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Asymptotic behavior of intermediate points in certain mean value theorems. III

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Abstract. The paper is devoted to the study of the asymptotic behavior of intermediate points in certain mean value theorems of integral and differential fractional calculus.

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1. Introduction

Let $a, b \in \mathbb{R}$ such that $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $g \in L^1[a, b]$ such that g does not change its sign in $[a, b]$. Then, according to the first mean value theorem of integral calculus (see, for instance, [11, Theorem 85.6], or [6]), for every $x \in (a, b]$ there exists $\xi_x \in (a, x)$ such that

$$\int_a^x f(t)g(t)dt = f(\xi_x) \int_a^x g(t)dt.$$

It was proved in [15, Theorem 2.2] that

$$\lim_{x \to a+} \frac{\xi_x - a}{x - a} = \sqrt[n]{\frac{k+1}{n+k+1}} \tag{1.1}$$

if, in addition, the functions f and g satisfy the following conditions:

(i) there exists a positive integer n such that f is n times differentiable at a, with $f^{(j)}(a) = 0$ for $1 \le j \le n-1$ and $f^{(n)}(a) \ne 0$;

(ii) $g \in C[a, b]$ and there exists a nonnegative integer k such that g is k times differentiable at a with $g^{(j)}(a) = 0$ for $0 \le j \le k-1$ and $g^{(k)}(a) \ne 0$.

Regarding (ii) we notice that the continuity of g is automatically assured if $k \ge 2$, at least on a small interval $[a, a+h] \subseteq [a, b]$ (this clearly suffices when dealing with the limit (1.1)).

Further, let n be a positive integer, and let $f : [a, b] \to \mathbb{R}$ be a function whose derivative $f^{(n)}$ exists on $[a, b]$. Then, according to the Lagrange-Taylor mean value theorem, for every $x \in (a, b]$ there exists some $\xi_x \in (a, x)$ such that

$$f(x) - T_{n-1}(f; a)(x) = \frac{f^{(n)}(\xi_x)}{n!}(x-a)^n,$$

where $T_m(h; a)$ denotes the mth Taylor polynomial associated with h and a

$$T_m(h; a)(x) := h(a) + h'(a)(x-a) + \cdots + \frac{h^{(m)}(a)}{m!}(x-a)^m,$$

provided that h is m times differentiable at a. It was proved by A. G. Azpeitia [4] that

$$\lim_{x \to a+} \frac{\xi_x - a}{x - a} = \binom{n+p}{p}^{-1/p} \tag{1.2}$$

if, in addition, f satisfies the following conditions:

(i) there exists a positive integer p such that $f \in C^{n+p}[a, b]$;
(ii) $f^{(n+j)}(a) = 0$ for $1 \le j < p$;
(iii) $f^{(n+p)}(a) \neq 0$.

This result was generalized by U. Abel [1], who derived for ξ_x a complete asymptotic expansion of the form

$$\xi_x = a + \sum_{k=1}^{\infty} \frac{c_k}{k!}(x-a)^k \quad (x \to a).$$

Azpeitia's result was generalized also by T. Trif [14], who obtained the asymptotic behavior of the intermediate point in the Cauchy-Taylor mean value theorem. For other results concerning the asymptotic behavior of the intermediate points in certain mean value theorems the reader is referred to [2, 7, 8, 9, 19].

The purpose of our paper is to establish asymptotic formulas, that are similar to (1.1) and (1.2), but in the framework of fractional calculus.

2. Fractional mean value theorems of integral calculus

K. Diethelm [6, Theorem 2.1] generalized the first mean value theorem of integral calculus to the framework of fractional calculus. Recall that given $\alpha > 0$, the Riemann-Liouville fractional primitive of order α of a function $f : [a, b] \to \mathbb{R}$ is defined by (see [12] or [5])

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt,$$

provided the right side is pointwise defined on $[a, b]$.

Theorem 2.1. ([6, Theorem 2.1]) *Let $\alpha > 0$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $g \in L^1[a, b]$ be a function which does not change its sign on $[a, b]$. Then for almost every $x \in (a, b]$ there exists some $\xi_x \in (a, x)$ such that*

$$J_a^\alpha(fg)(x) = f(\xi_x) J_a^\alpha g(x). \tag{2.1}$$

Moreover, if $\alpha \geq 1$ or $g \in C[a,b]$, then the existence of ξ_x is assured for every $x \in (a,b]$.

