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Greediness of higher rank Haar wavelet bases in $L^p_w(\mathbb{R})$ spaces

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Abstract. We prove that higher rank Haar wavelet systems are greedy in $L^p_w(\mathbb{R})$, $1 < p < \infty$ if and only if $w \in A^N_p$.

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1. Introduction

Let $\mathcal{X} = \{x_n : n \in \mathbb{N}\}$ be a semi-normalized basis in a Banach space X. This means that $\{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis and is semi-normalized i.e. $0 < \inf_{n\in\mathbb{N}} \|x_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. For an element $x \in X$ we define the error of the best $m-$term approximation as follows

$$\sigma_m(x, \mathcal{X}) = \inf\{\|x - \sum_{n\in A} \alpha_n x_n\|\},$$

where the inf is taken over all subset $A \subset \mathbb{N}$ of cardinality at most m and all possible scalars α_n. The main question in approximation theory concerns the construction of efficient algorithms for m-term approximation. A computationally efficient method to produce m-term approximations, which has been widely investigated in recent years, is the so called greedy algorithm. For $x \in X$ with $x = \sum_{n=1}^{\infty} a_n x_n$ and $m \in \mathbb{N}$, consider a subset $G(m, x) \subset \mathbb{N}$ of cardinality m such that

$$\min_{n\in G(m,x)} |a_n| \geq \max_{n\in\mathbb{N}\setminus G(m,x)} |a_n|.$$

There is some ambiguity in the choice of the set $G(m, x)$, but our considerations do not depend on the particular choice. Then the m-th greedy approximation of x with respect to the basis $\mathcal{X}$ is defined as

$$\mathcal{G}_m(x, \mathcal{X}) = \sum_{n\in G(m,x)} a_n x_n.$$

Clearly, $\sigma_m(x,\mathcal{X}) \leq \|x-\mathcal{G}_m(x,\mathcal{X})\|$. The basis $\mathcal{X}$ is called greedy if there is a constant $C>0$, independent of m, such that for each $m \in \mathbb{N}$ and $x \in X$,

$$\|x-\mathcal{G}_m(x,\mathcal{X})\| \leq C\sigma_m(x,\mathcal{X}).$$

Wavelet systems are well known examples of greedy bases for many function and distribution spaces. Indeed, V. N. Temlyakov showed in [13] that the classical dyadic Haar system (and any wavelet system L^p-equivalent to it) is greedy in the Lebesgye spaces $L^p(\mathbb{R}^n)$ for $1<p<\infty$.

When wavelets have sufficient smoothness and decay, they are also greedy bases for the more general Sobolev and Tribel-Lizorkin classes (see [3],[5]). Some example of greedy bases are given in [13], [14]. In most cases these bases are greedy simply because they are symmetric (e.g. Riesz bases for a Hilbert space), or because they are equivalent to the dyadic Haar basis or to a subsequence of the Haar basis (see [7]). S. V. Konyagin and V. N. Temlyakov [8] gave a very useful abstract characterization of greedy bases in a Banach spaces X as those which are unconditional and democratic, the last meaning that for some constant $C>0$

$$\left\|\sum_{n\in A}\frac{x_n}{\|x_n\|}\right\| \leq C\left\|\sum_{n\in A'}\frac{x_n}{\|x_n\|}\right\|$$

holds for all finite sets of indices A, $A'\subset\mathbb{N}$ with the same cardinality.

The purpose of this paper is to study the efficiency of greedy algorithms with respect higher rank Haar wavelet system in the spaces $L^p_w(\mathbb{R})$. We recall that, as M. Izuki proved in [6], that the dyadic Haar wavelet system ($N=2$) is greedy in $L^p(\mathbb{R}^n)$ for $1<p<\infty$ if and only if $w\in A^2_p$. Characterization of almost greedy uniformly bounded orthonormal bases in rearrangement invariant Banach function spaces are given [2].

By an N-adic ($N\in\mathbb{N}$, $N\geq 2$) lattice $\mathcal{D}$ we mean the collection of all N-adic intervals, i. e. the collection of all intervals of the form $[jN^{-k},(j+1)N^{-k})$, $j,k\in\mathbb{Z}$. If I is an interval, we denote by $|I|$ its length, and by χ_I its characteristic function. If I is an N-adic interval $[jN^{-k},(j+1)N^{-k})$ then we denote by $I^{(l)}$ the "children" intervals of $I:[jN^{-k}+lN^{-(k+1)},jN^{-k}+(l+1)N^{-(k+1)})$, $l=0,1,\cdots,N-1$.

