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Semilinear operator equations and systems with potential-type nonlinearities

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Abstract. The recent results of Precup [6] on the variational characterization of the fixed points of contraction-type operators are applied in this paper to semilinear operator equations and systems with linear parts given by positively defined operators, and nonlinearities of potential-type. Mihlin's variational theory is also involved. Applications are given to elliptic semilinear equations and systems.

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1. Introduction

In this paper, firstly we are concerning with semilinear operator equation of the type:

$$Au = J'(u), \tag{1.1}$$

where A is a positively defined linear operator and the nonlinear term is the Fréchet derivative of a functional J. Secondly we discuss the semilinear operator system

$$\begin{cases} A_1 u = J_{11}(u, v) \\ A_2 v = J_{22}(u, v), \end{cases} \tag{1.2}$$

associated to two positively defined linear operators A_1, A_2 and to two functionals J_1, J_2 where by $J_{11}(u, v), J_{22}(u, v)$ we mean the Fréchet derivatives of $J_1(., v), J_2(u, .)$, respectively. Our special interest in such kind of equations and systems is represented by semilinear elliptic equations of the type

$$-\Delta u = f(x, u), \tag{1.3}$$

and correspondingly, by the following elliptic system

$$\begin{cases} -\Delta u = f(x, u, v) \\ -\Delta v = g(x, u, v). \end{cases} \tag{1.4}$$

Recently in Precup [6], it was shown that the unique fixed point of a contraction T on a Hilbert space, in case that T has the variational form

$$Tu = u - E'(u),$$

minimizes the functional E. Also, the unique fixed point (u^*, v^*) of a Perov contraction $(T_1(u,v), T_2(u,v))$ with

$$\begin{cases} T_1(u,v) = u - E_{11}(u,v) \\ T_2(u,v) = u - E_{22}(u,v), \end{cases}$$

under some conditions, is a Nash-type equilibrium of the pair of functionals (E_1, E_2), that is:

$$\begin{aligned} E_1(u^*, v^*) &= \min_u E_1(u, v^*) \\ E_2(u^*, v^*) &= \min_v E_2(u^*, v). \end{aligned}$$

The goal of this paper is to apply the above results to the semilinear equation (1.1) and to the system (1.2). To this aim, we fully exploit Mihlin's theory [4], on linear operator equations. In particular we shall derive variational characterizations of the weak solutions of the Dirichlet problem for the equation (1.3) and the system (1.4).

The paper is organized as follows: Section 2 is devoted to preliminaires, and Section 3 contains the main results. More exactly, in Section 3.1 we discuss the case of the equation (1.1), while in Section 3.2 we obtain theoretical results for the system (1.2). Furthermore, in Section 3.3 we apply our first result to an elliptic equation of the type (1.3) and in Section 3.4 we apply our second result to the system (1.4).

2. Preliminaries

2.1. Variational theory of linear operator equations

In this section we sketch Mihlin's variational theory [4] (see also [5]) for linear equations associated to positively defined operators. Let H be a Hilbert space with the inner product denoted by $(.,.)_H$ and the norm $\|.\|_H$. Let $A : D(A) \to H$ be a symmetric, linear and densely defined operator. The operator A is said to be *positively defined*, if for some $\gamma > 0$,

$$(Au, u)_H \geq \gamma^2 \|u\|_H^2, \tag{2.1}$$

for every $u \in D(A)$. For such a linear operator, we endow the linear subspace $D(A)$ of H with the bilinear functional:

$$(u, v)_{H_A} = (Au, v)_H,$$

for every $u, v \in D(A)$. One can verify that $(.,.)_{H_A}$ is an inner product. Consequently, $D(A)$ endowed with the inner product $(.,.)_{H_A}$ is a pre-hilbertian space. This space may not be complete. The completion H_A of $(D(A), (.,.)_{H_A})$ is called the *energetic space* of A. By the construction, $D(A) \subset H_A \subset H$ with dense inclusions. We use the same symbol $(.,.)_{H_A}$ to denote the inner product of H_A. The corresponding norm

$$\|u\|_{H_A} = \sqrt{(u,u)_{H_A}},$$

is called the *energetic norm* associated to A.

If $u \in D(A)$, then $\|u\|_{H_A} = \sqrt{(Au,u)_H}$ and in view of (2.1) one has the *Poincaré inequality*

$$\|u\|_H \le \frac{1}{\gamma}\|u\|_{H_A}, \tag{2.2}$$

for every $u \in D(A)$. By density the above inequality extends to H_A. We use this inequality in order to identify the elements of H_A with elements from H.