In the special case when $g(x) \equiv 1$, the above mean value theorem takes the following form.

Corollary 2.2. ([6, Corollary 2.2]) *If $\alpha > 0$ and $f : [a,b] \to \mathbb{R}$ is a continuous function, then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$J_a^\alpha f(x) = f(\xi_x)\,\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}\,. \tag{2.2}$$

We point out that there is a misprint in the statement of [6, Corollary 2.2], where $\Gamma(\alpha)$ appears instead of $\Gamma(\alpha+1)$.

In what follows we intend to prove a fractional version of the second mean value theorem of integral calculus. For reader's convenience we recall first the second mean value theorem for Lebesgue integrals, which is usually stated for Riemann integrals (see, for instance, [3, Theorem 10.2.5] or [18, Theorem 1]).

Theorem 2.3. *Let $f : [a,b] \to [0,\infty)$ be a nondecreasing function, and let $g \in L^1[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in [a,x]$ such that*

$$\int_a^x f(t)g(t)dt = f(x-0)\int_{\xi_x}^x g(t)dt.$$

The fractional version of Theorem 2.3 can be formulated as follows.

Theorem 2.4. *Let $\alpha > 0$, let $f : [a,b] \to [0,\infty)$ be a nondecreasing function, and let $g \in L^1[a,b]$. Then for almost every $x \in (a,b]$ there exists some $\xi_x \in [a,x]$ such that*

$$J_a^\alpha(fg)(x) = f(x-0)J_{\xi_x}^\alpha g(x). \tag{2.3}$$

Moreover, if $\alpha \geq 1$ or $g \in C[a,b]$, then the existence of ξ_x is assured for every $x \in (a,b]$.

Proof. Let $x \in (a,b]$. Under the assumptions of the theorem we have

$$J_a^\alpha(fg)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)g(t)dt = \int_a^x f(t)h(t)dt,$$

where $h : (a,x) \to \mathbb{R}$ is the function defined by $h(t) := (x-t)^{\alpha-1}g(t)/\Gamma(\alpha)$.

If $\alpha \geq 1$ or $g \in C[a,b]$, then $h \in L^1[a,x]$. By Theorem 2.3 it follows that there exists $\xi_x \in [a,x]$ such that

$$J_a^\alpha(fg)(x) = \int_a^x f(t)h(t)dt = f(x-0)\int_{\xi_x}^x h(t)dt = f(x-0)J_{\xi_x}^\alpha g(x).$$

If $0 < \alpha < 1$ and g is supposed only Lebesgue integrable, then the above argument still works, but the integrability of h holds only for almost all $x \in (a,b]$ (see [17, Theorem 4.2 (d)]). $\square$

3. Asymptotic behavior of intermediate points in fractional mean value theorems of integral calculus

The purpose of this section is to investigate the asymptotic behavior of the point ξ_x in (2.1) and (2.2) as the interval $[a, x]$ shrinks to zero. More precisely, we prove that under certain additional assumptions on f and g, the limit $\lim_{x\to a+} \frac{\xi_x - a}{x - a}$ exists and we find its value. In the proof of the main result of this section we need the following

Lemma 3.1. *Let $\alpha > 0$, let p be a nonnegative integer, and let $\omega : [a, b] \to \mathbb{R}$ be a continuous function such that $\omega(x) \to 0$ as $x \to a+$. Then*

$$\int_a^x (x-t)^{\alpha-1}(t-a)^p \omega(t)dt = o\big((x-a)^{p+\alpha}\big) \quad (x \to a+).$$

