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Subclass of meromorphic functions with positive coefficients defined by convolution

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Abstract. In this paper we introduce and study new class of meromorphic functions defined by convolution. We obtain coefficients inequalities, distortion theorems, extreme points, closure theorems and some other results for the modified Hadamard products. Finally, we obtain application involving an integral operator.

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1. Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$. Let $g \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

A function $f \in \Sigma$ is meromorphic starlike of order β $(0 \leq \beta < 1)$ if

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \ (z \in U), \tag{1.4}$$

and the class of all such functions is denoted by $\Sigma^*(\beta)$. A function $f \in \Sigma$ is meromorphic convex of order β $(0 \le \beta < 1)$ if

$$-\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta \ (z \in U), \tag{1.5}$$

and the class of such functions is denoted by $\Sigma_k^*(\beta)$. The classes $\Sigma^*(\beta)$ and $\Sigma_k^*(\beta)$ are introduced and studied by Pommerenke [11], Miller [9], Mogra et al. [10], Cho [4], Cho et al. [5] and Aouf ([1] and [2]).

For $\alpha \ge 0,\ 0 \le \beta < 1,\ 0 \le \lambda < \frac{1}{2}$ and g is given by (1.2), with $b_k \ge 0$ $(k \ge 1)$, we denote by $M(f,g;\alpha,\beta,\lambda)$ the subclass of Σ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\mathrm{Re}\left\{\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+\beta\right\}$$
$$\ge \alpha\left|\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+1\right| (z \in U). \tag{1.6}$$

We note that for suitable choices of g , α and λ, we obtain the following subclasses:

$$M(f,\frac{1}{z(1-z)};0,\beta;0) = \Sigma^*(\beta) \ (0 \le \beta < 1)$$

and

$$M(f,\frac{1}{z(1-z)};0,\beta;1) = \Sigma_k^*(\beta) \ (0 \le \beta < 1)$$

(see Pommerenke [11]).

Also, we note that

(1) $M(f,g;\alpha,\beta,0) = N(f,g;\alpha,\beta)$

$$= \left\{f \in \Sigma : -\mathrm{Re}\left(\frac{z(f*g)'(z)}{(f*g)(z)}+\beta\right) \ge \alpha\left|\frac{z(f*g)'(z)}{(f*g)(z)}+1\right| \ (z \in U^*)\right\};$$

(2) Putting $g(z) = \frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k$ in (1.6), then the class

$$M(f,\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k;\alpha,\beta,\lambda)$$

reduces to the class

$$M_{\delta,\ell}(m;\alpha,\beta,\lambda) = \left\{f \in \Sigma : -\mathrm{Re}\left\{\frac{z(I^m(\delta,\ell)f(z))'+\lambda z^2(I^m(\delta,\ell)f(z))''}{(1-\lambda)(I^m(\delta,\ell)f(z))+\lambda z(I^m(\delta,\ell)f(z))'}+\beta\right\} \ge \alpha\right.$$
$$\left.\left|\frac{z(I^m(\delta,\ell)f(z))'+\lambda z^2(I^m(\delta,\ell)f(z))''}{(1-\lambda)(I^m(\delta,\ell)f(z))+\lambda z(I^m(\delta,\ell)f(z))'}+1\right| \ (\delta \ge 0;\ \ell > 0;\ m \in \mathbb{N}_0;\ z \in U)\right\},$$

where the operator

$$I^m(\delta,\ell)(z) = \frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k, \tag{1.7}$$

was introduced and studied by El-Ashwah [6, with $p = 1$] (see also Bulboacă et al. [3], El-Ashwah [7, with $p = 1$] and El-Ashwah et al. [8, with $p = 1$]).

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this paper that

$\alpha \geq 0,\ 0 \leq \beta < 1,\ 0 \leq \lambda < \frac{1}{2},$ g is given by (1.2) with $b_k > 0$ and $b_k \geq b_1$

$(k \geq 1)$.

