Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 2, 233–245

Ćirić type fixed point theorems

Adrian Petruşel

Abstract. The purpose of this paper is to review some fixed point and strict fixed point results for Ćirić type operators. The data dependence of the fixed point set, the well-posedness of the fixed point problem, the limit shadowing property, as well as, the fractal operator theory associated with these operators are also considered.

Mathematics Subject Classification (2010): 47H10, 54H25.

Keywords: fixed point, strict fixed point, metric space, data dependence, well-posed problem, limit shadowing property.

1. Preliminaries

Fixed points and strict fixed points (also called end-points) are important tools for the study of equilibrium problems in Mathematical Economics and Game Theory. Fixed points and the strict fixed points represent optimal preferences in some economical models and respectively different Nash type equilibrium points for some abstract games, see, for example, [1] and [23]. As a consequence, it is important aim to obtain fixed and strict fixed point theorems for multivalued operators in different contexts.

We shall begin by presenting some notions and notations that will be used throughout the paper.

Let us consider the following families of subsets of a metric space (X, d):

$$P(X) := \{Y \in \mathcal{P}(X) \,|Y \neq \emptyset\}\,; P_b(X) := \{Y \in P(X)|\ Y \text{ is bounded }\},$$

$$P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\}\,; P_{cp}(X) := \{Y \in P(X)|\ Y \text{ is compact}\}$$

Let us define the gap functional between the sets A and B in the metric space (X, d) as:

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$$

and the (generalized) Pompeiu-Hausdorff functional as:

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$H(A,B) = \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(A,b)\}.$$

The generalized diameter functional is denoted by $\delta : P(X)\times P(X) \to \mathbb{R}_+\cup\{\infty\}$, and defined by

$$\delta(A,B) = \sup\{d(a,b) \mid a \in A,\ b \in B\}.$$

Denote by $diam(A) := \delta(A,A)$ the diameter of the set A.

Let $T : X \to P(X)$ be a multivalued operator and

$$Graphic(T) := \{(x,y)|y \in T(x)\}$$

the graphic of T. An element $x \in X$ is called a *fixed point* for T if and only if $x \in T(x)$ and it is called a *strict fixed point* if and only if $\{x\} = T(x)$.

The set $Fix(T) := \{x \in X|\ x \in T(x)\}$ is called the fixed point set of T, while $SFix(T) = \{x \in X|\ \{x\} = T(x)\}$ is called the strict fixed point set of T. Notice that $SFix(T) \subseteq Fix(T)$.

We will also denote by

$$O(x_0,n) := \{x_0, t(x_0), t^2(x_0), \cdots, t^n(x_0)\}$$

the orbit of order n of the operator t corresponding to $x_0 \in X$, while

$$O(x_0) := \{x_0, t(x_0), t^2(x_0), \cdots, t^n(x_0), \cdots\}$$

is the orbit of f corresponding to $x_0 \in X$.

In this paper we will survey some fixed point and strict fixed point theorems for singlevalued and multivalued operators satisfying contractive conditions of Ćirić type. We will also present some new results for general classes of Ćirić type operators.

2. A survey of known results

The basic metric fixed point theorems for singlevalued, respectively multivalued operators are Banach's contraction principle (1922), respectively Nadler's contraction principle (1969). A lot of efforts were done to extend these results for so called generalized contractions.

Let (X,d) be a metric space and $f : X \to X$ be a singlevalued operator. Then, by definition (see I.A. Rus [22]), f is called a weakly Picard operator if:

(i) $F_f \neq \emptyset$;

(ii) for all $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}} \to x^*(x) \in F_f$ as $n \to \infty$.

In particular, if $F_f = \{x^*\}$, then f is called a Picard operator.

Moreover, if f is a weakly Picard operator and there exists $\tilde{c} > 0$ such that

$$d(x, x^*(x)) \leq \tilde{c}d(x, f(x)),\ \text{for all } x \in X,$$

then f is called a $\tilde{c}$-weakly Picard operator. Similarly, a Picard operator for which there exists $\tilde{c} > 0$ such that

$$d(x, x^*) \leq \tilde{c}d(x, f(x)),\ \text{for all } x \in X,$$

is called a $\tilde{c}$-Picard operator.