Proof. Indeed, for every $x \in (a, b]$ we have

$$\int_a^x (x-t)^{\alpha-1}(t-a)^p \omega(t)dt = \Gamma(\alpha) J_a^\alpha(\omega g)(x),$$

where $g : [a, b] \to [0, \infty)$ is defined by $g(t) := (t-a)^p$. According to Theorem 2.1 there exists $\xi_x \in (a, x)$ such that

$$\begin{aligned} J_a^\alpha(\omega g)(x) &= \omega(\xi_x) J_a^\alpha g(x) = \frac{\omega(\xi_x)}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}(t-a)^p dt \\ &= \omega(\xi_x) \frac{B(p+1, \alpha)}{\Gamma(\alpha)} (x-a)^{p+\alpha}, \end{aligned}$$

whence

$$\int_a^x (x-t)^{\alpha-1}(t-a)^p \omega(t)dt = \omega(\xi_x) B(p+1, \alpha)(x-a)^{p+\alpha}.$$

Since $\omega(x) \to 0$ as $x \to a+$, we obtain the conclusion. $\square$

Theorem 3.2. *Let α be a positive real number and let $f, g : [a, b] \to \mathbb{R}$ be functions satisfying the following conditions:*

(i) *$f \in C[a, b]$ and there is a positive integer n such that f is n times differentiable at a with $f^{(j)}(a) = 0$ for $1 \le j \le n-1$ and $f^{(n)}(a) \ne 0$;*

(ii) *$g \in C[a, b]$, g does not change its sign in some interval $[a, a+h] \subseteq [a, b]$, and there is a nonnegative integer k such that g is k times differentiable at a with $g^{(j)}(a) = 0$ for $0 \le j \le k-1$ and $g^{(k)}(a) \ne 0$.*

Then the point ξ_x in (2.1) satisfies

$$\lim_{x\to a+} \frac{\xi_x - a}{x - a} = \sqrt[n]{\frac{(k+1)(k+2)\cdots(k+n)}{(\alpha+k+1)(\alpha+k+2)\cdots(\alpha+k+n)}}.$$

Proof. Without loosing the generality we may assume that $f(a) = 0$. Indeed, otherwise we replace f by the function $t \in [a, b] \mapsto f(t) - f(a)$. Note that if ξ_x satisfies (2.1), then ξ_x satisfies also

$$J_a^\alpha\Big(\big(f - f(a)\big)g\Big)(x) = \big(f(\xi_x) - f(a)\big) J_a^\alpha g(x).$$

We notice also that (2.1) is equivalent to

$$\int_a^x (x-t)^{\alpha-1} f(t)g(t)dt = f(\xi_x)\int_a^x (x-t)^{\alpha-1} g(t)dt. \tag{3.1}$$

By the Taylor expansions of f and g we have

$$\begin{aligned} f(t) &= \frac{f^{(n)}(a)}{n!}(t-a)^n + \omega(t)(t-a)^n, \\ g(t) &= \frac{g^{(k)}(a)}{k!}(t-a)^k + \varepsilon(t)(t-a)^k, \end{aligned}$$

where ω and ε are continuous functions on $[a,b]$ satisfying $\omega(t)\to 0$ and $\varepsilon(t)\to 0$ as $t\to a+$. Therefore we have

$$\begin{aligned} (x-t)^{\alpha-1} f(t)g(t) &= \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}(x-t)^{\alpha-1}(t-a)^{n+k} \\ &\quad +(x-t)^{\alpha-1}(t-a)^{n+k}\gamma(t), \end{aligned}$$

where γ is continuous on $[a,b]$ and $\gamma(t)\to 0$ as $t\to a+$. By applying Lemma 3.1 we deduce that

$$\begin{aligned} &\int_a^x (x-t)^{\alpha-1} f(t)g(t)dt \\ &\quad = \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!} B(\alpha, n+k+1)\,(x-a)^{n+k+\alpha} + o\big((x-a)^{n+k+\alpha}\big) \end{aligned} \tag{3.2}$$

as $x\to a+$. On the other hand, since

$$(x-t)^{\alpha-1} g(t) = \frac{g^{(k)}(a)}{k!}(x-t)^{\alpha-1}(t-a)^k + (x-t)^{\alpha-1}(t-a)^k\varepsilon(t),$$

by Lemma 3.1 we get

$$\int_a^x (x-t)^{\alpha-1} g(t)dt = \frac{g^{(k)}(a)}{k!} B(\alpha, k+1)(x-a)^{k+\alpha} + o\big((x-a)^{k+\alpha}\big)$$

