Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 2, 191–198

Differential inequalities and criteria for starlike and convex functions

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Abstract. We, here, study a differential inequality involving a multiplier transformation. In particular, we obtain certain new criteria for starlikeness and convexity of normalized analytic functions. We also show that our results generalize some known results.

Mathematics Subject Classification (2010): 30C80, 30C45.

Keywords: Multiplier transformation, analytic function, starlike function, convex function.

1. Introduction

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ p \in \mathbb{N} = \{1, 2, 3, \cdots\},$$

analytic and multivalent in the open unit disk $\mathbb{E}$. Note that $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n, \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,\ (\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

The operator $I_p(n,\lambda)$ has been recently studied by Aghalary et al. [1]. $I_1(n,0)$ is the well-known Sălăgean [6] derivative operator D^n, defined for $f \in \mathcal{A}$ as under:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order α $(0 \leq \alpha < p)$ in $\mathbb{E}$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{E}.$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of all such functions. A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order α $(0 \leq \alpha < p)$ in $\mathbb{E}$, if it satisfies the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{E}.$$

We denote by $\mathcal{K}_p(\alpha)$, the class of all functions $f \in \mathcal{A}_p$ that are p-valent convex of order α $(0 \leq \alpha < p)$ in $\mathbb{E}$. Note that $\mathcal{S}^*(\alpha) = \mathcal{S}_1^*(\alpha)$ and $\mathcal{K}(\alpha) = \mathcal{K}_1(\alpha)$ are the usual classes of univalent starlike functions (w.r.t. the origin) of order α $(0 \leq \alpha < 1)$ and univalent convex functions of order α $(0 \leq \alpha < 1)$.

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1, \ z \in \mathbb{E}$ such that $f(z) = g(w(z)), \ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Liu [3], studied the differential operator $(1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \lambda \frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}$ to make certain estimates on $\left(\frac{f(z)}{z^p}\right)^{\alpha}$ where $\alpha > 0, \ \lambda \geq 0$ are some real numbers and $f \in \mathcal{A}_p$. As special cases of our main results, we also obtain the differential operators of above nature, but our estimations are on $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''(z)}{f(z)}$, consequently we get certain new criteria for starlikeness and convexity of $f \in \mathcal{A}_p$.

To prove our main result, we shall make use of following lemma of Hallenbeck and Ruscheweyh [2].

Lemma 1.1. *Let G be a convex function in $\mathbb{E}$, with $G(0) = a$ and let γ be a complex number, with $\Re(\gamma) > 0$. If $F(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, is analytic in $\mathbb{E}$ and $F \prec G$, then*

$$\frac{1}{z^{\gamma}} \int_0^z F(w) w^{\gamma - 1}\, dw \prec \frac{1}{n z^{\gamma/n}} \int_0^z G(w) w^{\frac{\gamma}{n} - 1}\, dw.$$

2. Main results

Theorem 2.1. *Let* $\alpha,\ \beta,\ \delta$ *be real numbers such that* $\alpha > \dfrac{2}{1-\delta},\ 0 \leq \delta < 1,\ \beta > 0$ *and let*

$$0 < M \equiv M(\alpha,\beta,\lambda,\delta,p) = \frac{[\alpha+\beta(p+\lambda)][\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)(p+\lambda)]}, \tag{2.1}$$

If $f \in \mathcal{A}_p$ *satisfies the differential inequality*

$$\left|\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\beta}\left[1-\alpha+\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right]-1\right| < M(\alpha,\beta,\lambda,\delta,p),\ z\in\mathbb{E}, \tag{2.2}$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ z\in\mathbb{E}.$$

Proof. Let us define

$$\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\beta} = u(z),\ z\in\mathbb{E}.$$

Differentiate logarithmically, we obtain

$$\frac{zI_p'(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} - p = \frac{zu'(z)}{\beta u(z)} \tag{2.3}$$

In view of the equality

$$zI_p'(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z),$$

(2.3) reduces to

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = 1 + \frac{zu'(z)}{\beta(p+\lambda)u(z)}$$