Let H'_A be the dual space of H_A. If we identify H with its dual, then from $H_A \subset H$ we have $H \subset H'_A$.

We attach to the operator A the following problem:

$$Au = f,\ u \in H_A, \tag{2.3}$$

where $f \in H'_A$. By a *weak solution* of (2.3) we mean an element $u \in H_A$ with:

$$(u,v)_{H_A} = (f,v) \tag{2.4}$$

for every $v \in H_A$, where the notation (f,v) stands for the value of the functional f on the element v. In case that $f \in H$, then $(f,v) = (f,v)_H$. Notice that if $u \in D(A)$, then (2.4) becomes $(Au,v)_H = (f,v)$.

Theorem 2.1. *For every $f \in H'_A$ there exists a unique weak solution $u \in H_A$ of the problem* (2.3)*.*

Proof. Consider the functional $F : H_A \to \mathbb{R}$ given by $F(v) = (f,v)$, for $v \in H_A$. Obviously, F is linear. Also

$$|F(v)| = |(f,v)| \le \|f\|_{H'_A}\|v\|_{H_A}.$$

Hence, F is a linear and continuous functional. By Riesz's theorem, there exists a unique $u \in H_A$ such that $F(v) = (u,v)_{H_A}$ for all $v \in H_A$. Clearly, u is the unique weak solution of (2.3). □

This result allows us to define the solution operator A^{-1} associated to operator A. Thus

$$A^{-1} : H'_A \to H_A\ ,$$
$$A^{-1}f = u, \tag{2.5}$$

where u is the unique weak solution of problem (2.3). The operator A^{-1} is well defined by the above theorem and one has

$$(A^{-1}f,v)_{H_A} = (f,v) \tag{2.6}$$

for all $v \in H_A$ and $f \in H'_A$.

The operator A^{-1} is an isometry between H'_A and H_A, i.e:

$$\|A^{-1}f\|_{H_A} = \|f\|_{H'_A} \tag{2.7}$$

for all $f \in H'_A$. Indeed, in order to show that the inequality $\|A^{-1}f\|_{H_A} \le \|f\|_{H'_A}$ holds, we replace v with $A^{-1}f$ in (2.6), to obtain $(A^{-1}f, A^{-1}f)_{H_A} = (f, A^{-1}f)$. Therefore

$$\|A^{-1}f\|^2_{H_A} = (f, A^{-1}f) \le \|f\|_{H'_A}\|A^{-1}f\|_{H_A}.$$

Hence, $\|A^{-1}f\|_{H_A} \le \|f\|_{H'_A}$. On the other hand, we have that

$$\begin{aligned}\|f\|_{H'_A} &= \sup_{\substack{v\in H_A\\ v\neq 0}} \frac{|(f,v)|}{\|v\|_{H_A}} = \sup_{\substack{v\in H_A\\ v\neq 0}} \frac{|(A^{-1}f,v)_{H_A}|}{\|v\|_{H_A}}\\ &\le \sup_{\substack{v\in H_A\\ v\neq 0}} \frac{\|A^{-1}f\|_{H_A}\|v\|_{H_A}}{\|v\|_{H_A}} = \|A^{-1}f\|_{H_A}.\end{aligned}$$

From the above inequalities, (2.7) follows.

We also mention Poincaré's inequality for the inclusion $H \subset H'_A$,

$$\|u\|_{H'_A} \le \frac{1}{\gamma}\|u\|_H, \qquad u \in H. \tag{2.8}$$

This can be proved as follows:

$$\|u\|_{H'_A} = \sup_{\substack{v\in H_A\\ v\neq 0}} \frac{|(u,v)_H|}{\|v\|_{H_A}} \le \sup_{\substack{v\in H_A\\ v\neq 0}} \frac{\|u\|_H\|v\|_H}{\|v\|_{H_A}}.$$

Now, using (2.2) we have:

$$\sup_{\substack{v\in H_A\\ v\neq 0}} \frac{\|u\|_H\|v\|_H}{\|v\|_{H_A}} \le \frac{1}{\gamma}\|u\|_H.$$

Therefore (2.8) holds.

