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Harmonic uniformly $\beta-$starlike functions of complex order defined by convolution and integral convolution

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Abstract. In this paper we introduce and study a subclass of harmonic univalent functions defined by convolution and integral convolution. Coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combinations are determined for functions in this family. Consequently, many of our results are either extensions or new approaches to those corresponding to previously known results.

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1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. Moreover,

$$h' = f_z = \frac{\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}}{2} \quad \text{and} \quad \bar{g}' = f_{\bar{z}} = \frac{\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}}{2}$$

are always analytic functions in D. A necessary and sufficient condition for f to be locally univalent and orientation-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [13]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the family of functions $f = h + \overline{g}$ which are harmonic, univalent and orientation-preserving in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, the functions

h and g analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \;\; g(z) = \sum_{k=1}^{\infty} b_k z^k \;\; (|b_1| < 1), \tag{1.1}$$

and f is then given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad (|b_1| < 1). \tag{1.2}$$

We note that the family $\mathcal{S_H}$ reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero ($g \equiv 0$).

Also, we denote by $T\mathcal{S_H}$ the subfamily of $\mathcal{S_H}$ consisting of harmonic functions $f = h + \overline{g}$ such that

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.3}$$

In [3] Clunie and Sheil-Small investigated the class $\mathcal{S_H}$ as well as its geometric subclasses and their properties. Since then, there have been several studies related to the class $\mathcal{S_H}$ and its subclasses. Following Clunie and Sheil-Small [3], Frasin [7], Jahangiri [9, 10], Silverman [17], Silverman and Silvia [18], Dixit and Porwal [4], Dixit et al. [5, 6] and others have investigated various subclasses of $\mathcal{S_H}$ and its properties.

Recently, Yalçin and Öztürk [20] introduced a new class of harmonic starlike functions of complex order $T\mathcal{S}^*_{\mathcal{H}}(b)$ subclass of $T\mathcal{S_H}$ consisting functions of the form (1.3) and satisfying the condition

$$\operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right)\right) > 0, \; z \in \mathcal{U}, \quad b \in \mathbb{C}\backslash\{0\}$$

and settled a conjecture. Further, Halim and Janteng [8] extended the study by introducing a new class $\mathcal{S}^*_{\mathcal{H}}(b, \alpha)$, $0 \le \alpha < 1$ of $\mathcal{S_H}$ consisting functions of the form (1.2) and satisfying the condition

$$\operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1\right)\right) > \alpha, \; z \in \mathcal{U}, \quad b \in \mathbb{C}\backslash\{0\}, \quad 0 \le \alpha < 1$$

and obtained following sufficient condition. If $f = h + \bar{g}$ is given by (1.2) and if

$$\sum_{n=2}^{\infty} \left(\frac{n - 1 + (1-\alpha)|b|}{(1-\alpha)|b|}\right) |a_n| + \sum_{n=1}^{\infty} \left(\frac{n + 1 - (1+\alpha)|b|}{(1-\alpha)|b|}\right) |b_n| \le 1$$

then $f \in \mathcal{S}^*_{\mathcal{H}}(b, \alpha)$. Also, they proved that the coefficient condition

$$\sum_{n=2}^{\infty} \left(\frac{n - 1 + (1-\alpha)|b|}{(1-\alpha)|b|}\right) |a_n| + \sum_{n=2}^{\infty} \left(\frac{n + 1 - (1+\alpha)|b|}{(1-\alpha)|b|}\right) |b_n| \le 1, \; b_1 = 0$$

is necessary for $f = h + \bar{g}$ is given by (1.3) and belongs to $T\mathcal{S}^*_{\mathcal{H}}(b, \alpha)$.