A nice extension of Banach's contraction principle was given by Ćirić, Reich and Rus (independently one to the others) in 1971-1972.

More precisely, if (X, d) is a complete metric space and $f : X \to X$ is an operator for which there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)), \text{ for all } x, y \in X,$$

then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-\beta}$, where $\beta := \min\{\frac{a+b}{1-c}, \frac{a+c}{1-b}\} < 1$.

An extension of this result was proved in a paper from 1973 by Hardy and Rogers. The result, in Picard operators language, is as follows.

If (X, d) is a complete metric space and $f : X \to X$ is an operator for which there exist $a, b, c, e, f \in \mathbb{R}_+$ with $a + b + c + e + f < 1$ such that

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)) + ed(x, f(y)) + fd(y, f(x)),$$

for all $x, y \in X$, then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-\beta}$, where

$$\beta := \min\{\frac{a+b+e}{1-c-e}, \frac{a+c+f}{1-b-f}\} < 1.$$

The idea of the proof in the above results is to prove that f is a contraction on the graphic of the operator. i.e.,

$$d(f(x), f^2(x)) \leq \beta d(x, f(x)), \text{ for all } x \in X.$$

Then in 1974, Ćirić proved the following very general result.

If (X, d) is a complete metric space and if $f : X \to X$ is an operator for which there exists $q \in (0, 1)$ such that, for all $x, y \in X$, we have

$$d(f(x), f(y)) \leq q \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}, \quad (2.1)$$

then f is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-q}$.

In this last case, the proof is based on some arguments involving the orbit of order n and the orbit of the operator f.

Remark 2.1. Notice that any Ćirić-Reich-Rus type operator is a Hardy-Rogers type operator and any Hardy-Rogers type operator is a Ćirić type operator. The reverse implications do not hold, as we can see from several examples given in [10], [21], [22].

There are also other generalizations of the above theorems. One of the most general one, was proved by I.A. Rus in 1979.

If (X, d) is a complete metric space and $f : X \to X$ is an operator for which there exists a generalized strict comparison function $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ (which means that φ is increasing in each variable and the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\Phi(t) := \varphi(t, t, t, t, t)$$

satisfy the conditions that $\Phi^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$ and $t - \Phi(t) \to +\infty$ as $t \to +\infty$) such that

$$d(f(x), f(y)) \leq \varphi(d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))), \text{ for all } x, y \in X,$$

then f is a Φ-Picard operator (i.e., f is a Picard operator and $d(x,x^*) \leq \Phi(d(x,f(x))$, for all $x \in X$).

Notice that if, in particular

$$\varphi(t_1,t_2,t_3,t_4,t_5) := at_1 + bt_2 + ct_3 + et_4 + ft_5,$$

(with $a,b,c,e,f \in \mathbb{R}_+$ and $a+b+c+e+f < 1$), then we obtain the Hardy-Rogers condition on f. Also, if we consider

$$\varphi(t_1,t_2,t_3,t_4,t_5) := q\max\{t_1,t_2,t_3,t_4,t_5\},$$

then we get the Ćirić type condition on f.

Finally, let us recall another nice generalization given, for the case of nonself operators, by Ćirić in 2006.

More precisely, if (X,d) is a complete metric space and $f : X \to X$ is an operator for which there exist five strict comparison functions $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e., φ_i is increasing, $\varphi_i^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$ and $t - \varphi_i(t) \to +\infty$ as $t \to +\infty$ for each $i \in \{1,2,3,4,5\}$) such that, for all $x,y \in X$ one have that

$$d(f(x),f(y)) \leq$$

$$\leq \max\{\varphi_1(d(x,y)), \varphi_2(d(x,f(x))), \varphi_3(d(y,f(y))), \varphi_4(d(x,f(y))), \varphi_5(d(y,f(x)))\},$$

then f is a Picard operator.
Notice that, in particular if we define $\varphi_i(t) := qt$ (where $q < 1$) for $i \in \{1,2,3,4,5\}$, then we obtain again the classical condition of Ćirić.