Using (2.7) and (2.8) we see that if $f \in H$, then

$$\|A^{-1}f\|_{H_A} = \|f\|_{H'_A} \le \frac{1}{\gamma}\|f\|_H. \tag{2.9}$$

For a fixed $f \in H'_A$, one considers the functional:

$$\begin{gathered}E : H_A \to \mathbb{R},\\ E(u) = \frac{1}{2}\|u\|^2{}_{H_A} - (f,u).\end{gathered}$$

The functional E is Fréchet differentiable and for any $u, v \in H_A$, we have:

$$(E'(u),v) = \lim_{t\to 0}\frac{E(u+tv)-E(u)}{t} = (u,v)_{H_A} - (f,v) = (u - A^{-1}f, v)_{H_A}. \tag{2.10}$$

Now (2.10) shows that $u \in H_A$ is a weak solution of (2.3) if and only if u is a critical point of E, i.e $E'(u) = 0$.

2.2. Variational properties for contraction-type operators

In this section and in the next one, we summarize the abstract results from the paper Precup [6], concerning the variational characterization of the fixed points of contraction-type operators. The first result refers to usual contractions on a Hilbert space.

Theorem 2.2. *([6]) Let X be a Hilbert space and $T : X \to X$ be a contraction with the unique fixed point u^* (guaranteed by Banach contraction theorem). If there exists a C^1 functional E bounded from below such that*

$$E'(u) = u - T(u) \tag{2.11}$$

for all $u \in X$, then u^* minimizes the functional E, i.e

$$E(u^*) = \inf_X E.$$

2.3. Nash-type equilibrium for Perov contractions

The next result from [6] is about systems of the type

$$\begin{cases} u = T_1(u,v) \\ v = T_2(u,v), \end{cases} \tag{2.12}$$

where $u \in X_1$, $v \in X_2$. In this case, instead of Lipschitz constants in the definition of contractions, we use matrices.

A square matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}^n)$ with nonnegative entries is said to be *convergent to zero* if

$$M^k \to 0, \text{ as } k \to \infty.$$

There are known the following characterizations of the convergent to zero matrices (see, e.g [7], [2]).

The following statements are equivalents:
(i) M is a matrix that is convergent to zero;
(ii) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + \ldots$ (where I stands for the unit matrix of the same order as M);
(iii) the eigenvalues of M are located inside the unit disc of the complex plane;
(iv) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative entries.

Refering to the system (2.12), we assume that $(X_i, |.|_i)$, $i = 1, 2$, are Hilbert spaces identified to their duals and we denote by $X = X_1 \times X_2$. Also, assume that each equation of the system has a variational form, i.e. that there exist the continuous functionals $E_1, E_2 : X \to \mathbf{R}$ such that $E_1(., v)$ is Fréchet differentiable for every $v \in X_2$, $E_2(u, .)$ is Fréchet differentiable for every $u \in X_1$, and

$$\begin{cases} E_{11}(u,v) = u - T_1(u,v) \\ E_{22}(u,v) = v - T_2(u,v). \end{cases} \tag{2.13}$$

Here $E_{11}(., v), E_{22}(u, .)$ are the Fréchet derivatives of $E_1(., v)$ and $E_2(u, .)$, respectively.

We say that the operator $T : X \to X$, $T(u,v) = (T_1(u,v), T_2(u,v))$ is a *Perov contraction* if there exists a matrix $M = [m_{ij}] \in \mathcal{M}_{2,2}(\mathbf{R}_+)$ which is convergent to zero such that the following matricial Lipschitz condition is satisfied

$$\begin{bmatrix} |T_1(u,v) - T_1(\overline{u},\overline{v})|_1 \\ |T_2(u,v) - T_2(\overline{u},\overline{v})|_2 \end{bmatrix} \le M \begin{bmatrix} |u - \overline{u}|_1 \\ |v - \overline{v}|_2 \end{bmatrix} \tag{2.14}$$

for every $u, \overline{u} \in X_1$ and $v, \overline{v} \in X_2$.

The next theorem gives us a variational characterization of the unique fixed point of a Perov contraction.

Theorem 2.3. *([6]) Assume that the above conditions are satisfied. In addition assume that $E_1(.,v)$ and $E_2(u,.)$ are bounded from below for every $u \in X_1$, $v \in X_2$, and that there are R, $c > 0$ such that one of the following conditions holds:*

$$\begin{aligned} E_1(u,v) &\geq \inf_{X_1} E_1(.,v) + c \quad \text{for } |u|_1 \geq R \text{ and } v \in X_2, \\ E_2(u,v) &\geq \inf_{X_2} E_2(u,.) + c \quad \text{for } |v|_2 \geq R \text{ and } u \in X_1. \end{aligned} \tag{2.15}$$

Then the unique fixed point (u^*, v^*) of T (guaranteed by Perov's fixed point theorem) is a Nash-type equilibrium of the pair of functionals (E_1, E_2), i.e.

