Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 2, 247–262

Iterative regularization methods for ill-posed Hammerstein-type operator equations in Hilbert scales

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Abstract. In this paper we report on a method for regularizing a nonlinear Hammerstein type operator equation in Hilbert scales. The proposed method is a combination of Lavrentieve regularization method and a Modified Newton's method in Hilbert scales . Under the assumptions that the operator F is continuously differentiable with a Lipschitz-continuous first derivative and that the solution of (1.1) fulfills a general source condition, we give an optimal order convergence rate result with respect to the general source function.

Mathematics Subject Classification (2010): 65J20, 65J10, 65R30, 47A52.

Keywords: Nonlinear ill-posed Hammerstein type equations, iterative regularization, adaptive choice, Hilbert scales.

1. Introduction

Let X and Y be Hilbert spaces. In this study we are concerned with the problem of approximately solving the operator equation

$$KF(x) = y, \tag{1.1}$$

where $K : X \to Y$ is a bounded linear operator with its range $R(K)$ not closed in Y and $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator (i.e., $\langle F(u)-F(v), u-v\rangle \geq 0,\ \forall u, v \in D$). We shall use the notations $\langle ., .\rangle_X,\ \langle ., .\rangle_Y$ and $\|.\|_X,\ \|.\|_Y$ for the inner product and the corresponding norm in the Hilbert spaces X, Y, respectively. The equation (1.1) is, in general, ill-posed, in the sense that a unique solution that depends continuously on the data does not exist.

A typical example of a Hammerstein type operator is the nonlinear integral operator

$$(KF(x))(t) := \int_0^1 k(s,t)f(s,x(s))ds$$

where $k(s,t) \in L^2([0,1] \times [0,1])$, $x \in L^2[0,1]$ and $t \in [0,1]$. Here $K : L^2[0,1] \to L^2[0,1]$ is a linear integral operator with kernel $k(t,s)$: defined as

$$Kx(t) = \int_0^1 k(t,s)x(s)ds$$

and $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$ is a nonlinear superposition operator (cf. [16]) defined as

$$Fx(s) = f(s, x(s)). \tag{1.2}$$

In [14], George and Nair studied a Modified NLR method for obtaining an approximation for the x_0-minimum norm solution (x_0-MNS) of the equation (1.1). Recall that a solution $\hat{x} \in D(F)$ of (1.1) is called an x_0-MNS of (1.1), if

$$\|F(\hat{x}) - F(x_0)\|_X = \min\{\|F(x) - F(x_0)\|_X : AF(x) = y,\, x \in D(F)\}. \tag{1.3}$$

In the following, we always assume the existence of an x_0-MNS for exact data y, i.e.,

$$KF(\hat{x}) = y.$$

Note that, due to the nonlinearity of F, the above solution need not be unique. The element $x_0 \in X$ in (1.3) plays the role of a selection criterion.

Further we assume throughout that X is a real Hilbert space, $y^\delta \in Y$ are the available noisy data with

$$\|y - y^\delta\|_Y \leq \delta \tag{1.4}$$

and $\|F'(x)\|_{X\to X} \leq M$ for all $x \in D$.

Since (1.1) is ill-posed, regularization methods are to be employed for obtaining a stable approximate solution for (1.1). See, for example [18], [24], [7], [9], [10] for various regularization methods for ill-posed operator equations.

In [6], we considered the sequence $\{x^\delta_{n,\alpha_k}\}$ defined iteratively by

$$x^\delta_{n+1,\alpha_k} = x^\delta_{n,\alpha_k} - R_\beta(x_0)^{-1}[F(x^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + \alpha_k(x^\delta_{n,\alpha_k} - x_0)] \tag{1.5}$$

where $x^\delta_{0,\alpha_k} := x_0$ is an initial guess and $R_\beta(x_0) := F'(x_0) + \beta I$, with $\beta > \alpha_k$ for obtaining an approximation of $\hat{x}$. Here $z^\delta_{\alpha_k} = (K^*K+\alpha_k I)^{-1}K^*(y^\delta - KF(x_0)) + F(x_0)$ and α_k is the regularization parameter chosen appropriately depending on the inexact data y^δ and the error level δ satisfying (1.4). For this we used the adaptive parameter selection procedure suggested by Pereverzev and Schock [20]. In order to improve the error estimate available in [14], in this paper we consider the Hilbert scale variant of (1.5).

Let $L : D(L) \subset X \to X$, be a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. We consider the Hilbert scale $(X_r)_{r\in\Re}$ (see [12], [13], [17] and [18]) generated by L for our analysis. Recall (c.f.[12])that the space X_t is the completion of $D := \cap_{k=0}^\infty D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u, v\rangle_t := \langle L^t u, L^t v\rangle, \qquad u, v \in D. \tag{1.6}$$

Moreover, if $\beta \leq \gamma$, then the embedding $X_\gamma \hookrightarrow X_\beta$ is continuous, and therefore the norm $\|.\|_\beta$ is also defined in X_γ and there is a constant $c_{\beta,\gamma}$ such that

$$\|x\|_\beta \leq c_{\beta,\gamma}\|x\|_\gamma,\ x \in X_\gamma.$$

In this paper we consider the sequence $\{x^\delta_{n,\alpha_k}\}$ in order to obtain stable approximate solution to (1.1), defined iteratively by

$$x^\delta_{n+1,\alpha_k,s} = x^\delta_{n,\alpha_k,s} - R_\beta(x_0)^{-1}[F(x^\delta_{n,\alpha_k,s}) - z^\delta_{\alpha_k,s} + \alpha_k L^{s/2}(x^\delta_{n,\alpha_k,s} - x_0)], \quad (1.7)$$

where $x^\delta_{0,\alpha_k,s} := x_0$ is an initial guess and $R_\beta(x_0) := F'(x_0) + \beta L^{s/2}$, with $\beta > \alpha_k$ for obtaining an approximation for $\hat{x}$. Here $z^\delta_{\alpha_k,s}$ be as in (2.2) with $\alpha = \alpha_k$ and α_k is the regularization parameter chosen appropriately depending on the inexact data y^δ and the error level δ satisfying (1.4). For this we use the adaptive parameter selection procedure suggested by Pereverzev and Schock [20].