Therefore, in view of (2.2), we have

$$u(z) + \frac{\alpha}{\beta(p+\lambda)}zu'(z) \prec 1 + Mz. \tag{2.4}$$

The use of Lemma 1.1 $\left(\text{taking } \gamma = \dfrac{\beta(p+\lambda)}{\alpha}\right)$ in (2.4) gives

$$u(z) \prec 1 + \frac{\beta(p+\lambda)Mz}{\alpha+\beta(p+\lambda)},$$

or

$$|u(z)-1| < \frac{\beta(p+\lambda)M}{\alpha+\beta(p+\lambda)} < 1,$$

therefore, we obtain

$$|u(z)| > 1 - \frac{\beta(p+\lambda)M}{\alpha+\beta(p+\lambda)} \tag{2.5}$$

Write $\dfrac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = (1-\delta)w(z)+\delta,\ 0\leq\delta<1$ and therefore (2.2) reduces to

$$|(1-\alpha)u(z) + \alpha u(z)[(1-\delta)w(z)+\delta] - 1| < M.$$

We need to show that $\Re(w(z)) > 0,\ z \in \mathbb{E}$. If possible, suppose that $\Re(w(z)) \not> 0,\ z \in \mathbb{E}$, then there must exist a point $z_0 \in \mathbb{E}$ such that $w(z_0) = ix, x \in \mathbb{R}$. To prove the required result, it is now sufficient to prove that

$$|(1-\alpha)u(z_0) + \alpha u(z_0)[(1-\delta)ix + \delta] - 1| \geq M. \tag{2.6}$$

By making use of (2.5), we have

$$\begin{aligned}
&|(1-\alpha)u(z_0) + \alpha u(z_0)[(1-\delta)ix + \delta] - 1| \\
\geq\ & |[1-\alpha(1-\delta) + \alpha(1-\delta)ix]\, u(z_0)| - 1 \\
=\ & \sqrt{[1-\alpha(1-\delta)]^2 + \alpha^2(1-\delta)^2x^2}\ |u(z_0)| - 1 \\
\geq\ & |1-\alpha(1-\delta)|\, |u(z_0)| - 1 \\
\geq\ & |1-\alpha(1-\delta)| \left(1 - \frac{\beta(p+\lambda)M}{\alpha + \beta(p+\lambda)}\right) - 1 \geq M.
\end{aligned} \tag{2.7}$$

Now (2.7) is true in view of (2.1) and therefore, (2.6) holds. Hence $\Re(w(z)) > 0$ i.e.

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ 0 \leq \delta < 1,\ z \in \mathbb{E}.$$

□

Remark 2.2. From Theorem 2.1, it follows, if $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \leq \delta < 1,\ \beta > 0$ and if $f \in \mathcal{A}_p$ satisfies

$$\left|\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\beta}\left[\frac{1}{\alpha} - 1 + \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] - \frac{1}{\alpha}\right| < \frac{[\alpha+\beta(p+\lambda)][\alpha(1-\delta)-2]}{\alpha^2[1+\beta(1-\delta)(p+\lambda)]},$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Letting $\alpha \to \infty$ in above remark, we get the following result.

Theorem 2.3. *Let $\beta,\ \delta$ be real numbers such that $\beta > 0,\ 0 \leq \delta < 1$ and let $f \in \mathcal{A}_p$ satisfy*

$$\left|\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^{\beta}\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} - 1\right)\right| < \frac{1-\delta}{1+\beta(1-\delta)(p+\lambda)},$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

For $p = 1$ and $\lambda = 0$ in Theorem 2.1, we get the following result involving Sălăgean operator.

Theorem 2.4. *If $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ and if $f \in \mathcal{A}$ satisfies the differential inequality*

$$\left|\left(\frac{D^n f(z)}{z}\right)^{\beta}\left[1-\alpha+\alpha\frac{D^{n+1}f(z)}{D^n f(z)}\right]-1\right| < \frac{(\alpha+\beta)[\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

Select $p = 1$ and $\lambda = 0$ in Theorem 2.3, we obtain:

Theorem 2.5. *If $\beta,\ \delta$ are real numbers such that $\beta > 0,\ 0 \le \delta < 1$ and $f \in \mathcal{A}$ satisfies*

$$\left|\left(\frac{D^n f(z)}{z}\right)^{\beta}\left(\frac{D^{n+1}f(z)}{D^n f(z)}-1\right)\right| < \frac{1-\delta}{1+\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \delta,\ z \in \mathbb{E}.$$

3. Criteria for starlikeness and convexity

Setting $\lambda = n = 0$ in Theorem 2.1, we obtain the following result.

Corollary 3.1. *Let $\alpha,\ \beta,\ \delta$ be real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ and let $f \in \mathcal{A}_p$ satisfy the differential inequality*

$$\left|(1-\alpha)\left(\frac{f(z)}{z^p}\right)^{\beta}+\alpha\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\beta}-1\right| < \frac{(\alpha+p\beta)[\alpha(1-\delta)-2]}{\alpha[1+p\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \le \gamma < p.$

Writing $\beta = 1$ in above corollary, we obtain:

Corollary 3.2. *Suppose that $\alpha,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1$ and suppose that $f \in \mathcal{A}_p$ satisfies*

$$\left|(1-\alpha)\frac{f(z)}{z^p}+\alpha\frac{f'(z)}{pz^{p-1}}-1\right| < \frac{(\alpha+p)[\alpha(1-\delta)-2]}{\alpha[1+p(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \le \gamma < p.$

Setting $n = 1$ and $\lambda = 0$ in Theorem 2.1, we obtain the following result.