Passing to the multivalued case, let (X,d) be a metric space and let $T : X \to P_b(X)$ be a multivalued operator with nonempty and bounded values. We will be interested in the study of strict fixed points of multivalued operators satisfying some contractive type conditions with respect to the functional δ.

In 1972, S. Reich proved the following very interesting strict fixed point theorem for multivalued operators.

If (X,d) is a complete metric space and if $T : X \to P_b(X)$ is a multivalued operator for which there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c < 1$ such that

$$\delta(T(x),T(y)) \leq ad(x,y) + b\delta(x,T(x)) + c\delta(y,T(y)), \text{ for all } x,y \in X,$$

then the following conclusions hold:

(i) $F_T = (SF)_T = \{x^*\}$;

(ii) for each $x_0 \in X$ there exists a sequence of successive approximations for T starting from x_0 (which means that $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$) convergent to x^*;

(iii) $d(x_n,x^*) \leq \frac{\beta^n}{1-\beta} d(x_0,x_1)$, for $n \in \mathbb{N}^*$ (where $\beta := \min\{\frac{a+b}{1-c}, \frac{a+c}{1-b}\} < 1$).

An important extension of the above result is the following theorem of Ćirić, given in 1972.

Let (X,d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator for which there exists $q \in \mathbb{R}_+$ with $q < 1$ such that, for all $x,y \in X$ the following condition holds

$$\delta(T(x),T(y)) \leq q\max\{d(x,y), \delta(x,T(x)), \delta(y,T(y)), D(x,T(y)), D(y,T(x))\}.$$

Then the following conclusions hold:

(i) $F_T = (SF)_T = \{x^*\}$;

(ii) for each $x_0 \in X$ there exists a sequence of successive approximations for T starting from x_0 convergent to x^*;

(iii) $d(x_n, x^*) \leq \frac{q^{(1-a)n}}{1-q^{1-a}} d(x_0, x_1)$, for $n \in \mathbb{N}^*$ (where $a \in (0,1)$ is an arbitrary real number).

In the above two results, the approach is based on the construction of a selection $t : X \to X$ of T which satisfies the corresponding fixed point theorem (given by Reich and respectively by Ćirić) for singlevalued operators.

A relevant generalization of the above theorems was given by I.A. Rus in 1982, as follows.

Let (X, d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued operator. Suppose that there exists an increasing function $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ for which there exists $p > 1$ such that the function $\psi : \mathbb{R}^5_+ \to \mathbb{R}_+$ defined by

$$\psi(t_1, t_2, t_3, t_4, t_5) := \varphi(t_1, pt_2, pt_3, t_4, t_5)$$

is a generalized strict comparison function. If, for all $x, y \in X$ the following assumption takes place:

$$\delta(T(x), T(y)) \leq \varphi(d(x, y), \delta(x, T(x)), \delta(y, T(y)), D(x, T(y)), D(y, T(x))),$$

then the following conclusions hold:

(i) $F_T = (SF)_T = \{x^*\}$;

(ii) for each $x_0 \in X$ there exists a sequence of successive approximations for T starting from x_0 convergent to x^*.

Remark 2.2. In none of the above cases, we cannot obtain (without additional assumptions) the conclusion that T is a multivalued Picard operator. Recall that, by definition, $T : X \to P(X)$ is called a multivalued Picard operator (see [15]) if and only if:

(i) $(SF)_T = F_T = \{x^*\}$;

(ii) $T^n(x) \stackrel{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

For example, if, in the case of Reich' strict fixed point theorem, we additionally impose the condition that

$$\min\left\{\frac{a+b}{1-b}, \frac{a+c}{1-c}\right\} < 1,$$

then we can prove that T is a multivalued Picard operator, see [14].

Remark 2.3. It is an open question if we can get similar results if we replace, in Ćirić' result (or more generally in Rus' theorem) the values $D(x, T(y))$ and $D(y, T(x))$ with $\delta(x, T(y))$ and, respectively $\delta(y, T(x))$.

3. Strict fixed point theorems in metric spaces endowed with a graph

A new research direction in fixed point theory was recently considered by J. Jachymski (see [11]) in the context of a metric space endowed with a graph.