This paper is organized as follows. Preparatory results are given in section 2 and section 3 comprises the proposed iterative method. Numerical examples are given in section 4. Finally the paper ends with a conclusion in section 5.

2. Preliminaries

We assume that the ill-posed nature of the operator K is related to the Hilbert scale $\{X_t\}_{t\in\mathbb{R}}$ according to the relation

$$c_1\|x\|_{-a} \le \|Kx\|_Y \le c_2\|x\|_{-a}, \qquad x \in X,$$

for some real numbers a, c_1, and c_2.

Observe that from the relation $\langle Kx, y\rangle_Y = \langle x, K^*y\rangle_X = \langle x, L^{-s}K^*y\rangle_s$ for all $x \in X$ and $y \in Y$, we conclude that $L^{-s}K^* : Y \to X$ is the adjoint of the operator K in X. Consequently $L^{-s}K^*K : X \to X$ is self-adjoint. Further we note that

$$(A_s^*A_s + \alpha I)^{-1}L^{s/2} = L^{s/2}(L^{-s}K^*K + \alpha I)^{-1}$$

where $A_s = KL^{-s/2}$.

One of the crucial results for proving the results in this paper is the following proposition, where f and g are defined by

$$f(t) = \min\{c_1^t, c_2^t\}, \qquad g(t) = max\{c_1^t, c_2^t\}, \qquad t \in \mathbb{R}, |t| \le 1.$$

Proposition 2.1. *(See* [23]*, Proposition 2.1) For $s \ge 0$ and $|\nu| \le 1$,*

$$f(\nu)\|x\|_{-\nu(s+a)} \le \|(A_s^*A_s)^{\nu/2}x\|_X \le g(\nu)\|x\|_{-\nu(s+a)}, \quad x \in H.$$

We make use of the relation

$$\|(A_s + \alpha I)^{-1}A_s^p\|_X \le \alpha^{p-1}, \;\; p > 0, \;\; 0 < p \le 1, \quad (2.1)$$

which follows from the spectral properties of the positive self-adjoint operator A_s, $s > 0$.

In this section we consider Tikhonov regularized solution $z^\delta_{\alpha,s}$ defined by

$$z^\delta_{\alpha,s} = (L^{-s}K^*K + \alpha I)^{-1}L^{-s}K^*(y^\delta - KF(x_0)) + F(x_0) \quad (2.2)$$

and obtain an a priori and an a posteriori error estimate for $\|F(\hat{x}) - z^\delta_{\alpha,s}\|_X$. The following assumption on source condition is based on a source function φ and a property of the source function φ. We will be using this assumption to obtain an error estimate for $\|F(\hat{x}) - z^\delta_{\alpha,s}\|_X$.

Assumption 2.2. *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, \|A_s^* A_s\|] \to (0, \infty)$ *such that the following conditions hold:*

- $\lim\limits_{\lambda \to 0} \varphi(\lambda) = 0,$
- $\sup\limits_{\lambda > 0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq \varphi(\alpha), \ \forall \lambda \in (0, \|A_s^* A_s\|]$ *and*
- *there exists* $v \in X$ *with* $\|v\| \leq \overline{E}, \ \overline{E} > 0$ *such that*

$$(A_s^* A_s)^{\frac{s}{2(s+a)}} L^{s/2}(F(\hat{x}) - F(x_0)) = \varphi(A_s^* A_s)v.$$

Remark 2.3. Note that if $F(\hat{x}) - F(x_0) \in X_t$ i.e., $\|F(\hat{x}) - F(x_0)\|_t \leq E$, for some $0 < t \leq 2s + a$, then the above assumption is satisfied. This can be seen as follows.

$$\begin{aligned}(A_s^* A_s)^{\frac{s}{2(s+a)}} L^{s/2}(F(\hat{x}) - F(x_0)) &= (A_s^* A_s)^{\frac{t}{2(s+a)}} (A_s^* A_s)^{\frac{(s-t)}{(2s+2a)}} L^{s/2}(F(\hat{x}) - F(x_0)),\\ &= \varphi(A_s^* A_s)v\end{aligned}$$

where $\varphi(\lambda) = \lambda^{\frac{t}{2(s+a)}}$ and $v = (A_s^* A_s)^{\frac{(s-t)}{(2s+2a)}} L^{s/2}(F(\hat{x}) - F(x_0))$.

Further note that

$$\begin{aligned}\|v\|_X &\leq g(\frac{s-t}{s+a})\|L^{s/2}(F(\hat{x}) - F(x_0))\|_{t-s}\\ &\leq g(\frac{s-t}{s+a})\|(F(\hat{x}) - F(x_0))\|_t\\ &\leq \overline{E}\end{aligned}$$

where $\overline{E} = g(\frac{s-t}{s+a})E$.

Theorem 2.4. ([22, Theorem 2.4]) *Suppose that Assumption* 2.2 *holds and let* $z_{\alpha,s} := z_{\alpha,s}^0$. *Then*

1.
$$\|z_{\alpha,s}^\delta - z_{\alpha,s}\|_X \leq \psi(s)\alpha^{\frac{-a}{2(s+a)}}\delta, \tag{2.3}$$
2.
$$\|F(\hat{x}) - z_{\alpha,s}\|_X \leq \phi(s)\varphi(\alpha), \tag{2.4}$$
3.
$$\|F(x_0) - z_{\alpha,s}\|_X \leq \psi_1(s)\|F(\hat{x}) - F(x_0)\|_X, \tag{2.5}$$

where $\psi(s) = \frac{1}{f(\frac{s}{s+a})}, \quad \phi(s) = \frac{\overline{E}}{f(\frac{s}{s+a})}$ *and* $\psi_1(s) = \frac{g(\frac{s}{s+a})}{f(\frac{s}{s+a})}$.