We denote by $L^2(\mathbb{R})$ the Hilbert space of square integrable (with respect to the Lebesgue measure) complex-valued functions on $\mathbb{R}$. We consider also $L^p_w(\mathbb{R})\,(1\leq p<\infty)$ spaces, where $w\in L^1_{loc}(\mathbb{R})$ is a positive function called a weight. The norm of a function $f:\mathbb{R}\to\mathbb{C}$ from the space $L^p_w(\mathbb{R})$ is

$$\|f\|_{L^p_w}=\left(\int_{\mathbb{R}}|f(x)|^p w(x)dx\right)^{1/p}.$$

Given a function f, we denote by $\langle f\rangle_I$ its average over the interval I,

$$\langle f\rangle_I=\frac{1}{|I|}\int_I f(x)dx.$$

We are concerned with a special class of weights, called A^N_p. We say that $w\in A^N_p$, $1<p<\infty$ if

$$A_w=\sup_{I\in\mathcal{D}}\langle w\rangle_I\langle w^{-1/(p-1)}\rangle_I^{p-1}<\infty.$$

We say that an $N \times N$ matrix is a Haar wavelet matrix of rank N if it is unitary and the elements of the first row are all equal to $1/\sqrt{N}$. Many classical examples of matrices used in mathematics and signal processing are Haar matrices of specific types. These include the discrete Fourier transform matrices, the discrete cosine transform matrices, Hadamard and Walsh matrices, Radmacher matrices, and Chebyshev matrices (see [12]).

Let $H = (g_{ki})_{k,i=0}^{N-1}$ be a $N \times N$ Haar matrix and $\varphi = \chi_{[0,1)}$. Define the functions

$$\psi^{(k)}(x) = \sqrt{N} \sum_{l=0}^{N-1} g_{kl}\varphi(Nx - l) \;\; k = 1, \cdots, N-1. \tag{1.1}$$

The collection of functions

$$\psi^{(k)}_{j,n} = N^{j/2}\psi^{(k)}(N^j x - n), \;\; j, n \in \mathbb{Z}, \;\; k = 1, \cdots, N-1$$

form an orthonormal basis of $L^2(\mathbb{R})$ (see [15]. Bellow we adopt the shorter notation $\psi^{(k)}_{j,n} = \psi^{(k)}_I$, where $I = [nN^{-j}, (n+1)N^{-j})$. The system $\mathcal{X} = \{\psi^{(k)}_I, I \in \mathcal{D}, k = 1, \cdots, N-1\}$, where the functions $\psi^{(k)}$, $k = 1, \cdots, N-1$ are defined by (1.1), is called the Haar wavelet system of rank N (corresponding to Haar matrix H). An important example of a higher rank Haar wavelet system is the system obtained by wavelet functions

$$\psi^{(k)}(x) = \sqrt{N} \sum_{l=0}^{N-1} e^{2\pi ikl/N}\varphi(Nx - l), \;\; k = 1, \cdots, N-1, \tag{1.2}$$

where φ is characteristic function of the interval $[0, 1)$.

Note that the wavelet system constructed by functions (1.2) became of interest in connection with some problems of p-adic (non-Archimedean) mathematical physics (see [10-11]).

For a Haar wavelet system $\mathcal{X}$ and $f \in L^1_{loc}(\mathbb{R})$, we define the Littlewood-Paley operator by

$$Pf(x) = \left(\sum_{k=1}^{N-1} \sum_{I \in \mathcal{D}} | < f, \psi^{(k)}_I > |^2 ||I|^{-1}\chi_I(x) \right)^{1/2},$$

where

$$< f, \psi^{(k)}_I >= \int_{\mathbb{R}} f(x)\overline{\psi^{(k)}_I}(x)dx.$$

The characterization of the spaces $L^p_w(\mathbb{R})$ $(w \in A_p, 1 < p < \infty)$ using higher rank Haar wavelet system $\mathcal{X}$ was given in [9].

Theorem 1.1. ([9]) *Let $\mathcal{X}$ be a Haar wavelet system of rank N and $1 < p < \infty$. The following conditions are equivalent: 1) The system $\mathcal{X}$ is unconditional basis of space $L^p_w(\mathbb{R})$; 2) There exist positive constants c and C such that $c\,\|f\|_{L^p_w} \le \|P(f)\|_{L^p_w} \le C\,\|f\|_{L^p_w}$ for all $f \in L^p_w(\mathbb{R})$; 3) $w \in A^N_p$.*

The purpose of this paper is to prove following theorem.