Let (X,d) be a metric space and Δ be the diagonal of $X\times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta\subseteq E(G)$, $E(G)$ being the set of the edges of the graph. Assuming that G has no parallel edges, we will suppose that G can be identified with the pair $(V(G),E(G))$.

If x and y are vertices of G, then a path in G from x to y of length $k\in\mathbb{N}$ is a finite sequence $(x_n)_{n\in\{0,1,2,\cdots,k\}}$ of vertices such that $x_0=x$, $x_k=y$ and $(x_{i-1},x_i)\in E(G)$, for $i\in\{1,2,\cdots,k\}$.

Let us denote by $\tilde{G}$ the undirected graph obtained from G by ignoring the direction of edges. Notice that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected.

We will write that $E(G)\in I(T\times T)$ if and only if $x,y\in X$ with $(x,y)\in E(G)$ implies $T(x)\times T(y)\subset E(G)$.

For the particular case of a singlevalued operator $t:X\to X$ the above notations should be considered accordingly. In particular, the condition $E(G)\in I(t\times t)$ means that the operator t is edge preserving (in the sense of the Jachymski's definition of a Banach contraction, see [11]), i.e., for each $x,y\in X$ with $(x,y)\in E(G)$ we have that $(t(x),t(y))\in E(G)$.

One of the main result of the paper [2] is a fixed point theorem for a singlevalued operator of Ćirić type in metric spaces endowed with a graph. An extended version of that theorem is the following.

Theorem 3.1. *Let (X,d) be a complete metric space and G be a directed graph such that the triple (X,d,G) satisfies the following property:*
(P) for any sequence $(x_n)_{n\in\mathbb{N}}\subset X$ $x_n\to x$ as $n\to\infty$, there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{k_n},x)\in E(G)$.

Let $t:X\to X$ be a singlevalued operator. Suppose the following assertions hold:

(i) *there exists $a\in[0,1[$ such that*

$$d(t(x),t(y))\le a\cdot\max\{d(x,y),d(x,t(x)),d(y,t(y)),d(x,t(y)),d(y,t(x))\},$$

for all $(x,y)\in E(G)$.

(ii) *there exists $x_0\in X$ such that $(x_0,t(x_0))\in E(G)$;*

(iii) *$E(G)\in I(t\times t)$;*

(iv) *if $(x,y)\in E(G)$ and $(y,z)\in E(G)$, then $(x,z)\in E(G)$.*

In these conditions we have:

(a) *(existence) $Fix(t)\neq\emptyset$.*

(b) *(uniqueness) If, in addition, the following implication holds*

$$x^*,y^*\in Fix(t)\ \Rightarrow (x^*,y^*)\in E(G),$$

then $Fix(t)=\{x^*\}$.

Moreover, the sequence $(t^n(x_0))_{n\in\mathbb{N}}$ converges to x^* in (X,d)

Recall now two important stability concepts for the case of fixed point inclusions.

Definition 3.2. *Let (X,d) be a metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. By definition, the fixed point problem is well-posed for T with respect to H if:*

(i) $SFixT = \{x^*\}$;

(ii) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H(x_n, T(x_n)) \to 0$, as $n \to \infty$, then $x_n \overset{d}{\to} x^*$, as $n \to \infty$.*

Definition 3.3. *Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. By definition T has the limit shadowing property if for any sequence $(y_n)_{n\in\mathbb{N}}$ from X such that $D(y_{n+1}, T(y_n)) \to 0$, as $n \to \infty$, there exists $(x_n)_{n\in\mathbb{N}}$ a sequence of successive approximation of T, such that $d(x_n, y_n) \to 0$, as $n \to \infty$.*

Another main result in [2] concerns with the case of multivalued operators satisfying a Ćirić type condition with respect to the functional δ. An extended version of that result is the following theorem.

Theorem 3.4. *Let (X,d) be a complete metric space and G be a directed graph such that the triple (X,d,G) satisfies the following property:*

(P) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ $x_n \to x$ as $n \to \infty$, there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{k_n}, x) \in E(G)$.