2.1. Error bounds and parameter choice in Hilbert scales

Let $C_s = \max\{\phi(s), \psi(s)\}$, then by (2.3), (2.4) and triangle inequality, we have

$$\|F(\hat{x}) - z_{\alpha,s}^\delta\|_X \quad \leq \quad C_s(\varphi(\alpha) + \alpha^{\frac{-a}{2(s+a)}}\delta). \tag{2.6}$$

The error estimate $\varphi(\alpha) + \alpha^{\frac{-a}{2(s+a)}}\delta$ in (2.6) attains minimum for the choice $\alpha := \alpha(\delta, s, a)$ which satisfies $\varphi(\alpha) = \alpha^{\frac{-a}{2(s+a)}}\delta$. Clearly $\alpha(\delta, s, a) = \varphi^{-1}(\psi_{s,a}^{-1}(\delta))$, where

$$\psi_{s,a}(\lambda) = \lambda[\varphi^{-1}(\lambda)]^{\frac{a}{2(s+a)}}, \qquad 0 < \lambda \leq \|A_s\|^2 \tag{2.7}$$

and in this case

$$\|F(\hat{x}) - z_{\alpha,s}^\delta\|_X \leq 2C_s\psi_{s,a}^{-1}(\delta),$$

which has at least optimal order with respect to δ, s and a (cf. [20]).

2.2. Adaptive scheme and stopping rule

In this paper we consider the adaptive scheme suggested by Pereverzev and Schock in [20] modified suitably, for choosing the parameter α which does not involve even the regularization method in an explicit manner.
Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu = \eta^{2(1+s/a)}, \eta > 1$ and $\alpha_0 = \delta^{2(1+s/a)}$.
Let

$$l := max\{i : \varphi(\alpha_i) \leq \alpha_i^{\frac{-a}{2(s+a)}} \delta\} < N \tag{2.8}$$

and

$$k := max\{i : \|z^\delta_{\alpha_{i,s}} - z^\delta_{\alpha_{j,s}}\|_X \leq 4\alpha_j^{\frac{-a}{2(s+a)}} \delta, j = 0, 1, 2, \cdots, i\}. \tag{2.9}$$

Analogous to the proof of Theorem 4.3 in [11], we have the following Theorem.

Theorem 2.5. ([22, Theorem 2.5]) *Let l be as in (2.8), k be as in (2.9), $\psi_{s,a}$ be as in (2.7) and $z^\delta_{\alpha_k,s}$ be as in (2.2) with $\alpha = \alpha_k$. Then $l \leq k$; and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k,s}\|_X \leq C_s(2 + \frac{4\eta}{\eta - 1})\eta\psi^{-1}_{s,a}(\delta)$$

where C_s is as in (2.6).

3. The method and convergence analysis

In the earlier papers [11, 15] the authors used the following Assumption:

Assumption 3.1. (cf. [21], Assumption 3 (A3)) *There exists a constant $K \geq 0$ such that for every $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\|_X \leq K\|v\|_X\|x - u\|_X$.*

Assumption 3.2. *For each $x \in B_r(x_0)$ there exists a bounded linear operator G such that*

$$F'(x) = F'(x_0)G(x, x_0)$$

with $\|G(x, x_0)\| \leq k$ where k is a constant.

One of the advantages of the proposed method is that we do not need the above assumption.

The hypotheses of Assumption 3.1 may not hold or may be very expensive or impossible to verify in general (see the numerical examples). In particular, as it is the case for well-posed nonlinear equations the computation of the Lipschitz constant K even if this constant exists is very difficult. Moreover, there are classes of operators for which Assumption 3.1 is not satisfied but the iterative method converges.

In the present paper, we expand the applicability of the method in [6] under less computational cost. We achieve this goal by introducing the following weaker Assumption.

Assumption 3.3. *There exists a constant $k_0 \geq 0$ such that for every $x \in D(F)$ and $v \in X$ there exists an element $\Phi(x, x_0, v) \in X$ such that*

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v), \|\Phi(x, x_0, v)\|_X \leq k_0\|v\|_X\|x - x_0\|_X.$$

Note that

$$k_0 \le K$$

holds in general and $\frac{K}{k_0}$ can be arbitrary large (see Example 4.3). The advantages of the new approach are:

(1) Assumption 3.3 is weaker than Assumption 3.1. Notice that there are classes of operators that satisfy Assumption 3.3 but do not satisfy Assumption 3.1 (see the numerical examples);
(2) The computational cost of finding the constant k_0 is less than that of constant K, even when $K = k_0$;
(3) The sufficient convergence criteria are weaker;
(4) The computable error bounds on the distances involved (including k_0) are less costly and more precise than the old ones (including K);
(5) The information on the location of the solution is more precise;

and

(6) The convergence domain of the iterative method is larger.

These advantages are also very important in computational mathematics since they provide under less computational cost a wider choice of initial guesses for iterative method and the computation of fewer iterates to achieve a desired error tolerance. Numerical examples for (1)-(6) are presented in Section 4.

In this section, we consider the method defined as (1.7) with α_k in place of α for approximating the zero $x^\delta_{\alpha_k,s}$ of the equation,

$$F(x) + \alpha_k L^{s/2}(x - x_0) = z^\delta_{\alpha_k,s} \tag{3.1}$$

and then we show that $x^\delta_{\alpha_k,s}$ is an approximation to the solution $\hat{x}$ of (1.1).