$$\begin{aligned} E_1(u^*, v^*) &= \inf_{X_1} E_1(., v^*) \\ E_2(u^*, v^*) &= \inf_{X_2} E_2(u^*, .). \end{aligned}$$

3. Main results

The main results of the paper are concerning with variational properties of the solutions of semilinear equations having the form

$$Au = J'(u),$$

with a positively defined linear operator A, and of semilinear systems of the type:

$$\begin{cases} A_1 u = J_{11}(u,v) \\ A_2 v = J_{22}(u,v). \end{cases}$$

We shall benefit of Mihlin's variational theory for linear operator equations and we shall apply the general results presented in Sections 2.2 and 2.3.

3.1. Semilinear operator equations with potential-type nonlinearities

First we consider the case of semilinear equations.

Let A be a symmetric, linear and densely defined operator as in Section 2.1 and let $J : H \to \mathbb{R}$ be a $C^1(H)$ functional. We look for weak solutions $u \in H_A$ for the semilinear equation

$$Au = J'(u). \tag{3.1}$$

This equation is equivalent to

$$u = A^{-1} J'(u), \tag{3.2}$$

this is, to the fixed point equation:

$$u = T(u), \tag{3.3}$$

where $T := A^{-1} J'$. We associate to the equation (3.1) the functional

$$E : H_A \to \mathbb{R}, \quad E(u) = \frac{1}{2} \|u\|^2{}_{H_A} - J(u). \tag{3.4}$$

The main result of this section is the following theorem.

Theorem 3.1. *Under the above conditions on A and J, if in addition $J' : H \to H$ satisfies the following conditions:*

$$\|J'(u) - J'(v)\|_H \le \alpha \|u - v\|_H \tag{3.5}$$

for all $u, v \in H$, and

$$J(u) \le a\|u\|^2{}_{H_A} + b, \tag{3.6}$$

for all $u \in H_A$, some $\alpha < \gamma^2$, $a \le \dfrac{1}{2}$ and $b \ge 0$, then there is a unique weak solution $u^* \in H_A$ of equation (3.1) such that

$$E(u^*) = \inf_{H_A} E.$$

Proof. We apply Theorem 2.2 to $X = H_A$, to the operator $T : H_A \to H_A$, $T := A^{-1}J'$ and to the functional given by (3.4). Since J is of class C^1 on H, it follows that E is of class C^1 on H_A. Indeed,

$$\begin{aligned}(E'(u), v) &= \lim_{t\to 0} \frac{E(u + tv) - E(u)}{t} \\ &= (u, v)_{H_A} - (J'(u), v) \\ &= (u - A^{-1}J'(u), v)_{H_A}.\end{aligned}$$

Therefore, if we identify H'_A to H_A, we have

$$E'(u) = u - T(u).$$

Hence the assumption (2.11) holds. Using (3.6) and $a \le \dfrac{1}{2}$, we obtain

$$E(u) = \frac{1}{2}\|u\|^2_{H_A} - J(u) \ge \left(\frac{1}{2} - a\right)\|u\|^2{}_{H_A} - b \ge -b > -\infty,$$

for all $u \in H_A$. Thus, E is bounded from below. It remains to show that T is a contraction on H_A. Using the hypothesis (3.5) and the Poincaré's inequality (2.2), for every $u, v \in H_A$, we have

$$\begin{aligned}\|J'(u) - J'(v)\|_H &\le \alpha\|u - v\|_H \\ &\le \frac{\alpha}{\gamma}\|u - v\|_{H_A}.\end{aligned}$$

Since A^{-1} is an isometry between H'_A and H_A, we then deduce that

$$\begin{aligned}\|T(u) - T(v)\|_{H_A} &= \|A^{-1}(J'(u) - J'(v))\|_{H_A} \\ &= \|J'(u) - J'(v)\|_{H'_A} \\ &\le \frac{1}{\gamma}\|J'(u) - J'(v)\|_H \\ &\le \frac{\alpha}{\gamma^2}\|u - v\|_{H_A}.\end{aligned}$$

This shows that T is a contraction on H_A, since α was assumed less than γ^2. Thus Theorem 2.2 applies and the result follows. $\square$

