t}\|P(s)x\|_\infty = Ne^{-\beta(t-s)}\|P(s)x\|_\infty. \qquad \square$$

Before stating the next result, we will need the following lemma, which gives us the sup-norm induced on $\mathcal{K}(t)$, $t \ge 0$.

Lemma 3.15. *For every $t \in \mathbb{R}_+$ and for every $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$, we have that*

$$\|Q(t)x\|_\infty = \max\{1, e^t - 1\} \cdot \sup_{n\ge 1} |x_{2n}|.$$

Proof. It follows from Proposition 3.6, (iii). $\square$

Proposition 3.16. *There exist $N, \beta > 0$ such that*

$$\|U_P(t,s)Q(s)x\|_\infty \ge Ne^{\beta(t-s)}\|Q(s)x\|_\infty, \quad \forall (t,s,x) \in \Delta \times l^\infty(\mathbb{N}^*, \mathbb{R}).$$

Proof. Choose $N = \beta = 1$. Let $(t,s) \in \Delta$ and $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$. We have that

$$\begin{aligned} \|U_P(t,s)Q(s)x\|_\infty &= e^{t-s}\|Q(t)x\|_\infty \\ &= e^{t-s}\max\{1, e^t - 1\} \sup_{n\ge 1}|x_{2n}| \\ &\ge e^{t-s}\max\{1, e^s - 1\} \sup_{n\ge 1}|x_{2n}| \\ &= e^{t-s}\|Q(s)x\|_\infty. \end{aligned}$$

$\square$

By synthesizing all of the above, we can state the following result which emphasizes the key properties of the evolution operator constructed in this section.

Theorem 3.17. *The following assertions hold:*

(i) $U_P : \Delta \to \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ *is a strongly continuous evolution operator on* $l^\infty(\mathbb{N}^*, \mathbb{R})$*;*
(ii) $U_P : \Delta \to \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ *has an exponential growth and* **does not have a uniform exponential growth***;*
(iii) (U_P, P) *is exponentially dichotomic.*
(iv) $\sup_{t\ge 0}\|P(t)\| = +\infty$.

Conclusion. In terms of Theorem 3.17, although the evolution operator U verifies all the conditions that makes it uniformly exponentially dichotomic, the property

$$\sup_{t\ge 0}\|P(t)\| < +\infty$$

fails, provided by the fact that the evolution operator does not admit a uniform exponential growth.

In the final part of this section, we will give a boundedness result of the dichotomic family of projections in the nonuniform case.

Theorem 3.18. *Let (U,P) be a dichotomy pair which is (e.d). If $U:\Delta\to\mathcal{B}(X)$ has an exponential growth then the family of projections $P:\mathbb{R}_+\to\mathcal{B}(X)$ is exponentially bounded.*

Proof. Let $M,\omega>0$ and $\varepsilon\geq 0$ given by the exponential growth and N,α,β given by the (e.d) property. Let $s\geq 0$, $x\in X$ and $t\geq s$. It follows that

$$\begin{aligned}
&(\|P(s)x\|-\|x\|)\frac{e^{\beta(t-s)}}{Ne^{\alpha s}}-Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|\\
&\leq\frac{e^{\beta(t-s)}}{N}e^{-\alpha s}\|Q(s)x\|-Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|\\
&\leq\|U(t,s)Q(s)x\|-\|U(t,s)P(s)x\|\leq\|U(t,s)x\|\leq Me^{\varepsilon s}e^{\omega(t-s)}\|x\|
\end{aligned}$$

from where

$$\left[\frac{e^{-\alpha s}e^{\beta(t-s)}}{N}-Ne^{\alpha s}e^{-\beta(t-s)}\right]\|P(s)x\|\leq Me^{\varepsilon s}e^{(\omega+\beta)(t-s)}\|x\|. \tag{3.8}$$

Consider

$$t=s+\frac{\alpha}{\beta}s+\frac{\ln N}{\beta}+1\geq s.$$

Then we have that

$$\frac{1}{N}e^{-\alpha s}e^{\beta(t-s)}=\frac{1}{N}e^{-\alpha s}Ne^{\beta}e^{\alpha s}=e^{\beta} \tag{3.9}$$

$$Ne^{\alpha s}e^{-\beta(t-s)}=Ne^{\alpha s}\frac{1}{N}e^{-\beta}e^{-\alpha s}=e^{-\beta} \tag{3.10}$$

$$e^{(\omega+\beta)(t-s)}=e^{\frac{\alpha(\omega+\beta)}{\beta}s}\cdot e^{(\omega+\beta)\left(\frac{\ln N}{\beta}+1\right)}. \tag{3.11}$$

By denoting

$$L=\frac{Me^{(\omega+\beta)\left(\frac{\ln N}{\beta}+1\right)}}{e^{\beta}-e^{-\beta}}\quad\text{şi}\quad\gamma=\frac{\alpha(\omega+\beta)}{\beta}+\varepsilon$$

from (3.9), (3.10), (3.11) and (3.8) we obtain that

$$\|P(s)x\|\leq Le^{\gamma s}\|x\|$$

which shows us that $P:\mathbb{R}_+\to\mathcal{B}(X)$ is exponentially bounded. □

As it was expected, the result from [15] can be obtained using the above theorem, which is pointed out below.

Corollary 3.19. *Let (U,P) be a dichotomy pair which is (u.e.d). If $U:\Delta\to\mathcal{B}(X)$ has a uniform exponential growth then the family of projections $P:\mathbb{R}_+\to\mathcal{B}(X)$ is bounded.*

Proof. It results from Theorem 3.18, by observing that if $\alpha=\varepsilon=0$ then $\gamma=0$. □

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