Let $F'(x_0) \in L(X)$ be a bounded positive self-adjoint operator on X and $B_s := L^{-s/4}F'(x_0)L^{-s/4}$. Usually, for the analysis of regularization methods in Hilbert scales, an assumption of the form (cf.[8], [19])

$$\|F'(\hat{x})x\|_X \sim \|x\|_{-b}, \ \ x \in X \tag{3.2}$$

on the degree of ill-posedness is used. In this paper instead of (3.2) we require only a weaker assumption;

$$d_1\|x\|_{-b} \le \|F'(x_0)x\|_X \le d_2\|x\|_{-b}, \qquad x \in D(F), \tag{3.3}$$

for some reals b, d_1, and d_2.

Note that (3.3) is simpler than that of (3.2). Next, we define f_1 and g_1 by

$$f_1(t) = \min\{d_1^t, d_2^t\}, \qquad g_1(t) = max\{d_1^t, d_2^t\}, \qquad t \in \mathbb{R}, \ |t| \le 1.$$

One of the crucial result for proving the results in this paper is the following Proposition.

Proposition 3.4. *(See.* [12], *Proposition 3.1) For $s > 0$ and $|\nu| \le 1$,*

$$f_1(\nu/2)\|x\|_{\frac{-\nu(s+b)}{2}} \le \|B_s^{\nu/2}x\|_X \le g_1(\nu/2)\|x\|_{\frac{-\nu(s+b)}{2}}, \ x \in H.$$

Let $\psi_2(s) := \frac{g_1(\frac{-s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}$, $\overline{\psi_2(s)} := \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}$.

Lemma 3.5. *Let Proposition* 3.4 *hold. Then for all* $h \in X$*, the following hold:*

(a) $\|(F'(x_0)+\beta L^{s/2})^{-1}F'(x_0)h\|_X \leq \overline{\psi_2(s)}\|h\|_X$

(b) $\|(F'(x_0)+\beta L^{s/2})^{-1}L^{s/2}h\|_X \leq \frac{\overline{\psi_2(s)}}{\beta}\|h\|_X$

(c) $\|(F'(x_0)+\beta L^{s/2})^{-1}h\|_X \leq \psi_2(s)\beta^{\frac{-b}{(s+b)}}\|h\|_X$

Proof. Observe that by Proposition 3.4,

$$\begin{aligned}
\|(F'(x_0)+\beta L^{s/2})^{-1}F'(x_0)h\|_X &= \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4}+\beta I)^{-1}L^{-s/4}\\
&\quad F'(x_0)L^{-s/4}L^{s/4}h\|_X\\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s+\beta I)^{-1}B_sL^{s/4}h\|_X\\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|(B_s+\beta I)^{-1}B_s\|\|B_s^{\frac{s}{2(s+b)}}L^{s/4}h\|_X\\
&\leq \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}\|L^{s/4}h\|_{-s/2}\\
&\leq \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}\|h\|_X.
\end{aligned}$$

This proves (a). To prove (b) and (c) we observe that

$$\begin{aligned}
\|(F'(x_0)+\beta L^{s/2})^{-1}L^{s/2}h\|_X &\leq \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4}+\beta I)^{-1}L^{s/4}h\|_X\\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s+\beta I)^{-1}L^{s/4}h\|_X\\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|(B_s+\beta I)^{-1}B_s^{\frac{s}{2(s+b)}}L^{s/4}h\|_X\\
&\leq \frac{g_1(\frac{s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}\beta^{-1}\|h\|_X\\
&\leq \overline{\psi_2(s)}\beta^{-1}\|h\|_X \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
\|(F'(x_0)+\beta L^{s/2})^{-1}h\|_X &\leq \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4}+\beta I)^{-1}L^{-s/4}h\|_X\\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s+\frac{\alpha_k}{c}I)^{-1}L^{-s/4}h\|_X\\
&\leq \frac{1}{f_1(\frac{s}{2(s+b)})}\|(B_s+\beta I)^{-1}B_s^{\frac{s}{(s+b)}}B_s^{\frac{-s}{2(s+b)}}L^{-s/4}h\|_X\\
&\leq \frac{g_1(\frac{-s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}\beta^{\frac{-b}{(s+b)}}\|h\|_X\\
&\leq \psi_2(s)\beta^{\frac{-b}{(s+b)}}\|h\|_X. \qquad (3.5)
\end{aligned}$$

□

Let
$$G(x) = x - R_\beta(x_0)^{-1}[F(x) - z^\delta_{\alpha_k,s} + \alpha_k L^{s/2}(x - x_0)]. \tag{3.6}$$
Note that with the above notation $G(x^\delta_{n,\alpha_k,s}) = x^\delta_{n+1,\alpha_k,s}$.

First we prove that $x^\delta_{n,\alpha_k,s}$ converges to the zero $x^\delta_{\alpha_k,s}$ of
$$F(x) + \alpha_k L^{s/2}(x - x_0) = z^\delta_{\alpha_k,s} \tag{3.7}$$
and then we prove that $x^\delta_{\alpha_k,s}$ is an approximation for $\hat{x}$.