Let $T : X \to P_b(X)$ be a multivalued operator. Suppose the following assertions hold:

(i) *there exists $a \in [0,1[$ such that*

$$\delta(T(x),T(y)) \le a \cdot max\{d(x,y), \delta(x,T(x)), \delta(y,T(y)), D(x,T(y)), D(y,T(x))\},$$

for all $(x,y) \in E(G)$.

(ii) *there exists $x_0 \in X$ such that, for all $y \in T(x_0)$ we have $(x_0,y) \in E(G)$;*

(iii) *$E(G) \in I(T \times T)$;*

(iv) *if $(x,y) \in E(G)$ and $(y,z) \in E(G)$, then $(x,z) \in E(G)$.*

In these conditions we have:

(a) $Fix(T) = SFix(T) \neq \emptyset$.

(b) *If, in addition, the following implication holds*

$$x^*, y^* \in Fix(T) \ \Rightarrow (x^*, y^*) \in E(G),$$

then $Fix(T) = SFix(T) = \{x^*\}$. Moreover, there exists a selection $t : X \to X$ of T satisfying the condition (2.1) on $E(G)$, such that the sequence x^* is a fixed point for t and $(t^n(x_0))_{n\in\mathbb{N}}$ converges to x^* as $n \to +\infty$.

(c) *If T has closed graphic and if, for any sequence $(x_n)_{n\in\mathbb{N}}$ in X for which*

$$H(x_n, T(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$(x_n, x^*) \in E(G) \text{ for all } n \in \mathbb{N},$$

then the fixed point problem is well-posed for T with respect to H.

(d) *If* $a < \frac{1}{3}$ *and if, for all sequences* $(y_n)_{n\in\mathbb{N}}$ *in* X *for which*

$$D\left(y_{n+1}, T\left(y_n\right)\right) \to 0 \text{ as } n \to \infty,$$

it follows that

$$(y_n, x^*) \in E(G) \text{ for all } n \in \mathbb{N},$$

then T *has the limit shadowing property.*

A data dependence result for the fixed point of a multivalued operator satisfying a Ćirić type condition with respect to the functional δ is the following.

Theorem 3.5. *Let* (X, d) *be a complete metric space and* G *be a directed graph such that the triple* (X, d, G) *satisfies the following property:*
(P) for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ *with* $x_n \to x$ *as* $n \to \infty$, *there exists a subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$ *such that* $(x_{k_n}, x) \in E(G)$.
Let $T, S : X \to P_b(X)$ *be two multivalued operators. Suppose the following assertions hold:*

(i) *there exists* $a \in [0, 1[$ *such that*

$$\delta(T(x), T(y)) \le a \cdot max\{d(x, y), \delta(x, T(x)), \delta(y, T(y)), D(x, T(y)), D(y, T(x))\},$$

for all $(x, y) \in E(G)$.

(ii) *there exists* $x_0 \in X$ *such that, for all* $y \in T(x_0)$ *we have* $(x_0, y) \in E(G)$;

(iii) $E(G) \in I(T \times T)$;

(iv) *if* $(x, y) \in E(G)$ *and* $(y, z) \in E(G)$, *then* $(x, z) \in E(G)$.

(v) *if* $x^*, y^* \in Fix(T)$ *then* $(x^*, y^*) \in E(G)$.

(vi) $Fix(S) \neq \emptyset$.

(vii) *if* $x^* \in Fix(T)$, *then* $(x^*, y) \in E(G)$, *for each* $y \in Fix(S)$.

(viii) *there exists* $\eta > 0$ *such that* $\delta(T(x), S(x)) \le \eta$, *for all* $x \in X$.

Then

$$\delta(x^*, Fix(S)) \le \frac{\eta}{1-a},$$

where x^* *is the unique fixed point of* T.