3.2. Semilinear operator systems with potential-type nonlinearities

This subsection is devoted to the study of systems of the type:

$$\begin{cases} A_1 u = J_{11}(u,v) \\ A_2 v = J_{22}(u,v), \end{cases} \tag{3.7}$$

where A_1, A_2 are symmetric, linear and densely defined operators on two Hilbert spaces H_1, H_2. Denote $H = H_1 \times H_2$. Also, $J_1, J_2 : H \to \mathbb{R}$ are two $C^1(H)$ functionals and by $J_{11}(u,v)$ we mean the partial derivative of J_1 with respect to u and by $J_{22}(u,v)$ the partial derivative of J_2 with respect to v. We express the above system as a fixed point equation of the type

$$w = T(w) \tag{3.8}$$

for the nonlinear operator $T = (T_1, T_2)$, where $w = (u,v)$, $T_1 : H_{A_1} \times H_{A_2} \to H_{A_1}$, $T_1(u,v) = A_1^{-1} J_{11}$ and $T_2 : H_{A_1} \times H_{A_2} \to H_{A_2}$, $T_2(u,v) = A_2^{-1} J_{22}$. Hence (3.8) can be rewritten explicitly as follows

$$\begin{cases} u = T_1(u,v) \\ v = T_2(u,v). \end{cases} \tag{3.9}$$

This vectorial structure of (3.8) allows the two terms T_1 and T_2 to behave differently one from another and also with respect to the two variables. Also, this requires the use of matrices instead of constants, when Lipschitz conditions are imposed to T_1 and T_2. Each component equation of (3.9) has a variational form. We associate to the equations of (3.9) the functionals $E_1, E_2 : H_{A_1} \times H_{A_2} \to \mathbb{R}$ defined by

$$\begin{aligned} E_1(u,v) &= \frac{1}{2}\|u\|^2_{H_{A_1}} - J_1(u,v) \\ E_2(u,v) &= \frac{1}{2}\|v\|^2_{H_{A_2}} - J_2(u,v). \end{aligned} \tag{3.10}$$

One has

$$\begin{aligned} E_{11}(u,v) &= u - T_1(u,v) \\ E_{22}(u,v) &= v - T_2(u,v), \end{aligned} \tag{3.11}$$

where $E_{11}(.,v), E_{22}(u,.)$ are the Fréchet derivatives of $E_1(.,v)$ and $E_2(u,.)$, respectively.

The main result of this subsection is the following theorem.

Theorem 3.2. *Let the above conditions on A_1, A_2 and J_1, J_2 hold. In addition assume that $J_{11} : H_1 \times H_2 \to H_1$ and $J_{22} : H_1 \times H_2 \to H_2$ satisfy the following conditions:*
(i) there exist $m_{ij} \in \mathbb{R}_+$ $(i,j = 1,2)$ such that

$$\begin{aligned} \|J_{11}(u,v) - J_{11}(\bar{u},\bar{v})\|_{H_1} &\le m_{11}\|u-\bar{u}\|_{H_1} + m_{12}\|v-\bar{v}\|_{H_2} \\ \|J_{22}(u,v) - J_{22}(\bar{u},\bar{v})\|_{H_2} &\le m_{21}\|u-\bar{u}\|_{H_1} + m_{22}\|v-\bar{v}\|_{H_2} \end{aligned} \tag{3.12}$$

for all $u, \bar{u} \in H_1$ *and* $v, \bar{v} \in H_2$*, and the matrix*

$$M = \begin{bmatrix} \dfrac{m_{11}}{\gamma_1^2} & \dfrac{m_{12}}{\gamma_1^2} \\ \dfrac{m_{21}}{\gamma_2^2} & \dfrac{m_{22}}{\gamma_2^2} \end{bmatrix} \tag{3.13}$$

is convergent to zero;
(ii)

$$\begin{aligned} J_1(u,v) &\le a_1 \|u\|^2_{H_{A_1}} + b_1 \\ J_2(u,v) &\le a_2 \|v\|^2_{H_{A_2}} + b_2 \end{aligned} \tag{3.14}$$

for all $u \in H_{A_1}$*,* $v \in H_{A_2}$ *and some* $a_1,\, a_2 \le \dfrac{1}{2}$ *and* $b_1,\, b_2 \ge 0$*;*
(iii) there are $R,\, c > 0$ *such that one of the following conditions holds:*

$$\begin{aligned} E_1(u,v) &\ge \inf_{H_{A_1}} E_1(.,v) + c \quad \text{for } \|u\|_{H_{A_1}} \ge R \text{ and } v \in H_{A_2}, \\ E_2(u,v) &\ge \inf_{H_{A_2}} E_2(u,.) + c \quad \text{for } \|v\|_{H_{A_2}} \ge R \text{ and } u \in H_{A_1}. \end{aligned} \tag{3.15}$$