Hereafter we assume that $\|\hat{x} - x_0\|_X < \rho$ where
$$\rho < \frac{1}{\psi_1(s)M}\left(\frac{\beta^{\frac{b}{s+b}}[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})]^2}{4k_0\overline{\psi_2(s)}^2} - \psi(s)\frac{\delta_0}{\alpha_0^{\frac{a}{2(s+a)}}}\right)$$
with $\delta_0 < \frac{\beta^{\frac{b}{s+b}}[1-\overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})]^2}{4k_0\psi(s)\overline{\psi_2(s)}^2}\alpha_0^{\frac{-a}{2(s+a)}}$. Let
$$\gamma_\rho := \psi_2(s)\beta^{\frac{-b}{(s+b)}}[\psi_1(s)M\rho + \psi(s)\alpha_0^{\frac{-a}{2(s+a)}}\delta_0].$$
and we define
$$q = \overline{\psi_2(s)}[k_0 r + \frac{\beta - \alpha_k}{\beta}], \;\; r \in (r_1, r_2) \tag{3.8}$$
where
$$r_1 = \frac{[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})] - \sqrt{[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})]^2 - 4k_0\overline{\psi_2(s)}\gamma_\rho}}{2k_0\overline{\psi_2(s)}}$$
and
$$\begin{aligned} r_2 \;\; = \;\; \min\Bigg\{ & \frac{1 - (1 - c)\psi_2(s)}{k_0\overline{\psi_2(s)}}, \frac{1}{k_0}[\frac{1}{\overline{\psi_2(s)}} - \frac{\beta - \alpha_k}{\beta}], \\ & \frac{[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})] + \sqrt{[1 - \overline{\psi_2(s)}(\frac{\beta-\alpha_k}{\beta})]^2 - 4k_0\overline{\psi_2(s)}\gamma_\rho}}{2k_0\overline{\psi_2(s)}}\Bigg\} \end{aligned}$$
where $0 < c < \alpha_k < 1$ is a constant.

Remark 3.6. Note that for $r \in (r_1, r_2)$ we have $q < 1$ and $\gamma_\rho < \frac{\gamma_\rho}{1-q} \le r$.

Theorem 3.7. *Let $r \in (r_1, r_2)$ and Assumption 3.3 be satisfied. Then the sequence $(x^\delta_{n,\alpha,s})$ defined in (1.7) is well defined and $x^\delta_{n,\alpha,s} \in B_r(x_0)$ for all $n \ge 0$. Further $(x^\delta_{n,\alpha,s})$ is Cauchy sequence in $B_r(x_0)$ and hence converges to $x^\delta_{\alpha_k,s} \in \overline{B_r(x_0)}$ and $F(x^\delta_{\alpha_k,s}) + \alpha_k L^{s/2}(x^\delta_{\alpha_k,s} - x_0) = z^\delta_{\alpha_k}$.*

Moreover, the following estimate holds for all $n \ge 0$,
$$\|x^\delta_{n,\alpha,s} - x^\delta_{\alpha_k,s}\|_X \le \frac{\gamma_\rho q^n}{1 - q}. \tag{3.9}$$

Proof. Let G be as in (3.6). Then for $u, v \in B_r(x_0)$,

$$\begin{aligned}
G(u) - G(v) &= u - v - R_\beta(x_0)^{-1}[F(u) - z^\delta_{\alpha_k} + \alpha_k L^{s/2}(u - x_0)] \\
&\quad + R_\beta(x_0)^{-1}[F(v) - z^\delta_{\alpha_k,s} + \alpha_k L^{s/2}(v - x_0)] \\
&= R_\beta(x_0)^{-1}[R_\beta(x_0)(u - v) - (F(u) - F(v))] \\
&\quad + \alpha_k R_\beta(x_0)^{-1} L^{s/2}(v - u) \\
&= R_\beta(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v)) + \beta L^{s/2}(u - v)] \\
&\quad + \alpha_k R_\beta(x_0)^{-1} L^{s/2}(v - u) \\
&= R_\beta(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v)) + (\beta - \alpha_k) L^{s/2}(u - v)] \\
&= R_\beta(x_0)^{-1}\int_0^1 [F'(x_0) - F'(v + t(u - v)]dt(u - v) \\
&\quad + R_\beta(x_0)^{-1}(\beta - \alpha_k) L^{s/2}(u - v)].
\end{aligned}$$

Thus by Assumption 3.3 and Lemma 3.5 we have

$$\|G(u) - G(v)\|_X \le q\|u - v\|_X. \tag{3.10}$$

Now we shall prove that $x^\delta_{n,\alpha_k,s} \in B_r(x_0)$, for all $n \ge 0$. Note that

$$\begin{aligned}
\|x^\delta_{1,\alpha_k,s} - x_0\|_X &= \|(F'(x_0) + \beta L^{s/2})^{-1}(F(x_0) - z^\delta_{\alpha_k,s})\|_X \\
&\le \|L^{-s/4}(L^{-s/4}F'(x_0)L^{-s/4} + \beta I)^{-1} L^{-s/4} \\
&\quad (F(x_0) - z^\delta_{\alpha_k,s})\|_X \\
&\le \frac{1}{f_1(\frac{s}{2(s+b)})}\|B_s^{\frac{s}{2(s+b)}}(B_s + \frac{\alpha_k}{c}I)^{-1}L^{-s/4} \\
&\quad (F(x_0) - z^\delta_{\alpha_k,s})\|_X \\
&\le \frac{1}{f_1(\frac{s}{2(s+b)})}\|(B_s + \beta I)^{-1}B_s^{\frac{s}{(s+b)}}B_s^{\frac{-s}{2(s+b)}} \\
&\quad L^{-s/4}(F(x_0) - z^\delta_{\alpha_k,s})\|_X \\
&\le \frac{g_1(\frac{-s}{2(s+b)})}{f_1(\frac{s}{2(s+b)})}\beta^{\frac{-b}{(s+b)}}\|F(x_0) - z^\delta_{\alpha_k,s}\|_X \\
&\le \psi_2(s)\beta^{\frac{-b}{(s+b)}}[\|F(x_0) - z_{\alpha_k,s}\|_X \\
&\quad + \|z_{\alpha_k,s} - z^\delta_{\alpha_k,s}\|_X]
\end{aligned} \tag{3.11}$$