Theorem 1.2. *Let $\mathcal{X} = \{\psi_I^{(k)};\ I \in \mathcal{D},\ k = 1, \cdots, N-1\}$ be a Haar wavelet system of rank N. Suppose $w \in A_p^N$ $(1 < p < \infty)$. Then the system $\{\psi_I^{(k)}/\|\psi_I^{(k)}\|_p;\ I \in \mathcal{D},\ k = 1, \cdots, N-1\}$ forms a greedy basis in the space $L_w^p(\mathbb{R})$.*

2. Proof of Theorem 1.2

For simplicity we shall denote the *normalized characteristic function* of a set of indices $\Gamma \subset \{1, 2, \cdots, N-1\} \times \mathcal{D}$ by

$$\mathbf{1}_\Gamma = \sum_{(k,I)\in\Gamma} \frac{\psi_I^k}{\|\psi_I^k\|_p}.$$

Observe that $\mathcal{X}$ is democratic in $L_w^p(\mathbb{R})$ if and only if there exists a function $h : \mathbb{N} \to \mathbb{R}^+$ for which

$$\frac{1}{C}\, h(\mathrm{Card}(\Gamma)) \le \|\mathbf{1}_\Gamma\|_{L_w^p} \le C\, h(\mathrm{Card}(\Gamma)),\ \forall\, \Gamma \subset \{1, 2, \cdots, N-1\} \times \mathcal{D} \tag{2.1}$$

for some $C \ge 1$.

Observe that from Theorem 1.1 for a single element ψ_I^k

$$\|\psi_I^k\|_{L_w^p} \asymp \frac{w(I)^{1/p}}{|I|^{1/2}},$$

where $w(I) = \int_I w(x)dx$ and the constants involved in $\asymp$ are independent of ψ_I^k. Thus, using again the expression of the norm $\|\cdot\|_{L_w^p}$ it follows that

$$\|\mathbf{1}_\Gamma\|_{L_w^p} \asymp \left\| \left(\sum_{(k,I)\in\Gamma} \frac{\chi_I}{w(I)^{2/p}} \right)^{1/2} \right\|_{L_w^p} \asymp \left\| \left(\sum_{I\in\widetilde{\Gamma}} \frac{\chi_I}{w(I)^{2/p}} \right)^{1/2} \right\|_{L_w^p}, \tag{2.2}$$

where $\widetilde{\Gamma} = \{I :\ (k, I) \in \Gamma\}$. Note that $\mathrm{Card}(\widetilde{\Gamma}) \asymp \mathrm{Card}(\Gamma)$.

Given a finite set of intervals $\Gamma \subset \mathcal{D}$, we shall denote

$$S_\Gamma(x) = \left(\sum_{I\in\Gamma} \frac{\chi_I(x)}{w(I)^{2/p}} \right)^{1/2}.$$

We "linearize" the square function $S_\Gamma(x)$. Note that this lineralization procedure has been used by other authors in the context of $m-$term approximation (see e.g. [3-5]).

For every $x \in \cup_{I\in\Gamma} I$, we define I_x as the smallest (hence unique) interval in Γ containing x. It is clear that

$$S_\Gamma(x) \ge \frac{\chi_{I_x}(x)}{w(I_x)^{1/p}} \quad \forall\ x \in \cup_{I\in\Gamma} I. \tag{2.3}$$

We now show that the reverse inequality holds with some universal constant.

Note that if $w \in A_p^N$, then there exist C_1, $C_2 > 0$ and $\delta > 0$ such that

$$C_1(|A|/|I|)^p \le w(A)/w(I) \le C_2(|A|/|I|)^\delta \tag{2.4}$$

for all $I \in \mathcal{D}$ and all subsets $A \subset I$ (see [1]).

If we enlarge the sum to include all $N-$adic intervals containing I_x we have

$$S_\Gamma(x)^2 = \sum_{I\in\Gamma} \frac{\chi_I(x)}{w(I)^{2/p}} \leq \sum_{I\in\mathcal{D},\, I\supset I_x} \frac{1}{w(I)^{2/p}}.$$

If $I_x \subset I$ and $|I| = N^j|I_x|$, then by (2.4) we have, $w(I) \geq C_2^{-1} w(I_x) N^{-j\delta}$. Thus,

$$S_\Gamma(x)^2 \leq C \frac{\chi_{I_x}(x)}{w(I_x)^{2/p}}.$$

This and (2.3) show that $S_\Gamma(x) \asymp \frac{\chi_{I_x}(x)}{w(I_x)^{1/p}}$.

Observe that $S_\Gamma(x) \asymp S_{\Gamma_{\min}}(x)$, where $\Gamma_{\min}$ denotes the family of minimal intervals in Γ, that is, $\Gamma_{\min} = \{I_x : \; x \in \cup_{I\in\Gamma} I\}$. Note that the intervals in $\Gamma_{\min}$ are not necessarily pairwise disjoint.