Proof. By Theorem 3.4 the operator T has a unique fixed point, i.e.,

$$Fix(T) = SFix(T) = \{x^*\}.$$

Let $y \in Fix(S)$ be arbitrary. Denote by t the selection of T which exists as in the text of Theorem 3.4. Then, we have

$$d(x^*, y) \le d(x^*, t(y)) + d(t(y), y)) \le d(t(x^*), t(y)) + \delta(T(y), S(y))$$
$$\le a \cdot \max\{d(x^*, y), d(x^*, t(x^*)), d(y, t(y)), d(x^*, t(y)), d(y, t(x^*))\} + \eta$$
$$= a \cdot \max\{d(x^*, y), d(y, t(y)), d(x^*, t(y))\} + \eta$$
$$\le a \cdot \max\{d(x^*, y), \delta(S(y), T(y)), d(x^*, t(y))\} + \eta \le a \cdot \max\{d(x^*, y), \eta, d(x^*, t(y))\} + \eta.$$

We have the following cases:

1) If the above maximum is $d(x^*, y)$, then we obtain that

$$d(x^*, y) \le \frac{\eta}{1-a}.$$

2) If the above maximum is η, then we obtain that

$$d(x^*, y) \leq \eta(1 + a).$$

3) If the above maximum is $d(x^*, t(y))$, then

$$d(x^*, y) \leq ad(x^*, t(y)) + \eta = ad(t(x^*), t(y)) + \eta \leq ad(x^*, y) + \eta.$$

Hence

$$d(x^*, y) \leq \frac{\eta}{1 - a}.$$

As a conclusion, from the above cases we get that

$$\delta(x^*, Fix(S)) \leq \frac{\eta}{1 - a}.$$ □

For other results in the context of metric spaces endowed with a graph or the case of ordered metric spaces we refer to [2], [3], [8], [9], [13], [12], [16], [19], etc.

4. A fractal operator theory for Ćirić-type operators

We will present now an existence and uniqueness result for the multivalued fractal operator generated by a multivalued operator of Ćirić type.

Let (X, d) be a metric space and $F : X \to P(X)$ be a multivalued operator. The multi-fractal operator generated by F is denoted by $\hat{F} : P_{cp}(X) \to P_{cp}(X)$ and is defined by $Y \mapsto F(Y)$

$$F(Y) := \bigcup_{x \in Y} F(x), \text{ for each } Y \in P_{cp}(X)$$

A fixed point for $\hat{F}$ is a fixed set for F, i.e., a nonempty compact set A^* with the property $\hat{F}(A^*) = A^*$.

Concerning the above problem, we have the following result.

Theorem 4.1. *Let (X, d) be a complete metric space and let $F : X \to P_{cl}(X)$ be an upper semicontinuous multivalued operator. Suppose that there exists a continuous and increasing (in each variable) function $\varphi : \mathbb{R}^3_+ \to \mathbb{R}_+$ such that the function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\Psi(t) := \varphi(t, t, t)$$

satisisfies the following properties:

(i) $\Psi^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$;

(ii) $t - \Psi(t) \to +\infty$ as $t \to +\infty$.

Suppose also that

$$H(F(x), F(y)) \leq \varphi(d(x, y), D(x, F(y)), D(y, F(x))), \text{ for all } x, y \in X.$$

Then the multi-fractal $\hat{F} : P_{cp}(X) \to P_{cp}(X)$ generated by F has a unique fixed point, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that

$$\hat{F}(A^*) = A^*.$$

Proof. We will prove that $\hat{F}$ satisfies the assumptions of Rus' Theorem for singlevalued operators, i.e.,

$$H(\hat{F}(A),\hat{F}(B)) \le \varphi(H(A,B),H(A,F(B)),H(B,F(A))), \text{ for all } A,B \in P_{cp}(X).$$

Indeed we have:

$$\rho(F(A),F(B)) = \sup_{a\in A}\rho(F(a),F(B)) = \sup_{a\in A}(\inf_{b\in B}\rho(F(a),F(b))) \le$$

$$\le \sup_{a\in A}(\inf_{b\in B} H(F(a),F(b))) \le \sup_{a\in A}(\inf_{b\in B}(\varphi(d(a,b),D(a,F(b)),D(b,F(a)))))$$

$$\le \sup_{a\in A}\varphi(\inf_{b\in B} d(a,b), \inf_{b\in B} D(a,F(b)), \inf_{b\in B} D(b,F(a)))$$

$$= \sup_{a\in A}\varphi(D(a,B),D(a,F(B)),D(F(a),B))$$

$$= \varphi(\sup_{a\in A} D(a,B), \sup_{a\in A} D(a,F(B)), \sup_{a\in A} D(F(a),B))$$

$$= \varphi(\rho(A,B),\rho(A,F(B)),\rho(F(A),B)) \le \varphi(H(A,B),H(A,F(B)),H(B,F(A))).$$