Now using (2.3) and (2.5) in (3.5), one can see that

$$\begin{aligned}
\|x^\delta_{1,\alpha_k,s} - x_0\|_X &\le \psi_2(s)\beta^{\frac{-b}{(s+b)}}[\psi_1(s)\|F(\hat{x}) - F(x_0)\|_X + \psi(s)\alpha^{\frac{-a}{2(s+a)}}\delta] \\
&\le \psi_2(s)\beta^{\frac{-b}{(s+b)}}[\psi_1(s)M\rho + \psi(s)\alpha_0^{\frac{-a}{2(s+a)}}\delta_0] = \gamma_\rho.
\end{aligned}$$

Assume that $x^\delta_{k,\alpha_k,s} \in B_r(x_0)$, for some k. Then

$$\begin{aligned}
\|x^\delta_{k+1,\alpha_k,s} - x_0\|_X &= \|x^\delta_{k+1,\alpha_k,s} - x^\delta_{k,\alpha_k,s} + x^\delta_{k,\alpha_k,s} - x^\delta_{k-1,\alpha_k,s} \\
&\quad + \cdots + x^\delta_{1,\alpha_k,s} - x_0\|_X \\
&\leq \|x^\delta_{k+1,\alpha_k,s} - x^\delta_{k,\alpha_k,s}\|_X + \|x^\delta_{k,\alpha_k,s} - x^\delta_{k-1,\alpha_k,s}\|_X \\
&\quad + \cdots + \|x^\delta_{1,\alpha_k,s} - x_0\|_X \\
&\leq (q^k + q^{k-1} + \cdots + 1)\gamma_\rho \\
&\leq \frac{\gamma_\rho}{1-q} \leq r.
\end{aligned}$$

So $x^\delta_{k+1,\alpha_k,s} \in B_r(x_0)$ and hence, by induction $x^\delta_{n,\alpha_k,s} \in B_r(x_0), \quad \forall n \geq 0$. Next we shall prove that $(x^\delta_{k+1,\alpha_k,s})$ is a Cauchy sequence in $B_r(x_0)$.

$$\begin{aligned}
\|x^\delta_{n+m,\alpha_k,s} - x^\delta_{n,\alpha_k,s}\|_X &\leq \sum_{i=0}^{m} \|x^\delta_{n+i+1,\alpha_k,s} - x^\delta_{n+i,\alpha_k,s}\|_X \qquad (3.12) \\
&\leq \sum_{i=0}^{m} q^{n+i}\gamma_\rho \\
&\leq \frac{q^n}{1-q}\gamma_\rho. \qquad (3.13)
\end{aligned}$$

Thus $(x^\delta_{n,\alpha_k,s})$ is a Cauchy sequence in $B_r(x_0)$ and hence converges to some $x^\delta_{\alpha_k,s} \in \overline{B_r(x_0)}$. Now by $n \to \infty$ in (1.7) we obtain $F(x^\delta_{\alpha_k,s}) + \alpha_k L^{s/2}(x^\delta_{\alpha_k,s} - x_0) = z^\delta_{\alpha_k,s}$. This completes the proof of the Theorem.

In addition to the Assumption 2.2, we use the following assumption to obtain the error estimate for $\|\hat{x} - x^\delta_{\alpha_k,s}\|$.

Assumption 3.8. *There exists a continuous, strictly monotonically increasing function* $\varphi_1 : (0, \|B_s\|] \to (0, \infty)$ *such that the following conditions hold:*

- $\lim_{\lambda \to 0} \varphi_1(\lambda) = 0$,
- $\sup_{\lambda > 0} \frac{\alpha\varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha)\ \forall \lambda \in (0, \|B_s\|]$ *and*
- *there exists* $w \in X$ *with* $\|w\|_X \leq E_2$, *such that*

$$B_s^{\frac{s}{2(s+b)}} L^{s/4}(x_0 - \hat{x}) = \varphi_1(B_s)w$$

Remark 3.9. If $x_0 - \hat{x} \in X_{t_1}$ i.e., $\|x_0 - \hat{x}\|_{t_1} \leq E_1$ for some positive constant E_1 and $0 \leq t_1 \leq s+b$. Then as in Remark 2.3, we have $B_s^{\frac{s}{2(s+b)}} L^{s/4}(x_0 - \hat{x}) = \varphi_1(B_s)w$ where $\varphi_1(\lambda) = \lambda^{t_1/(s+b)}$, $w = B_s^{\frac{s-2t_1}{2(s+b)}} L^{s/4}(\hat{x} - x_0)$ and $\|w\| \leq g_1(\frac{s-2t_1}{2(s+b)})E_1 := E_2$.

Hereafter we assume that $\varphi_1(\alpha_k) \leq \varphi(\alpha_k)$.

Theorem 3.10. *Suppose* $x^\delta_{\alpha_k,s}$ *is the solution of (*3.1*) and Assumptions* 3.3 *and* 3.8 *hold. Then*

$$\|\hat{x} - x^\delta_{\alpha_k,s}\|_X = O(\psi^{-1}(\delta)).$$

Proof. Note that $(F(x^\delta_{\alpha_k,s}) - z^\delta_{\alpha_k,s}) + \alpha_k L^{s/2}(x^\delta_{\alpha_k,s} - x_0) = 0$, so

$$(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})(x^\delta_{\alpha_k,s} - \hat{x}) = (F'(x_0) + \frac{\alpha_k}{c}L^{s/2})(x^\delta_{\alpha_k,s} - \hat{x})$$

$$-(F(x^\delta_{\alpha_k,s}) - z^\delta_{\alpha_k,s}) - \alpha_k L^{s/2}(x^\delta_{\alpha_k,s} - x_0)$$