Theorem 2.1. *Let the function f defined by (1.1). Then $f \in M(f, g; \alpha, \beta, \lambda)$ if and only if*

$$\sum_{k=1}^{\infty} [1 + \lambda(k-1)]\,[k(1+\alpha) + (\alpha+\beta)]\,a_k b_k \leq (1-\beta)(1-2\lambda). \tag{2.1}$$

Proof. Let the condition (2.1) holds true and using the fact that

$$-\mathrm{Re}(w) \geq \alpha\,|w+1| + \beta \text{ if and only if } -\mathrm{Re}\{(1+\alpha e^{i\theta})w + \alpha e^{i\theta}\} \geq \beta,$$

we have

$$-\mathrm{Re}\left\{\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} + \beta\right\}$$

$$\geq \alpha\left|\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} + 1\right|.$$

Hence

$$-\mathrm{Re}\left\{(1+\alpha e^{i\theta})\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} + \alpha e^{i\theta}\right\} \geq \beta,$$

or, equivalently,

$$-\mathrm{Re}\left\{\frac{(1+\alpha e^{i\theta})\left[z(f*g)'(z)+\lambda z^2(f*g)''(z)\right] + \alpha e^{i\theta}\left[(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)\right]}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}\right\} \geq \beta,$$

where $-\pi \leq \theta < \pi$. Suppose that

$$G(z) = -(1+\alpha e^{i\theta})\left[z(f*g)'(z) + \lambda z^2(f*g)''(z)\right]$$

$$-\alpha e^{i\theta}\left[(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)\right],$$

$$H(z) = (1-\lambda)(f*g)(z) + \lambda z(f*g)'(z),$$

and using the fact that

$$\mathrm{Re}(w) \geq \beta \text{ if and only if } |w-(1+\beta)| \leq |w+(1-\beta)| \text{ where } w = -(u+iv),$$

we need to prove that

$$|G(z) + (1-\beta)H(z)| \geq |G(z) - (1+\beta)H(z)| \text{ for } 0 \leq \beta < 1.$$

Then

$$|G(z) + (1-\beta)H(z)| - |G(z) - (1+\beta)H(z)|$$

$$= \left|(2-\beta)(1-2\lambda)\frac{1}{z} - \sum_{k=1}^{\infty}[k-(1-\beta)][1+\lambda(k-1)]a_k b_k z^k - \alpha e^{i\theta}. \right.$$
$$\left. .\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k z^k \right| - \left|\beta(1-2\lambda)\frac{1}{z} - \sum_{k=1}^{\infty}[k+(1+\beta)]. \right.$$
$$\left. .[1+\lambda(k-1)]a_k b_k z^k - \alpha e^{i\theta}\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k z^k \right|$$
$$\geq (2-\beta)(1-2\lambda)\frac{1}{|z|} - \sum_{k=1}^{\infty}[k-(1-\beta)][1+\lambda(k-1)]a_k b_k |z|^k - \alpha.$$
$$.\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k |z|^k - \beta(1-2\lambda)\frac{1}{|z|} - \sum_{k=1}^{\infty}[k+(1+\beta)].$$
$$.[1+\lambda(k-1)]a_k b_k |z|^k - \alpha\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k |z|^k$$
$$= 2(1-\beta)(1-2\lambda)\frac{1}{|z|} - 2\sum_{k=1}^{\infty}(k+\beta)[1+\lambda(k-1)]a_k b_k |z|^k - 2\alpha.$$
$$.\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k |z|^k \geq 0.$$

On simplification we easily arrive at the inequality (2.1).
Conversely, suppose that f is in the class $M(f,g;\alpha,\beta,\lambda)$. Then

$$-\mathrm{Re}\left\{\frac{(1+\alpha e^{i\theta})\left[z(f*g)'(z)+\lambda z^2(f*g)''(z)\right]+\alpha e^{i\theta}\left[(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)\right]}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}\right\} \geq \beta,$$

Hence

$$\mathrm{Re}\left\{\frac{(1-2\lambda)(1-\beta)\frac{1}{z} - \sum_{k=1}^{\infty}\{k+\alpha e^{i\theta}(k+1)+\beta\}[1+\lambda(k-1)]a_k b_k z^k}{(1-2\lambda)\frac{1}{z} + \sum_{k=1}^{\infty}[1+\lambda(k-1)]a_k b_k z^k}\right\} \geq 0,$$