By the above inequality and the similar one for $\rho(F(A),F(B))$, we obtain that

$$H(F(A),F(B)) \le \varphi(H(A,B),H(A,F(B)),H(B,F(A))).$$

As a consequence, by Rus' theorem applied for $\hat{F}$, we get that $\hat{F}$ has a unique fixed point in $P_{cp}(X)$, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that $\hat{F}(A^*) = A^*$. □

Moreover, if (X,d) is a metric space and $F_1,...,F_m : X \to P(X)$ are multivalued operators, then the system $F = (F_1,...,F_m)$ is called an iterated multifunction system (IMS).

If the system $F = (F_1,...,F_m)$ is such that, for each $i \in \{1,2,\cdots m\}$, the multivalued operators $F_i : X \to P_{cp}(X)$ are upper semicontinuos, then the operator T_F defined as

$$T_F(Y) = \bigcup_{i=1}^{m} F_i(Y), \text{ for each } Y \in P_{cp}(X)$$

has the property that $T_F : P_{cp}(X) \to P_{cp}(X)$ and it is called the multi-fractal operator generated by the IMS $F = (F_1,...,F_m)$.

A nonempty compact subset $A^* \subset X$ is said to be a multivalued fractals with respect to the iterated multifunction system $F = (F_1,...,F_m)$ if and only if it is a fixed point for the associated multifractal operator, i.e., $T_F(A^*) = A^*$.

In particular, if F_i are singlevalued continuous operators from X to X, then $f = (f_1,...,f_m)$ is called an iterated function system (briefly IFS) and the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ given by

$$T_f(Y) = \bigcup_{i=1}^{m} f_i(Y), \text{ for each } Y \in P_{cp}(X)$$

is called the fractal operator generated by the IFS f. A fixed point of T_f is called a fractal generated by the IFS f.

An existence and uniqueness result for the multivalued fractal is the following.

Theorem 4.2. *Let (X,d) be a complete metric space and let $F_i : X \to P_{cl}(X)$ $(i \in \{1,2,\cdots,m\})$ be upper semicontinuous multivalued operators. Suppose that there exists continuous and increasing (in each variable) functions $\varphi_i : \mathbb{R}^3_+ \to \mathbb{R}_+$ $(i \in \{1,2,\cdots,m\})$ such that the functions $\Psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$\Psi_i(t) := \varphi_i(t,t,t), (i \in \{1,2,\cdots,m\})$$

satisfy, for each $i \in \{1,2,\cdots,m\}$, the following properties:

(i) $\Psi_i^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$;

(ii) $t - \Psi_i(t) \to +\infty$ as $t \to +\infty$.

Suppose also that, for each $i \in \{1,2,\cdots,m\}$, we have that

$$H(F_i(x), F_i(y)) \leq \varphi_i(d(x,y), D(x,F(y)), D(y,F(x))), \text{ for all } x,y \in X.$$

Then the multi-fractal $T_F : P_{cp}(X) \to P_{cp}(X)$ generated by IMS $F := (F_1,...,F_m)$ has a unique fixed point, i.e., there exists a unique $A^* \in P_{cp}(X)$ such that $T_F(A^*) = A^*$.

Proof. For $A, B \in P_{cp}(X)$ and using the proof of the previous theorem, we have

$$\begin{aligned} H(T_F(A), T_F(B)) &= H(\bigcup_{i=1}^m F_i(A), \bigcup_{i=1}^m F_i(B)) \leq \max_{i\in\{1,2,\cdots,m\}} H(F_i(A), F_i(B)) \\ &\leq \max_{i\in\{1,2,\cdots,m\}} \varphi_i(H(A,B), H(A,F_i(B)), H(B,F_i(A))) \\ &\leq \max_{i\in\{1,2,\cdots,m\}} \varphi_i(H(A,B), H(T_F(B),A), H(T_F(A),B)) \\ &= \bar{\varphi}(H(A,B), H(T_F(B),A), H(T_F(A),B)), \end{aligned}$$

where $\bar{\varphi}(t_1,t_2,t_3) := \max_{i\in\{1,2,\cdots,m\}} \varphi_i(t_1,t_2,t_3)$. The conclusion follows again by Rus's Theorem applied for T_F. □

It is an open question to prove a similar result to Theorem 4.1 or Theorem 4.2 for the multifractal operator T_F generated by an IMS $F = (F_1,...,F_m)$ of multivalued operators of Ćirić type.