$$= (\frac{\alpha_k}{c} - \alpha_k)L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x}) + \alpha_k L^{s/2}(x_0 - \hat{x})$$

$$+F'(x_0)(x^\delta_{\alpha_k,s} - \hat{x}) - [F(x^\delta_{\alpha_k,s}) - z^\delta_{\alpha_k,s}]$$

$$= (\frac{\alpha_k}{c} - \alpha_k)L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x}) + \alpha_k L^{s/2}(x_0 - \hat{x})$$

$$+F'(x_0)(x^\delta_{\alpha_k,s} - \hat{x}) - [F(x^\delta_{\alpha_k,s}) - F(\hat{x}) + F(\hat{x}) - z^\delta_{\alpha_k,s}]$$

$$= (\frac{\alpha_k}{c} - \alpha_k)L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x}) + \alpha_k L^{s/2}(x_0 - \hat{x}) - (F(\hat{x}) - z^\delta_{\alpha_k,s})$$

$$+F'(x_0)(x^\delta_{\alpha_k,s} - \hat{x}) - [F(x^\delta_{\alpha_k,s}) - F(\hat{x})].$$

Thus

$$\|x^\delta_{\alpha_k,s} - \hat{x}\|_X \leq \|(\frac{\alpha_k}{c} - \alpha_k)(F'(x_0 + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x})\|_X$$

$$+\|\alpha_k(F'(x_0 + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x_0 - \hat{x})\|_X + \|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}$$

$$(F(\hat{x}) - z^\delta_{\alpha_k,s})\|_X + \|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}[F'(x_0)(x^\delta_{\alpha_k,s} - \hat{x})$$

$$-(F(x^\delta_{\alpha_k,s}) - F(\hat{x}))]\|_X$$

$$\leq \|(\frac{\alpha_k}{c} - \alpha_k)(F'(x_0 + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x})\|_X$$

$$+\|\alpha_k(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x_0 - \hat{x})\|_X$$

$$+\psi_2(s)(\frac{\alpha_k}{c})^{-1}\|F(\hat{x}) - z^\delta_{\alpha_k,k}\|_X + \Gamma \tag{3.14}$$

where $\Gamma := \|(F'(x_0) + \frac{\alpha_k}{c}L^{s/2})^{-1}\int_0^1[F'(x_0) - F'(\hat{x} + t(x^\delta_{\alpha_k,s} - \hat{x})](x^\delta_{\alpha_k,s} - \hat{x})dt\|_X$. Note that

$$\begin{aligned} &\|(\frac{\alpha_k}{c} - \alpha_k)(F'(x_0 + \frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x^\delta_{\alpha_k,s} - \hat{x})\|_X \\ \leq\ & \frac{\frac{\alpha_k}{c} - \alpha_k}{\frac{\alpha_k}{c}}\overline{\psi_2(s)}\|x^\delta_{\alpha_k,s} - \hat{x}\|_X \\ \leq\ & (1-c)\overline{\psi_2(s)}\|x^\delta_{\alpha_k,s} - \hat{x}\|_X, \end{aligned} \tag{3.15}$$

and by Assumption 3.8, we obtain

$$\begin{aligned}
& \|\alpha_k(F'(x_0)+\frac{\alpha_k}{c}L^{s/2})^{-1}L^{s/2}(x_0-\hat{x})\|_X \\
= & \|\alpha_k L^{-s/4}(B_s+\frac{\alpha_k}{c})^{-1}L^{s/4}(x_0-\hat{x})\|_X \\
\leq & \frac{1}{f_1(\frac{s}{2(s+b)})}\|\alpha_k(B_s+\frac{\alpha_k}{c})^{-1}B_s^{\frac{s}{2(s+b)}}L^{s/4}(x_0-\hat{x})\|_X \\
\leq & \frac{1}{f_1(\frac{s}{2(s+b)})}\sup_{\lambda\in\sigma(F'(x_0))}\frac{\alpha_k\varphi_1(\lambda)}{\lambda+\frac{\alpha_k}{c}} \\
\leq & \sup_{\lambda\in\sigma(F'(x_0))}\frac{\alpha_k\varphi_1(\lambda)}{\lambda+\alpha_k} \\
\leq & \varphi_1(\alpha_k)
\end{aligned} \tag{3.16}$$

and by Assumption 3.3, and Lemma 3.5 we obtain

$$\begin{aligned}
\Gamma \leq & \|(F'(x_0)+\frac{\alpha_k}{c}L^{s/2})^{-1}\int_0^1[F'(x_0)-F'(\hat{x}+t(x^\delta_{\alpha_k,s}-\hat{x})] \\
& (x^\delta_{\alpha_k,s}-\hat{x})dt\|_X \\
\leq & \overline{\psi_2(s)}k_0 r\|x^\delta_{\alpha_k,s}-\hat{x}\|_X
\end{aligned} \tag{3.17}$$

and hence by (3.15), (3.16), (3.17) and (3.14) we have

$$\begin{aligned}
\|x^\delta_{\alpha_k,s}-\hat{x}\|_X \leq & \frac{\varphi_1(\alpha_k)+C_s\psi_2(s)(2+\frac{4\eta}{\eta-1})\eta\psi^{-1}_{s,a}(\delta)}{1-(1-c)\psi_2(s)-\overline{\psi_2(s)}k_0 r} \\
= & O(\psi^{-1}_{s,a}(\delta)).
\end{aligned}$$

This completes the proof of the Theorem.

The following Theorem is a consequence of Theorem 3.7 and Theorem 3.10.