as $x\to a+$. Taking into account that

$$f(\xi_x) = \frac{f^{(n)}(a)}{n!}(\xi_x-a)^n + \omega(\xi_x)(\xi_x-a)^n$$

and that $0<\xi_x-a<x-a$, we obtain

$$\begin{aligned} &f(\xi_x)\int_a^x (x-t)^{\alpha-1} g(t)dt \\ &\quad = \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!} B(\alpha, k+1)(\xi_x-a)^n(x-a)^{k+\alpha} \\ &\qquad +o\big((x-a)^{n+k+\alpha}\big) \end{aligned} \tag{3.3}$$

as $x \to a+$. By (3.1), (3.2) and (3.3) we conclude that

$$\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}\,B(\alpha,k+1)(\xi_x-a)^n(x-a)^{k+\alpha}$$
$$=\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}\,B(\alpha,n+k+1)\,(x-a)^{n+k+\alpha}+o\big((x-a)^{n+k+\alpha}\big)$$

as $x \to a+$. Multiplying both sides by $n!\,k!(x-a)^{-n-k-\alpha}/\big(f^{(n)}(a)g^{(k)}(a)\big)$ we get

$$\begin{aligned}\left(\frac{\xi_x-a}{x-a}\right)^n &= \frac{B(\alpha,n+k+1)}{B(\alpha,k+1)}+o(1)\\ &= \frac{(k+1)(k+2)\cdots(k+n)}{(\alpha+k+1)(\alpha+k+2)\cdots(\alpha+k+n)}+o(1)\end{aligned}$$

as $x \to a+$, whence the conclusion. □

Corollary 3.3. *Let $\alpha > 0$, and let $f : [a,b] \to \mathbb{R}$ be a function satisfying the condition* (i) *in Theorem* 3.2. *Then the point ξ_x in* (2.2) *satisfies*

$$\lim_{x\to a+}\frac{\xi_x-a}{x-a}=\sqrt[n]{\frac{n!}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}}\,.$$

In the special case when $\alpha = 1$, then Theorem 3.2 and Corollary 3.3 coincide with earlier results obtained by T. Trif [15, Theorem 2.2] and B. Zhang [20, Theorem 4], respectively.

Unfortunately, we were not able to prove a result similar to those stated in Theorem 3.2 and Corollary 3.3, but concerning the asymptotic behavior of the point ξ_x in formula (2.3).

4. Fractional mean value theorems of differential calculus

K. Diethelm [6] and P. Guo, C. P. Li, and G. R. Chen [10] extended recently also the classical Lagrange and Lagrange-Taylor mean value theorems to the framework of fractional calculus. Let $\alpha > 0$, and let $f : [a,b] \to \mathbb{R}$ be a given function. The Caputo fractional derivative of order α of f is defined by

$$D^\alpha_{*a}f := D^\alpha_a\Big(f - T_{\lceil\alpha\rceil-1}(f;a)\Big),$$

where $\lceil\cdot\rceil$ denotes the ceiling function that rounds up to the nearest integer, while D^α_a is the Riemann-Liouville differential operator, defined by

$$D^\alpha_a f := D^{\lceil\alpha\rceil}J_a^{\lceil\alpha\rceil-\alpha}f.$$

In the above formula D^m denotes the classical differential operator of order m, and $J^0_a f := f$.

Theorem 4.1. ([6, Theorem 2.3], [10, Theorem 3]) *Let $\alpha > 0$, and let $f \in C^{\lceil\alpha\rceil-1}[a,b]$ be a function such that $D^\alpha_{*a} f \in C[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$\frac{f(x) - T_{\lceil\alpha\rceil-1}(f;a)(x)}{(x-a)^\alpha} = \frac{1}{\Gamma(\alpha+1)}\, D^\alpha_{*a} f(\xi_x). \tag{4.1}$$

In the special case when $0 < \alpha \le 1$, then $T_{\lceil\alpha\rceil-1}(f;a)(x) = f(a)$, and Theorem 4.1 takes the following form.