The convolution of two power series

$$\Phi(z)=z+\sum_{k=2}^{\infty}\lambda_k z^k, \text{ and } \Psi(z)=z+\sum_{k=2}^{\infty}\mu_k z^k \tag{1.4}$$

is defined by

$$(\Phi*\Psi)(z)=z+\sum_{k=2}^{\infty}\lambda_k\mu_k z^k, \tag{1.5}$$

where $\lambda_k\geq 0$ and $\mu_k\geq 0$. Also the integral convolution is defined by

$$(\Phi\diamond\Psi)(z)=z+\sum_{k=2}^{\infty}\frac{\lambda_k\mu_k}{k}z^k. \tag{1.6}$$

Motivated by the works of Yalçin and Öztürk [20], Halim and Janteng [8], Janteg and Halim [11] and Magesh and Mayilvaganan [14], we consider the subclass $\mathcal{G}_{\mathcal{H}}(\Phi,\Psi;\beta,\gamma,b;t)$ of functions of the form (1.2) satisfying the condition

$$\operatorname{Re}\left(1+\frac{1}{b}\left((1+\beta e^{i\alpha})\frac{h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}}{h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}}-\beta e^{i\alpha}-1\right)\right)>\gamma,\ z\in\mathcal{U}, \tag{1.7}$$

where $b\in\mathbb{C}\backslash\{0\}$, $\beta\geq 0$, $0\leq\gamma<1$, $\alpha\in\mathbb{R}$, $h_t(z)=(1-t)z+th(z)$, $g_t(z)=tg(z)$, $0\leq t\leq 1$, Φ and Ψ are of the form (1.4). We further let $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ denote the subclass of $\mathcal{G}_{\mathcal{H}}(\Phi,\Psi,\beta,\gamma,b;t)$ consisting of functions $f=h+\overline{g}\in\mathcal{S}_{\mathcal{H}}$ such that h and g are of the form (1.3).

We note that by specializing the functions Φ, Ψ and parameters β, γ and t we obtain well-known harmonic univalent functions as well as many new ones.

For example,

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2};0,0,b;1)=T\mathcal{S}^*_{\mathcal{H}}(b)$$

was introduced by Yalçin and Öztürk [20] and studied by Halim and Janteng [8],

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2};1,\gamma,b;1)=T\mathcal{S}^*_{\mathcal{H}}(\gamma,b)$$

was introduced by Stephen et al. [19]. Furthermore,

$$\mathcal{G}_{\mathcal{H}}(\frac{z+z^2}{(1-z)^3},\frac{z+z^2}{(1-z)^3};1,\gamma,1;1)=\mathcal{HC}(\gamma)$$

was studied by Kim et al. [12] and

$$\mathcal{G}_{\mathcal{H}}(z+\sum_{k=2}^{\infty}k^{n+1}z^k,z+\sum_{k=2}^{\infty}k^{n+1}z^k;1,\gamma,1;1)=RS(\gamma)$$

was studied by Yalcin et al. [21]. Also,

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2},\frac{z}{(1-z)^2};1,\gamma,1;1)=G_{\mathcal{H}}(\gamma)$$

was studied by Rosy et al. [16],

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \beta, \gamma, 1; t) = \mathcal{G}_{\overline{\mathcal{H}}}(\beta, \gamma; t)$$

was considered by Ahuja et al. [1]. Also, the class

$$\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi, \beta, \gamma, 1; t) = \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, t)$$

was studied by Magesh and Porwal [15],

$$\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; 0, \gamma, 1; 1) = \overline{\mathcal{HS}}(\Phi, \Psi; \gamma)$$

was studied by Dixit et al. [5],

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 0, \gamma, 1; 1) = S^*_{H}(\gamma)$$

and

$$\mathcal{G}_{\overline{\mathcal{H}}}(\frac{z+z^2}{(1-z)^3}, \frac{z+z^2}{(1-z)^3}; 0, \gamma, 1; 1) = K(\gamma)$$

were introduced and studied by Jahangiri [10]. For $\gamma = 0$ the classes $S^*_{\mathcal{H}}(\gamma)$ and $K(\gamma)$ were studied by Silverman and Silvia [18], for $\gamma = 0$ and $b_1 = 0$ see [2, 17].

If we set $\beta = 1$ and $\alpha = 0$ in the above definition we define the unified class of harmonic starlike functions of complex order satisfying the following analytic criteria:

$$\mathrm{Re}\left(1 + \frac{2}{b}\left(\frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}} - 1\right)\right) > \gamma, \ z \in \mathcal{U},$$

where $b \in \mathbb{C}\backslash\{0\}$, $0 \le \gamma < 1$, $h_t(z) = (1-t)z + th(z)$, $g_t(z) = tg(z)$, $0 \le t \le 1$, Φ and Ψ are of the form (1.4).