Acknowledgment. This work benefits of the financial support of a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.

References

[1] Border, K., *Fixed Point Theorems with Applications to Economic and Game Theory*, Cambridge Univ. Press, London, 1985.

[2] Chifu, C., Petruşel, A., *Ćirić-type δ-contractions in metric spaces endowed with a graph*, J. Ineq. Appl. (to appear).

[3] Chifu, C., Petruşel, G., Bota, M., *Fixed points and strict fixed points for multivalued contractions of Reich type on metric spaces endowed with a graph*, Fixed Point Theory and Applications 2013, 2013:203, doi:10.1186/1687-1812-2013-203.

[4] Ćirić, L.B., *Generalized contraction and fixed point theorems*, Publ. Inst. Math., **12**(1971), 19-26.

[5] Ćirić, L.B., *Fixed points for generalized multi-valued contractions*, Math. Vesnik, **9**(24)(1972), 265-272.

[6] Ćirić, L.B., *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45**(1974), 267-273.

[7] Ćirić, L.B., *Contractive type non-self mappings on metric spaces of hyperbolic type*, J. Math. Anal. Appl., **317**(2006), 28-42.

[8] Dinevari, T., Frigon, M., *Fixed point results for multivalued contractions on a metric space with a graph*, J. Math. Anal. Appl.,**405**(2013), 507-517.

[9] Gwóźdź-Lukawska, G., Jachymski, J., *IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem*, J. Math. Anal. Appl., **356**(2009), 453-463.

[10] Hardy, G.E., Rogers, T.D., *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16**(1973), 201-206.

[11] Jachymski, J., *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136**(2008), 1359-1373.

[12] Nicolae, A., O'Regan, D., Petruşel, A., *Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph*, J. Georgian Math. Soc., **18**(2011), 307-327.

[13] Nieto, J.J., Pouso, R.L., Rodríguez-López, R., *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc., **135**(2007), 2505-2517.

[14] Petruşel, A., Petruşel, G., *Multivalued Picard operator,* J. Nonlinear Convex Anal., **13**(2012), 157-171.

[15] Petruşel, A., Rus, I.A., *Multivalued Picard and weakly Picard operators*, Proc. Sixth International Conference on Fixed Point Theory and Applications, Valencia, Spain, July 19-26, 2003 (E. Llorens Fuster, J. Garcia Falset, B. Sims-Eds.), Yokohama Publ., 2004, 207-226.

[16] Petruşel, A., Rus, I.A., *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134**(2006), 411-418.

[17] Petruşel, A., Rus, I.A., *Well-posedness of the fixed point problem for multivalued operators*, Applied Analysis and Differential Equations (O. Cârjă and I.I. Vrabie-Eds.), World Scientific 2007, pp. 295-306.

[18] Petruşel, A., Rus, I.A., Yao, J.C., *Well-posedness in the generalized sense of the fixed point problems for multivalued operators*, Taiwanese J. Math., **11**(2007), 903-914.

[19] Ran, A.C.M., Reurings, M.C., *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), 1435-1443.

[20] Reich, S., *Fixed point of contractive functions*, Boll. Un. Mat. Ital., **5**(1972), 26-42.

[21] Rhoades, B.E., *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226**(1977), 257-288.

[22] Rus, I.A., *Generalized Contractions and Applications*, Transilvania Press, Cluj-Napoca, 2001.

[23] Yuan, G.X.-Z. *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, New York, 1999.

Adrian Petruşel
Babeş-Bolyai University
Department of Mathematics
Kogalniceanu Street, 1, 400084
Cluj-Napoca, Romania
e-mail: `petrusel@math.ubbcluj.ro`