If we now choose z to be real and $z \to 1^-$, we write

$$\sum_{k=1}^{\infty}[1+\lambda(k-1)]\left[k(1+\alpha)+(\alpha+\beta)\right]a_k b_k \leq (1-\beta)(1-2\lambda),$$

which completes the proof of Theorem 2.1.
Corollary 2.2. *Let the function f defined by (1.1) be in the class $M(f,g;\alpha,\beta,\lambda)$. Then*

$$a_k \leq \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\left[k(1+\alpha)+(\alpha+\beta)\right]b_k}. \tag{2.2}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\left[k(1+\alpha)+(\alpha+\beta)\right]b_k}z^k. \tag{2.3}$$

3. Distortion theorems

Theorem 3.1. *Let the function f defined by (1.1) be in the class $M(f,g;\alpha,\beta,\lambda)$, then for $0<|z|=r<1$, we have*

$$\frac{1}{|z|}-\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}|z| \le |f(z)| \le \frac{1}{|z|}+\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}|z|. \tag{3.1}$$

The result is sharp for the function f given by

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}z. \tag{3.2}$$

Proof. It is easy to see from Theorem 2.1 that

$$\begin{aligned}(2\alpha+\beta+1)b_1\sum_{k=1}^{\infty}a_k &\le \sum_{k=1}^{\infty}[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,a_kb_k\\ &\le (1-\beta)(1-2\lambda).\end{aligned}$$

Then

$$\sum_{k=1}^{\infty}a_k \le \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}. \tag{3.3}$$

Making use of (3.3), we have

$$\begin{aligned}|f(z)| &\ge \frac{1}{|z|}-|z|\sum_{k=1}^{\infty}a_k\\ &\ge \frac{1}{|z|}-\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}|z|,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}|f(z)| &\le \frac{1}{|z|}+|z|\sum_{k=1}^{\infty}a_k\\ &\le \frac{1}{|z|}+\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}|z|,\end{aligned} \tag{3.5}$$

which proves the assertion (3.1), and this completes the proof of Theorem 3.1.

Theorem 3.2. *Let the function f defined by (1.1) be in the class $M(f,g;\alpha,\beta,\lambda)$, then for $0<|z|=r<1$, we have*

$$\frac{1}{|z|^2}-\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1} \le |f'(z)| \le \frac{1}{|z|^2}+\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}. \tag{3.6}$$

The result is sharp for the function f given by (3.2).

Proof. From Theorem 2.1 and (3.3), we have

$$\sum_{k=1}^{\infty}ka_k \le \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}. \tag{3.7}$$

Since the remaining part of the proof is similar to the proof of Theorem 3.1, we omit the details.

4. Closure theorems

Let the functions f_j be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \ (a_{k,j} \geq 0). \tag{4.1}$$

Theorem 4.1. *Let the functions f_j $(j = 1, 2, ..., m)$ defined by (4.1) be in the class $M(f, g; \alpha, \beta, \lambda)$. Then the function h defined by*

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) z^k, \tag{4.2}$$

also belongs to the class $M(f, g; \alpha, \beta, \lambda)$.
Proof. Since f_j $(j = 1, 2, ..., m)$ are in the class $M(f, g; \alpha, \beta, \lambda)$, it follows from Theorem 2.1, that

$$\sum_{k=1}^{\infty} [1 + \lambda(k-1)] [k(1+\alpha) + (\alpha+\beta)] a_{k,j} b_k \leq (1-\beta)(1-2\lambda),$$

for every $j = 1, 2, ..., m$. Hence

$$\begin{aligned} & \sum_{k=1}^{\infty} [1 + \lambda(k-1)] [k(1+\alpha) + (\alpha+\beta)] b_k \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) \\ = & \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} [1 + \lambda(k-1)] [k(1+\alpha) + (\alpha+\beta)] a_{k,j} b_k \right) \\ \leq & (1-\beta)(1-2\lambda). \end{aligned}$$