Then there is a unique solution $(u^*, v^*) \in H_{A_1} \times H_{A_2}$ *of the system* (3.7) *which is a Nash-type equilibrium of the pair of functionals* (E_1, E_2)*, i.e:*

$$\begin{aligned} E_1(u^*,v^*) &= \inf_{H_{A_1}} E_1(.,v^*) \\ E_2(u^*,v^*) &= \inf_{H_{A_2}} E_2(u^*,.). \end{aligned} \tag{3.16}$$

Proof. We apply the Theorem 2.3 to $X_1 = H_{A_1}$, and $X_2 = H_{A_2}$. Using (3.12) we show that T is a Perov contraction. Indeed, for $(u,v) \in X$ we have

$$\begin{aligned} \|T_1(u,v) - T_1(\bar{u},\bar{v})\|_{H_{A_1}} &= \|A_1^{-1}(J_{11}(u,v) - J_{11}(\bar{u},\bar{v}))\|_{H_{A_1}} \\ &= \|J_{11}(u,v) - J_{11}(\bar{u},\bar{v})\|_{H'_{A_1}} \\ &\le \frac{1}{\gamma_1}\|J_{11}(u,v) - J_{11}(\bar{u},\bar{v})\|_{H_1} \\ &\le \frac{m_{11}}{\gamma_1}\|u-\bar{u}\|_{H_1} + \frac{m_{12}}{\gamma_1}\|v-\bar{v}\|_{H_2} \\ &\le \frac{m_{11}}{\gamma_1^2}\|u-\bar{u}\|_{H_{A_1}} + \frac{m_{12}}{\gamma_1^2}\|v-\bar{v}\|_{H_{A_2}}. \end{aligned}$$

A similar inequality holds for T_2. Using (3.14) and $a_1, a_2 \le \dfrac{1}{2}$ we deduce that

$$E_1(u,v) = \frac{1}{2}\|u\|^2_{H_{A_1}} - J_1(u,v) \ge \left(\frac{1}{2} - a_1\right)\|u\|^2{}_{H_{A_1}} - b_1 \ge -b_1 > -\infty.$$

Thus, E_1 is bounded from below. Analogously, E_2 is bounded from below. Thus Theorem 2.3 is applicable and the result yields. □

3.3. Application to elliptic equations

In this subsection we present an application of Theorem 3.1 to elliptic equations. More exactly, we deal with the Dirichlet problem:

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{3.17}$$

Here Ω is a bounded open subset of $\mathbb{R}^n$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and Δ is the Laplacian. In this specific case $H = L^2(\Omega)$ and $A = -\Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Also, $H_A = H_0^1(\Omega)$ with the inner product

$$(u,v)_{H_0^1} = \int_\Omega \nabla u \nabla v \mathrm{d}x$$

and the norm

$$\|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

The functional $J : L^2(\Omega) \to \mathbb{R}$ is given by

$$J(u) = \int_\Omega F(x,u(x))dx,$$

where $F(x,t) = \displaystyle\int_0^t f(x,s)ds$. Also $\gamma = \sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the Dirichlet problem for $-\Delta$ (see, e.g [8], [1], [3]). Hence the energy functional associated to (3.17) is the following one:

$$E : H_0^1(\Omega) \to \mathbb{R},$$
$$E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(x,u(x))\right) \mathrm{d}x,$$

Problem (3.17) is equivalent to the fixed point equation:

$$u = (-\Delta)^{-1} N_f(u), \tag{3.18}$$

where N_f is the Nemytskii superposition operator assumed to act from $L^2(\Omega)$ to itself, $N_f(u)(x) = f(x,u(x))$ (see, e.g [8], [9]). Notice that the functional J is C^1 on $L^2(\Omega)$, $J' = N_f$ and

$$E'(u) = u - (-\Delta)^{-1} N_f(u).$$

Theorem 3.3. *Assume that the folllowing conditions are satisfied:*
(i) f satisfies the Carathéodory conditions, i.e
$f(.,y) : \Omega \to \mathbb{R}$ is measurable for each $y \in \mathbb{R}$ and $f(x,.) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e $x \in \Omega$;
(ii) $f(.,0) = 0$ on Ω;
(iii) exists $\alpha \in [0, \lambda_1)$ such that

$$|f(x,u) - f(x,\bar{u})| \leq \alpha |u - \bar{u}|$$

for all $u, \bar{u} \in \mathbb{R}$ and a.e $x \in \Omega$.
Then (3.17) has a unique weak solution $u^* \in H_0^1(\Omega)$ and