Corollary 4.2. ([6, Corollary 2.4]) *Let $0 < \alpha \le 1$, and let $f \in C[a,b]$ be a function such that $D^\alpha_{*a} f \in C[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$\frac{f(x) - f(a)}{(x-a)^\alpha} = \frac{1}{\Gamma(\alpha+1)}\, D^\alpha_{*a} f(\xi_x). \tag{4.2}$$

Theorem 4.3. ([10, Theorem 4]) *Let $\alpha > 0$, and let $f, g \in C^{\lceil\alpha\rceil-1}[a,b]$ be functions such that $D^\alpha_{*a} f, D^\alpha_{*a} g \in C[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$D^\alpha_{*a} f(\xi_x)\Big(g(x) - T_{\lceil\alpha\rceil-1}(g;a)(x)\Big) = D^\alpha_{*a} g(\xi_x)\Big(f(x) - T_{\lceil\alpha\rceil-1}(f;a)(x)\Big). \tag{4.3}$$

Corollary 4.4. ([10, Corollary 3.6]) *Let $0 < \alpha \le 1$, and let $f, g \in C[a,b]$ be functions such that $D^\alpha_{*a} f, D^\alpha_{*a} g \in C[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$D^\alpha_{*a} f(\xi_x)\big(g(x) - g(a)\big) = D^\alpha_{*a} g(\xi_x)\big(f(x) - f(a)\big). \tag{4.4}$$

Remark 4.5. Let $\alpha > 0$, and let $f \in C^{\lceil\alpha\rceil-1}[a,b]$ such that $D^\alpha_{*a} f \in C[a,b]$. Further, let $g : [a,b] \to \mathbb{R}$ be the function defined by $g(t) := (t-a)^\alpha$. If $n := \lceil\alpha\rceil$, then $n-1 < \alpha \le n$, and $T_{\lceil\alpha\rceil-1}(g;a)(x) = T_{n-1}(g;a)(x) = 0$. On the other hand, for every $y \in (a,b)$ one has

$$\begin{aligned} D^\alpha_{*a} g(y) &= \frac{d^n}{dy^n}\, J^{n-\alpha}_a \Big(g - T_{\lceil\alpha\rceil-1}(g;a)\Big)(y) = \frac{d^n}{dy^n}\, J^{n-\alpha}_a g(y) \\ &= \Gamma(\alpha+1). \end{aligned}$$

Therefore, in this case (4.3) reduces to (4.1), while (4.4) reduces to (4.2). In other words, Theorem 4.3 coincides with Theorem 4.1, while Corollary 4.4 coincides with Corollary 4.2 in the special case when $g(t) := (t-a)^\alpha$.

It should be mentioned that similar fractional mean value theorems, but involving the Riemann-Liouville fractional derivative instead of the Caputo fractional derivative have been obtained by other authors (see, for instance, [10], [13], [16]).

5. Asymptotic behavior of intermediate points in fractional mean value theorems of differential calculus

Theorem 5.1. *Let $\alpha > 0$ be a non-integer number, let $n := \lceil\alpha\rceil$, and let $f : [a,b] \to \mathbb{R}$ be a function satisfying the following conditions:*

(i) *there exists a nonnegative integer p such that $f \in C^{n+p}[a,b]$;*

(ii) $f^{(n+j)}(a) = 0$ *for* $0 \leq j < p$;
(iii) $f^{(n+p)}(a) \neq 0$.

Then the point ξ_x *in* (4.1) *satisfies*

$$\lim_{x \to a+} \frac{\xi_x - a}{x - a} = \big((n + p - \alpha)B(\alpha + 1, n + p - \alpha)\big)^{\frac{1}{n+p-\alpha}}.$$

Proof. We note first that since $f \in C^{n+p}[a, b]$, the derivative $f^{(n-1)}$ must be absolutely continuous on $[a, b]$. By [5, Theorem 3.1] it follows that

$$D^{\alpha}_{*a} f = J^{n-\alpha}_{a} D^{n} f. \tag{5.1}$$

Due to (ii), by the Taylor expansion of f we have

$$f(x) - T_{n-1}(f; a)(x) = \frac{f^{(n+p)}(a)}{(n + p)!} (x - a)^{n+p} + \omega(x)(x - a)^{n+p}, \tag{5.2}$$

where ω is continuous on $[a, b]$ and satisfies $\omega(x) \to 0$ as $x \to a+$. On the other hand, by the Taylor expansion of $f^{(n)}$ we have

$$f^{(n)}(t) = \frac{f^{(n+p)}(a)}{p!} (t - a)^{p} + \varepsilon(t)(t - a)^{p}, \tag{5.3}$$

where ε is continuous on $[a, b]$ and satisfies $\varepsilon(t) \to 0$ as $t \to a+$.