From Theorem 2.1, it follows that $h \in M(f, g; \alpha, \beta, \lambda)$. This completes the proof of Theorem 4.1.
Theorem 4.2. *The class $M(f, g; \alpha, \beta, \lambda)$ is closed under convex linear combinations.*
Proof. Let the functions f_j $(j = 1, 2)$ defined by (4.1) be in the class $M(f, g; \alpha, \beta, \lambda)$. Then it is sufficient to show that the function

$$h(z) = \eta f_1(z) + (1-\eta) f_2(z) \ (0 \leq \eta \leq 1), \tag{4.3}$$

is in the class $M(f, g; \alpha, \beta, \lambda)$. Since for $0 \leq \eta \leq 1$,

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} [\eta a_{k,1} + (1-\eta) a_{k,2}] z^k, \tag{4.4}$$

with the aid of Theorem 2.1, we have

$$\begin{aligned} & \sum_{k=1}^{\infty} [1 + \lambda(k-1)] [k(1+\alpha) + (\alpha+\beta)] b_k [\eta a_{k,1} + (1-\eta) a_{k,2}] \\ \leq & \ \eta(1-\beta)(1-2\lambda) + (1-\eta)(1-\beta)(1-2\lambda) \\ = & \ (1-\beta)(1-2\lambda), \end{aligned}$$

which implies that $h \in M(f, g; \alpha, \beta, \lambda)$.

Theorem 4.3. *Let* $\sigma \geq 0,$ *then*

$$M(f, g; \alpha, \beta, \lambda) \subseteq N(f, g; \alpha, \sigma),$$

where

$$\sigma = 1 - \frac{2(1-\beta)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)+(1-\beta)(1-2\lambda)}. \tag{4.5}$$

Proof. If $f \in M(f, g; \alpha, \beta, \lambda)$, then

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_k \leq 1. \tag{4.6}$$

We need to find the value of σ such that

$$\sum_{k=1}^{\infty} \frac{[k(1+\alpha)+(\alpha+\sigma)]\,b_k}{(1-\sigma)} a_k \leq 1. \tag{4.7}$$

Thus it is sufficient to show that

$$\frac{[k(1+\alpha)+(\alpha+\sigma)]}{(1-\sigma)} \leq \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]}{(1-\beta)(1-2\lambda)}.$$

Then

$$\sigma \leq 1 - \frac{(k+1)(1-\beta)(1-2\lambda)(1+\alpha)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\lambda)}.$$

Since

$$D(k) = 1 - \frac{(k+1)(1-\beta)(1-2\lambda)(1+\alpha)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\lambda)},$$

is an increasing function of k $(k \geq 1)$, we obtain

$$\sigma \leq D(1) = 1 - \frac{2(1-\beta)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)+(1-\beta)(1-2\lambda)}.$$

Theorem 4.4. *Let* $f_0(z) = \dfrac{1}{z}$ *and*

$$f_k(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} z^k \ (k \geq 1). \tag{4.8}$$

Then f *is in the class* $M(f, g; \alpha, \beta, \lambda)$ *if and only if can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z), \tag{4.9}$$

where $\mu_k \geq 0$ *and* $\displaystyle\sum_{k=0}^{\infty} \mu_k = 1.$

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \mu_k f_k(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} \mu_k z^k. \end{aligned} \tag{4.10}$$

Then it follows that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \cdot \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} \mu_k$$

$$= \sum_{k=1}^{\infty} \mu_k = 1 - \mu_0 \leq 1.$$

which implies that $f \in M(f, g; \alpha, \beta, \lambda)$.
Conversely, assume that the function f defined by (1.1) be in the class $M\ (f, g;$ $\alpha, \beta, \lambda)$. Then

$$a_k \leq \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}.$$

Setting

$$\mu_k = \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_k,$$

where

$$\mu_0 = 1 - \sum_{k=1}^{\infty} \mu_k\ ,$$

we can see that f can be expressed in the form (4.9).
Corollary 4.5. *The extreme points of the class $M(f, g; \alpha, \beta, \lambda)$ are the functions $f_0(z) = \frac{1}{z}$ and*

$$f_k(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} z^k \ (k \geq 1). \tag{4.11}$$