$$E(u^*) = \inf_{H_0^1(\Omega)} E.$$

Proof. We shall apply Theorem 3.1. From (*iii*) we deduce that

$$\|N_f(u) - N_f(v)\|_{L^2} \leq \alpha \|u - v\|_{L^2}$$

for $u, v \in L^2(\Omega)$. Hence (3.5) holds. Also, since

$$|f(x,t)| = |f(x,t) - f(x,0)| \leq \alpha |t|,$$

for $u \in H_0^1(\Omega)$, one has

$$\begin{aligned}
|J(u)| &\leq \int_\Omega |F(x,u(x))|\mathrm{d}x \leq \int_\Omega \left| \int_0^{u(x)} f(x,s)\mathrm{d}s \right| \mathrm{d}x \\
&\leq \int_\Omega \left| \int_0^{u(x)} |f(x,s)|\mathrm{d}s \right| \mathrm{d}x \leq \int_\Omega \left| \int_0^{u(x)} \alpha |s| \mathrm{d}s \right| \mathrm{d}x \\
&= \frac{\alpha}{2} \int_\Omega u(x)^2 \mathrm{d}x = \frac{\alpha}{2} \|u\|_{L^2}^2 \leq \frac{\alpha}{2\lambda_1} \|u\|_{H_0^1}^2.
\end{aligned}$$

Therefore (3.6) holds with $a = \dfrac{\alpha}{2\lambda_1} \leq \dfrac{1}{2}$ and $b = 0$. Thus Theorem 3.1 can be applied and the result follows. $\square$

3.4. Application to elliptic systems

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$. We consider the following system:

$$\begin{cases} -\Delta u = f(x,u,v) & \text{in } \Omega \\ -\Delta v = g(x,u,v) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \\ \quad v = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{3.19}$$

This problem is equivalent to the system:

$$\begin{cases} u = (-\Delta)^{-1} f(.,u,v) \\ v = (-\Delta)^{-1} g(.,u,v). \end{cases}$$

Also a pair $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of (3.19) if and only if

$$\begin{cases} E_{11}(u,v) = 0 \\ E_{22}(u,v) = 0, \end{cases} \tag{3.20}$$

where $E_1, E_2 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ are defined by

$$\begin{aligned}
E_1(u,v) &= \frac{1}{2}\|u\|_{H_0^1}^2 - \int_\Omega F(x,u(x),v(x))\mathrm{d}x \qquad (3.21) \\
E_2(u,v) &= \frac{1}{2}\|v\|_{H_0^1}^2 - \int_\Omega G(x,u(x),v(x))\mathrm{d}x,
\end{aligned}$$

and

$$F(x,t,s)=\int_0^t f(x,\tau,s)\mathrm{d}\tau,\quad G(x,t,s)=\int_0^s g(x,t,\tau)\mathrm{d}\tau. \tag{3.22}$$

The functionals $E_1(.,v)$ and $E_2(u,.)$ are C^1 for any fixed u and v, respectively, and

$$\begin{aligned} E_{11}(u,v) &= u-(-\Delta)^{-1}f(.,u,v) \\ E_{22}(u,v) &= v-(-\Delta)^{-1}g(.,u,v). \end{aligned} \tag{3.23}$$

Here again $E_{11}(.,v), E_{22}(u,.)$ are the Fréchet derivatives of $E_1(.,v)$ and $E_2(u,.)$, respectively.

We shall say that a function $H:\Omega\times\mathbb{R}\to\mathbb{R}$ is of *coercive type* if the functional $\Phi: H_0^1(\Omega)\to\mathbb{R}$,

$$\Phi(v)=\frac{1}{2}\|v\|^2_{H_0^1}-\int_\Omega H(x,v(x))\mathrm{d}x \tag{3.24}$$

is coercive, i.e $\Phi(v)\to+\infty$ as $\|v\|_{H_0^1}\to\infty$.

The main result of this subsection is the following theorem.