Given a fixed $\Gamma \subset \mathcal{D}$, for any $I \in \Gamma$ we define the set $S(I)$ as the union of all intervals from Γ strictly contained in I. We define also the set $L(I) = I \backslash S(I)$. It is clear that $I \in \Gamma_{\min}$ if and only if $L(I) \neq \emptyset$, and moreover $\cup_{I\in\Gamma} I = \cup_{I\in\Gamma_{\min}} L(I)$, where the sets in the last union are pairwise disjoint. Therefore we can write

$$S_\Gamma(x) \asymp \sum_{I\in\Gamma_{\min}} \frac{\chi_{L(I)}(x)}{w(I)^{1/p}} \tag{2.5}$$

where in the last sum there is a most one non-zero term.

Denote by Γ_S the collection of all intervals I from Γ with property: $|S(I)| > (1-1/N)|I|$. Denote by Γ_L the collection of all intervals I from Γ with property: $|L(I)| \geq |I|/N$. Observe that $\Gamma_L \subset \Gamma_{\min}$. We have

$$(1-1/N)\mathrm{Card}(\Gamma) \leq \mathrm{Card}(\Gamma_L) \leq \mathrm{Card}(\Gamma_{\min}) \leq \mathrm{Card}(\Gamma), \;\; \forall\, \Gamma \subset \mathcal{D}. \tag{2.6}$$

Clearly $\mathrm{Card}(\Gamma_L) \leq \mathrm{Card}(\Gamma_{\min}) \leq \mathrm{Card}(\Gamma)$. Thus, we need to prove only the inequality from the left hand side of (2.6). Given $I \in \mathcal{D}$, we write $I^{(k)}$, $k = 1, \cdots, N$ for the N-adic intervals contained in I of size $N^{-1}|I|$. For $I \in \Gamma_S$ and $k = 1, \cdots, N$ let $I_0^{(k)}$ be the biggest interval from Γ with $I_0^{(k)} \subset I^{(k)}$. Note that the intervals $I_0^{(k)}$ exist for every $I \in \Gamma_S$; otherwise, if for some $k_0 \in \{1, \cdots, N\}$ there is no interval from Γ contained in $I^{(k_0)}$ we have $I^{(k_0)} \subset L(I)$ and then

$$|S(I)| \leq |I \backslash I^{(k_0)}| = (1-1/N)|I|,$$

contradicting the definition of Γ_S.

We claim that if $I, R \in \Gamma_S$ and $I \neq R$, then we necessarily have $I_0^{(k)} \neq R_0^{(l)}$ for all $1 \leq k, l \leq N$. This is trivially true if $I \cap R = \emptyset$. Without loss of generality we may assume $I \subset R$ and also $I \subset R^{(1)}$. It follows that $I_0^{(k)} \neq R_0^{(l)}$ for all $k = 1, \cdots, N$ and all $l = 2, \cdots, N$. Moreover, as $R_0^{(1)}$ is the biggest interval in Γ contained in $R^{(1)}$ and $I \subset R^1$ we have that $I \subset R_0^{(1)} \subset R^{(1)}$. Hence, for all $k = 1, \cdots, N$ we have $I_0^{(k)} \subset I \subset R_0^{(1)}$ and thus $I_0^{(k)} \neq R_0^{(1)}$. In short, to each $I \in \Gamma_S$ we have assigned N different intervals in Γ and these are not associated to any other interval in Γ_S. We conclude that $N\mathrm{Card}(\Gamma_S) \leq \mathrm{Card}(\Gamma)$. Consequently we have

$$\mathrm{Card}(\Gamma_L) = \mathrm{Card}(\Gamma_L) - \mathrm{Card}(\Gamma_S) \geq \mathrm{Card}(\Gamma) - \mathrm{Card}(\Gamma)/N = (1-1/N)\mathrm{Card}(\Gamma).$$

Note that $|I|/N \leq |L(I)| \leq |I|$ and by (2.4) we have $\|\chi_{L(I)}\|_{L^p_w} \asymp \|\chi_I\|_{L^p_w}$. Using this estimate we can write

$$\|S_\Gamma\|_{L^p_w} \asymp \left\| \sum_{I\in\Gamma_{\min}} \frac{\chi_{L(I)}(x)}{w(I)^{1/p}} \right\|_{L^p_w} \asymp (\mathrm{Card}(\Gamma_{\min}))^{1/p} \asymp (\mathrm{Card}(\Gamma))^{1/p}. \tag{2.7}$$

From (2.2), (2.7) one obtains the estimates (2.1), with $h(n) = n^{1/p}$. □

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