Corollary 3.3. *Let $\alpha,\ \beta,\ \delta$ be real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \leq \delta < 1,\ \beta > 0$ and let $f \in \mathcal{A}_p$ satisfy the differential inequality*

$$\left|(1-\alpha)\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta} + \frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta} - 1\right| < \frac{(\alpha+p\beta)[\alpha(1-\delta)-2]}{\alpha[1+p\beta(1-\delta)]},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \leq \gamma < p.$

Writing $\beta = 1$ in above corollary, we obtain:

Corollary 3.4. *If $\alpha,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \leq \delta < 1$ and if $f \in \mathcal{A}_p$ satisfies*

$$\left|(1-\alpha)\frac{f'(z)}{pz^{p-1}} + \frac{\alpha}{p^2}\frac{f'(z)}{z^{p-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) - 1\right| < \frac{(\alpha+p)[\alpha(1-\delta)-2]}{\alpha[1+p(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \leq \gamma < p.$

Writing $\lambda = n = 0$ in Theorem 2.3, we get:

Corollary 3.5. *If $\beta,\ \delta$ are real numbers such that $\beta > 0,\ 0 \leq \delta < 1$ and if $f \in \mathcal{A}_p$ satisfies*

$$\left|\left(\frac{f(z)}{z^p}\right)^{\beta}\left(\frac{zf'(z)}{pf(z)} - 1\right)\right| < \frac{1-\delta}{1+p\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}_p^*(\gamma),\ 0 \leq \gamma < p.$

Note that for $\beta = p = 1$, the above corollary gives the result of Oros [5].
Setting $\lambda = 0$ and $n = 1$ in Theorem 2.3, we obtain:

Corollary 3.6. *Assume that $\beta,\ \delta$ be real numbers such that $\beta > 0,\ 0 \leq \delta < 1$ and assume that $f \in \mathcal{A}_p$ satisfies*

$$\left|\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta}\left[\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) - 1\right]\right| < \frac{1-\delta}{1+p\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > p\delta = \gamma,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}_p(\gamma),\ 0 \leq \gamma < p.$

Note that for $\beta = p = 1$ and $\delta = 0$, the above corollary deduces to the result of Mocanu [4].

Taking $p = 1$ in Corollary 3.1, we get:

Corollary 3.7. *If $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ and if $f \in \mathcal{A}$ satisfies*

$$\left|(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha\frac{(f(z))^{\beta-1}f'(z)}{z^{\beta-1}} - 1\right| < \frac{(\alpha+\beta)[\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}^*(\delta)$.

Setting $p = 1$ in Corollary 3.3, we get:

Corollary 3.8. *If $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ and if $f \in \mathcal{A}$ satisfies*

$$\left|(f'(z))^{\beta}\left[1-\alpha+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] - 1\right| < \frac{(\alpha+\beta)[\alpha(1-\delta)-2]}{\alpha[1+\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \delta,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{K}(\delta)$.

Put $\lambda = p = 1$ and $n = 0$ in Theorem 2.1, we get:

Corollary 3.9. *Suppose that $\alpha,\ \beta,\ \delta$ are real numbers such that $\alpha > \dfrac{2}{1-\delta},\ 0 \le \delta < 1,\ \beta > 0$ and suppose that $f \in \mathcal{A}$ satisfies*

$$\left|\left(1-\frac{\alpha}{2}\right)\left(\frac{f(z)}{z}\right)^{\beta} + \frac{\alpha}{2}\frac{(f(z))^{\beta-1}f'(z)}{z^{\beta-1}} - 1\right| < \frac{(\alpha+2\beta)[\alpha(1-\delta)-2]}{\alpha[1+2\beta(1-\delta)]},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 2\delta - 1,\ z \in \mathbb{E}.$$

Put $\lambda = p = 1$ and $n = 0$ in Theorem 2.3, we obtain the following result.

Corollary 3.10. *If $f \in \mathcal{A}$ satisfies*

$$\left|\left(\frac{f(z)}{z}\right)^{\beta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < \frac{2(1-\delta)}{1+2\beta(1-\delta)},\ z \in \mathbb{E},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 2\delta - 1,\ z \in \mathbb{E},$$

where $\beta,\ \delta$ are real numbers such that $\beta > 0,\ 0 \le \delta < 1$.

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