Taking into account (5.1), equality (4.1) can be rewritten as

$$f(x) - T_{n-1}(f; a)(x) = \frac{(x - a)^{\alpha}}{\Gamma(\alpha + 1)\Gamma(n - \alpha)} \int_{a}^{\xi_x} (\xi_x - t)^{n-\alpha-1} f^{(n)}(t)dt. \tag{5.4}$$

By (5.3) and Lemma 3.1 we find that

$$\begin{aligned}
&\int_{a}^{\xi_x} (\xi_x - t)^{n-\alpha-1} f^{(n)}(t)dt \\
&\quad = \frac{f^{(n+p)}(a)}{p!} \int_{a}^{\xi_x} (\xi_x - t)^{n-\alpha-1} (t - a)^{p} dt \\
&\qquad + \int_{a}^{\xi_x} (\xi_x - t)^{n-\alpha-1} (t - a)^{p} \varepsilon(t)dt \\
&\quad = \frac{\Gamma(n - \alpha)}{\Gamma(n + p + 1 - \alpha)} f^{(n+p)}(a)(\xi_x - a)^{n+p-\alpha} + o\big((\xi_x - a)^{n+p-\alpha}\big),
\end{aligned}$$

whence

$$\begin{aligned}
&\frac{(x - a)^{\alpha}}{\Gamma(\alpha + 1)\Gamma(n - \alpha)} \int_{a}^{\xi_x} (\xi_x - t)^{n-\alpha-1} f^{(n)}(t)dt \qquad (5.5) \\
&\quad = \frac{f^{(n+p)}(a)(x - a)^{\alpha}(\xi_x - a)^{n+p-\alpha}}{\Gamma(\alpha + 1)\Gamma(n + p + 1 - \alpha)} + o\big((x - a)^{n+p}\big)
\end{aligned}$$

as $x \to a+$, because $0 < \xi_x - a < x - a$. Taking now into account (5.2), (5.4), and (5.5), we find that

$$\frac{f^{(n+p)}(a)(x-a)^{n+p}}{\Gamma(n+p+1)} = \frac{f^{(n+p)}(a)(x-a)^{\alpha}(\xi_x - a)^{n+p-\alpha}}{\Gamma(\alpha+1)\Gamma(n+p+1-\alpha)} + o\big((x-a)^{n+p}\big)$$

as $x \to a+$. Multiplying both sides by

$$\Gamma(\alpha+1)\Gamma(n+p+1-\alpha)(x-a)^{-n-p}/f^{(n+p)}(a)$$

we get

$$\begin{aligned}\left(\frac{\xi_x - a}{x-a}\right)^{n+p-\alpha} &= \frac{\Gamma(\alpha+1)\Gamma(n+p+1-\alpha)}{\Gamma(n+p+1)} + o(1) \\ &= (n+p-\alpha)B(\alpha+1, n+p-\alpha) + o(1) \qquad (x \to a+),\end{aligned}$$

whence the conclusion. □

Corollary 5.2. *Let $0 < \alpha < 1$, and let $f : [a,b] \to \mathbb{R}$ be a function satisfying the following conditions:*

(i) *there exists a positive integer p such that $f \in C^p[a,b]$;*
(ii) $f^{(j)}(a) = 0$ *for* $1 \leq j < p$ *;*
(iii) $f^{(p)}(a) \neq 0$.

Then the point ξ_x in (4.2) *satisfies*

$$\lim_{x \to a+} \frac{\xi_x - a}{x - a} = \big((p-\alpha)B(\alpha+1, p-\alpha)\big)^{\frac{1}{p-\alpha}}.$$

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