5. Modified Hadamard products

Let the functions f_j $(j = 1, 2)$ defined by (4.1). The modified Hadamard product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{5.1}$$

Theorem 5.1. *Let the functions f_j $(j = 1, 2)$ defined by (4.1) be in the class $M(f, g; \alpha, \beta, \lambda)$. Then $f_1 * f_2 \in M(f, g; \alpha, \varphi, \lambda)$, where*

$$\varphi = 1 - \frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2 b_1 + (1-\beta)^2(1-2\lambda)}. \tag{5.2}$$

The result is sharp for the functions f_j $(j=1,2)$ *given by*

$$f_j(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}z\ (j=1,2). \tag{5.3}$$

Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest real parameter φ such that

$$\sum_{k=1}^{\infty}\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\varphi)]\,b_k}{(1-\varphi)(1-2\lambda)}a_{k,1}a_{k,2}\le 1. \tag{5.4}$$

Since $f_j\in M(f,g;\alpha,\beta,\lambda)$ $(j=1,2)$, we readily see that

$$\sum_{k=1}^{\infty}\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)}a_{k,1}\le 1, \tag{5.5}$$

and

$$\sum_{k=1}^{\infty}\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)}a_{k,2}\le 1. \tag{5.6}$$

By the Cauchy-Schwarz inequality we have

$$\sum_{k=1}^{\infty}\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)}\sqrt{a_{k,1}a_{k,2}}\le 1. \tag{5.7}$$

Thus it is sufficient to show that

$$\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\varphi)]\,b_k}{(1-\varphi)(1-2\lambda)}a_{k,1}a_{k,2}\le$$
$$\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)}\sqrt{a_{k,1}a_{k,2}}, \tag{5.8}$$

or equivalently, that

$$\sqrt{a_{k,1}a_{k,2}}\le\frac{[k(1+\alpha)+(\alpha+\beta)]\,(1-\varphi)}{[k(1+\alpha)+(\alpha+\varphi)]\,(1-\beta)}. \tag{5.9}$$

Hence, in light of the inequality (5.7), it is sufficient to prove that

$$\frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}\le\frac{[k(1+\alpha)+(\alpha+\beta)]\,(1-\varphi)}{[k(1+\alpha)+(\alpha+\varphi)]\,(1-\beta)}. \tag{5.10}$$

It follows from (5.10) that

$$\varphi\le 1-\frac{(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]^2\,b_k+(1-\beta)^2(1-2\lambda)}. \tag{5.11}$$

Now defining the function $E(k)$ by

$$E(k)=1-\frac{(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]^2\,b_k+(1-\beta)^2(1-2\lambda)}. \tag{5.12}$$

We see that $E(k)$ is an increasing function of k $(k\ge 1)$. Therefore, we conclude that

$$\varphi\le E(1)=1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2b_1+(1-\beta)^2(1-2\lambda)}, \tag{5.13}$$

which evidently completes the proof of Theorem 5.1.
Using arguments similar to those in the proof of Theorem 5.1, we obtain the following theorem:

Theorem 5.2. *Let the function f_1 defined by (4.1) be in the class $M(f,g;\alpha,\beta,\lambda)$. Suppose also that the function f_2 defined by (4.1) be in the class $M(f,g;\alpha,\rho,\lambda)$. Then $f_1 * f_2 \in M(f,g;\alpha,\zeta,\lambda)$ where*

$$\zeta = 1 - \frac{2(1-\beta)(1-\rho)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)(2\alpha+\rho+1)b_1 + (1-\beta)(1-\rho)(1-2\lambda)}. \tag{5.14}$$

The result is sharp for the functions f_j $(j=1,2)$ given by

$$f_1(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1} z\ , \tag{5.15}$$

and

$$f_2(z) = \frac{1}{z} + \frac{(1-\rho)(1-2\lambda)}{(2\alpha+\rho+1)b_1} z\ . \tag{5.16}$$

Theorem 5.3. *Let the functions f_j $(j = 1,2)$ defined by (4.1) be in the class $M(f,g;\alpha,\beta,\lambda)$. Then the function*

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{5.17}$$

belong to the class $M(f,g;\alpha,\varepsilon,\lambda)$, where

$$\varepsilon = 1 - \frac{4(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2 b_1 + 2(1-\beta)^2(1-2\lambda)}. \tag{5.18}$$

The result is sharp for the functions f_j $(j=1,2)$ defined by (5.3).