Theorem 3.4. *Let $f,g:\Omega\times\mathbb{R}^2\to\mathbb{R}$, $f=f(x,y,z)$, $g=g(x,y,z)$ satisfy the Carathédory conditions, i.e $f(.,y,z)$, $g(.,y,z)$ are measurable for each $(y,z)\in\mathbb{R}^2$ and $f(x,.)$, $g(x,.)$ are continuous for a.e $x\in\Omega$. Assume that $f(.,0,0)$, $g(.,0,0)\in L^2(\Omega)$ and that the following conditions hold:*
(i) there exist $m_{ij}\in\mathbb{R}_+$ $(i,j=1,2)$ with:

$$\begin{cases} |f(x,u,v)-f(x,\bar{u},\bar{v})|\le m_{11}|u-\bar{u}|+m_{12}|v-\bar{v}| \\ |g(x,u,v)-g(x,\bar{u},\bar{v})|\le m_{21}|u-\bar{u}|+m_{22}|v-\bar{v}|, \end{cases} \tag{3.25}$$

for all $u,\bar{u},v,\bar{v}\in\mathbb{R}$ and a.e $x\in\Omega$;
(ii) there exist $H,H_1:\Omega\times\mathbb{R}\to\mathbb{R}$ with

$$H_1(x,v)\le G(x,u,v)\le H(x,v), \tag{3.26}$$

for all $u,v\in\mathbb{R}$ and a.e. $x\in\Omega$, where H and H_1 are of coercive type.

If the matrix

$$M=\frac{1}{\lambda_1}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix} \tag{3.27}$$

is convergent to zero, then (3.19) *has a unique solution $(u^*,v^*)\in H_0^1(\Omega)\times H_0^1(\Omega)$ which is a Nash-type equilibrium of the pair of energy functionals (E_1,E_2) associated to the problem* (3.19).

Proof. We shall apply Theorem 3.2. Here $H_1=H_2=L^2(\Omega)$, $A_1=A_2=-\Delta$ and $J_1,J_2:H\to\mathbb{R}$ are given by

$$J_1(u,v)=\int_\Omega F(x,u(x),v(x))\mathrm{d}x,\quad J_2(u,v)=\int_\Omega G(x,u(x),v(x))\mathrm{d}x.$$

Also $\gamma_1 = \gamma_2 = \sqrt{\lambda_1}$. Using (3.25), in the same way as for a single equation, we have that $E_1(.,v)$ and $E_2(u,.)$ are bounded from below for any fixed u and v. In addition, we use the second inequality from (3.26) to obtain:

$$\begin{aligned} E_2(u,v) &= \frac{1}{2}\|v\|_{H_0^1}^2 - \int\limits_{\Omega} G(x,u(x),v(x))\mathrm{d}x \\ &\geq \frac{1}{2}\|v\|_{H_0^1}^2 - \int_{\Omega} H(x,v(x))\mathrm{d}x =: \Phi(v). \end{aligned}$$

Since H is of coercive type, Φ is bounded from below. Hence

$$E_2(u,v) \geq \Phi(v) \geq c > -\infty,$$

for all $u, v \in H_0^1(\Omega)$, that is $E_2(.,v)$ is bounded from below uniformly with respect to v. Denote

$$\Phi_1(v) = \frac{1}{2}\|v\|_{H_0^1}^2 - \int_{\Omega} H_1(x,v(x))\mathrm{d}x.$$

Since Φ is coercive, for each $\lambda > 0$, there is R_λ such that $\Phi(v) \geq \lambda$ for $\|v\|_{H_0^1} \geq R_\lambda$. Take $c > 0$ and $\lambda = \inf \Phi_1 + c$. Then for $\|v\|_{H_0^1} \geq R_\lambda$ and any $u \in H_0^1(\Omega)$ we have

$$E_2(u,v) \geq \Phi(v) \geq \inf \Phi_1 + c.$$

From the first inequality of (3.26), we have $\Phi_1(v) \geq E_2(u,v)$. It follows that

$$E_2(u,v) \geq \inf E_2(u,.) + c$$

for $\|v\|_{H_0^1} \geq R_\lambda$ and all $u \in H_0^1(\Omega)$. This shows that E_2 satisfies the condition (3.15). Furthermore,

$$\begin{aligned} \|J_{11}(u,v) - J_{11}(\bar{u},\bar{v})\|_{L^2} &= \|f(.,u,v) - f(.,\bar{u},\bar{v})\|_{L^2} \\ &\leq m_{11}\|u-\bar{u}\|_{L_2} + m_{12}\|v-\bar{v}\|_{L_2}, \end{aligned}$$

and similarly

$$\begin{aligned} \|J_{22}(u,v) - J_{22}(\bar{u},\bar{v})\|_{L^2} &= \|g(.,u,v) - g(.,\bar{u},\bar{v})\|_{L^2} \\ &\leq m_{21}\|u-\bar{u}\|_{L_2} + m_{22}\|v-\bar{v}\|_{L_2}. \end{aligned}$$

Therefore the hypothesis of Theorem 3.2 are fulfilled and the result follows. □

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