Theorem 3.11. *Let $x^\delta_{n,\alpha_k,s}$ be as in (1.7) with $\alpha=\alpha_k$ and $\delta\in(0,\delta_0]$, assumptions in Theorem 3.7 and Theorem 3.10 hold. Then*

$$\|\hat{x}-x^\delta_{n,\alpha_k,s}\|_X \leq \frac{\gamma_\rho}{1-q}q^n+O(\psi^{-1}_{s,a}(\delta)).$$

Theorem 3.12. *Let $x^\delta_{n,\alpha_k,s}$ be as in (1.7) with $\alpha=\alpha_k$ and $\delta\in(0,\delta_0]$, and assumptions in Theorem 3.11 hold. Let*

$$n_k := \min\{n : \tilde{q}^n \leq \alpha_k^{\frac{-a}{2(s+a)}}\delta\}.$$

Then

$$\|\hat{x}-x^\delta_{n_k,\alpha_k,s}\|_X = O(\psi^{-1}_{s,a}(\delta)).$$

4. Numerical examples

In the next two cases, we present examples for nonlinear equations where Assumption 3.3 is satisfied but not Assumption 3.1.

Example 4.1. Let $X = Y = \mathbb{R}$, $D = [0, \infty), x_0 = 1$ and define function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1 x + c_2, \tag{4.1}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. Hence, Assumption 3.1 is not satisfied. However central Lipschitz condition Assumption 3.3 holds for $k_0 = 1$.

Indeed, we have

$$\begin{aligned} \|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}} \end{aligned}$$

so

$$\|F'(x) - F'(x_0)\| \leq k_0 |x - x_0|.$$

Example 4.2. We consider the integral equations

$$u(s) = f(s) + \lambda \int_a^b G(s,t) u(t)^{1+1/n} dt, \quad n \in \mathbb{N}. \tag{4.2}$$

Here, f is a given continuous function satisfying $f(s) > 0, s \in [a, b], \lambda$ is a real number, and the kernel G is continuous and positive in $[a, b] \times [a, b]$.

For example, when $G(s, t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$\begin{aligned} u'' &= \lambda u^{1+1/n} \\ u(a) &= f(a), u(b) = f(b). \end{aligned}$$

These type of problems have been considered in [1]- [5].

Equation of the form (4.2) generalize equations of the form

$$u(s) = \int_a^b G(s,t) u(t)^n dt \tag{4.3}$$

studied in [1]-[5]. Instead of (4.2) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a, b] \to C[a, b], \Omega = \{u \in C[a, b] : u(s) \geq 0, s \in [a, b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t) u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \lambda(1 + \frac{1}{n}) \int_a^b G(s,t) u(t)^{1/n} v(t) dt, \quad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b]=[0,1], G(s,t)=1$ and $y(t)=0$. Then $F'(y)v(s)=v(s)$ and

$$\|F'(x)-F'(y)\| = |\lambda|(1+\frac{1}{n})\int_a^b x(t)^{1/n}dt.$$

If F' were a Lipschitz function, then

$$\|F'(x)-F'(y)\| \leq L_1\|x-y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n}dt \leq L_2 \max_{x\in[0,1]} x(s), \tag{4.4}$$

would hold for all $x\in\Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t)=\frac{t}{j}, \quad j\geq 1, \quad t\in[0,1].$$

If these are substituted into (4.4)

$$\frac{1}{j^{1/n}(1+1/n)} \leq \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leq L_2(1+1/n), \quad \forall j\geq 1.$$

This inequality is not true when $j\to\infty$.

Therefore, condition (4.4) is not satisfied in this case. Hence Assumption 3.1 is not satisfied. However, condition Assumption 3.3 holds. To show this, let $x_0(t)=f(t)$ and $\gamma=\min_{s\in[a,b]} f(s), \alpha>0$ Then for $v\in\Omega$,

$$\begin{aligned}\|[F'(x)-F'(x_0)]v\| &= |\lambda|(1+\frac{1}{n})\max_{s\in[a,b]}|\int_a^b G(s,t)(x(t)^{1/n}-f(t)^{1/n})v(t)dt| \\ &\leq |\lambda|(1+\frac{1}{n})\max_{s\in[a,b]} G_n(s,t)\end{aligned}$$

where $G_n(s,t)=\frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n}+x(t)^{(n-2)/n}f(t)^{1/n}+\cdots+f(t)^{(n-1)/n}}\|v\|$.

Hence,

$$\begin{aligned}\|[F'(x)-F'(x_0)]v\| &= \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}}\max_{s\in[a,b]}\int_a^b G(s,t)dt\|x-x_0\| \\ &\leq k_0\|x-x_0\|,\end{aligned}$$

where $k_0=\frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}}N$ and $N=\max_{s\in[a,b]}\int_a^b G(s,t)dt$. Then Assumption 3.3 holds for sufficiently small λ.

In the last example, we show that $\frac{K}{k_0}$ can be arbitrarily large in certain nonlinear equation.

Example 4.3. Let $X=D(F)=\mathbb{R}, x_0=0$, and define function F on $D(F)$ by

$$F(x)=d_0x+d_1+d_2\sin e^{d_3x}, \tag{4.5}$$

where $d_i, i=0,1,2,3$ are given parameters. Then, it can easily be seen that for d_3 sufficiently large and d_2 sufficiently small, $\frac{K}{k_0}$ can be arbitrarily large.

5. Conclusion

In this paper we present an iterative regularization method for obtaining an approximate solution of an ill-posed Hammerstein type operator equation $KF(x) = y$ in the Hilbert scale setting where K is a bounded linear operator and F is a nonlinear monotone operator. It is assumed that the available data is y^δ in place of exact data y. We considered the Hilbert space $(X_t)_{t\in\mathbb{R}}$ generated by L for the analysis where $L : D(L) \to X$ is a linear, unbounded, self-adjoint, densely defined and strictly positive operator on X. For choosing the regularization parameter α we used the adaptive scheme of Pereverzev and Schock (2005).

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