Proof. By using Theorem 2.1, we obtain

$$\sum_{k=1}^{\infty} \left\{ \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \right\}^2 a_{k,1}^2$$

$$\leq \left\{ \sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_{k,1} \right\}^2 \leq 1, \tag{5.19}$$

and

$$\sum_{k=1}^{\infty} \left\{ \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \right\}^2 a_{k,2}^2$$

$$\leq \left\{ \sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_{k,2} \right\}^2 \leq 1. \tag{5.20}$$

It follows from (5.19) and (5.20) that

$$\sum_{k=1}^{\infty} \frac{1}{2} \left\{ \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \right\}^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \tag{5.21}$$

Therefore, we need to find the largest ε such that

$$\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\varepsilon)]\,b_k}{(1-\varepsilon)(1-2\lambda)} \leq \frac{1}{2}\left\{\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)}\right\}^2, \tag{5.22}$$

that is

$$\varepsilon \leq 1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]^2\,b_k+2(1-\beta)^2(1-2\lambda)}. \tag{5.23}$$

Since

$$G(k)=1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]^2\,b_k+2(1-\beta)^2(1-2\lambda)}, \tag{5.24}$$

is an increasing function of k $(k \geq 1)$, we obtain

$$\varepsilon \leq G(1)=1-\frac{4(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2 b_1+2(1-\beta)^2(1-2\lambda)}, \tag{5.25}$$

and hence the proof of Theorem 5.3 is completed.

6. Integral operators

Theorem 6.1. *Let the functions f given by* (1.1) *be in the class $M(f,g;\alpha,\beta,\lambda)$. Then the integral operator*

$$F(z)=c\int_0^1 u^c\, f(uz)du \;\; (0<u\leq 1;\;\; c>0), \tag{6.1}$$

is in the class $M(f,g;\alpha,\xi,\lambda)$, where

$$\xi=1-\frac{2c(1-\beta)(1+\alpha)}{(c+2)(2\alpha+\beta+1)+c(1-\beta)}. \tag{6.2}$$

The result is sharp for the function f given by (3.2).

Proof. Let $f \in M(f,g;\alpha,\beta,\lambda)$, then

$$\begin{aligned} F(z) &= c\int_0^1 u^c\, f(uz)du \\ &= \frac{1}{z}+\sum_{k=1}^{\infty}\frac{c}{k+c+1}a_k z^k. \end{aligned} \tag{6.3}$$

Thus it is sufficient to show that

$$\sum_{k=1}^{\infty}\frac{c[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\xi)]\,b_k}{(k+c+1)(1-\xi)(1-2\lambda)}a_k \leq 1. \tag{6.4}$$

Since $f \in M(f,g;\alpha,\beta,\lambda)$, then

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_k \leq 1. \tag{6.5}$$

From (6.4) and (6.5), we have

$$\frac{c\,[k(1+\alpha)+(\alpha+\xi)]}{(k+c+1)(1-\xi)} \leq \frac{[k(1+\alpha)+(\alpha+\beta)]}{(1-\beta)}.$$

Then

$$\xi \leq 1 - \frac{c(1-\beta)(k+1)(1+\alpha)}{(c+k+1)[k(1+\alpha)+(\alpha+\beta)]+c(1-\beta)}.$$

Since

$$Y(k) = 1 - \frac{c(1-\beta)(k+1)(1+\alpha)}{(c+k+1)[k(1+\alpha)+(\alpha+\beta)]+c(1-\beta)},$$

is an increasing function of k $(k \geq 1)$, we obtain

$$\xi \leq Y(1) = 1 - \frac{2c(1-\beta)(1+\alpha)}{(c+2)(2\alpha+\beta+1)+c(1-\beta)},$$

and hence the proof of Theorem 6.1 is completed.

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