In this paper we give a sufficient condition for $f = h + \bar{g}$ given by (1.2) to be in $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$ and it is shown that this condition is also necessary for functions to be in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. We also obtain extreme points, distortion bounds, convolution and convex combination properties. Further, we obtain the closure property of this class under integral operator. We remark that the results so obtained for these general families can be viewed as extensions and generalizations for various subclasses of $\mathcal{S}_{\mathcal{H}}$ as listed previously in this section.

2. Coefficient bounds

Our first theorem gives a sufficient condition for functions to be in $\mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1.1). If*

$$\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}|b_k| \le 1, \tag{2.1}$$

where $\beta \ge 0$, $0 \le \gamma < 1$, $0 \le t \le 1$, $k^2(1-\gamma) \le [(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k$ and $k^2(1-\gamma) \le [(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k$ for $k \ge 2$. Then $f \in \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$.

Proof. To prove that $f \in \mathcal{G}_{\mathcal{H}}(\Phi,\Psi;\beta,\gamma,b;t)$, we only need to show that if (2.1) holds, then the required condition (1.7) is satisfied. For (1.7), we can write

$$\operatorname{Re}\left(1+\frac{1}{b}\left((1+\beta e^{i\alpha})\frac{h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}}{h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}}-\beta e^{i\alpha}-1\right)\right)=\operatorname{Re}\frac{A(z)}{B(z)}>\gamma,\ z\in\mathcal{U}.$$

Using the fact that $\operatorname{Re}\{\omega\}\geq\gamma$ if and only if $|1-\gamma+\omega|\geq|1+\gamma-\omega|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\geq 0,\ z\in\mathcal{U}, \tag{2.2}$$

where

$$\begin{aligned}A(z)&=(1+\beta e^{i\alpha})[h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}]\\&\quad+[b-(1+\beta e^{i\alpha})][h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}]\end{aligned}$$

and

$$B(z)=b[h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}].$$

Substituting A and B in (2.2) and making use of (2.1), we obtain

$$\begin{aligned}&|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\\&=\Big|(1+\beta e^{i\alpha})[h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}]\\&\quad+[b-(1+\beta e^{i\alpha})][h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}]\\&\quad+(1-\gamma)b\left(h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}\right)\Big|\\&\quad-\Big|(1+\beta e^{i\alpha})[h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}]\\&\quad+[b-(1+\beta e^{i\alpha})][h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}]\\&\quad-(1+\gamma)b\left(h_t(z)\diamond\Phi(z)+\overline{g_t(z)\diamond\Psi(z)}\right)\Big|\\&\geq 2(1-\gamma)|b||z|-\sum_{k=2}^{\infty}2\left[\frac{(k-t)(1+\beta)+(1-\gamma)t|b|}{k}\right]\lambda_k|a_k||z|^k\\&\quad-\sum_{k=1}^{\infty}2\left[\frac{(k+t)(1+\beta)-(1-\gamma)t|b|}{k}\right]\mu_k|b_k||z|^k\\&=2(1-\gamma)|b||z|\left\{1-\sum_{k=2}^{\infty}\left[\frac{(k-t)(1+\beta)+(1-\gamma)t|b|}{k(1-\gamma)|b|}\right]\lambda_k|a_k||z|^{k-1}\right.\\&\quad\left.-\sum_{k=1}^{\infty}\left[\frac{(k+t)(1+\beta)-(1-\gamma)t|b|}{k(1-\gamma)|b|}\right]\mu_k|b_k||z|^{k-1}\right\}\\&>2(1-\gamma)|b|\left\{1-\sum_{k=2}^{\infty}\left[\frac{(k-t)(1+\beta)+(1-\gamma)t|b|}{k(1-\gamma)|b|}\right]\lambda_k|a_k|\right.\\&\quad\left.-\sum_{k=1}^{\infty}\left[\frac{(k+t)(1+\beta)-(1-\gamma)t|b|}{k(1-\gamma)|b|}\right]\mu_k|b_k|\right\}\geq 0\end{aligned}$$

which implies that $f \in \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$.

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{k(1-\gamma)|b|}{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k} x_k z^k + \sum_{k=1}^{\infty} \frac{k(1-\gamma)|b|}{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k} \overline{y_k z^k},$$

where

$$\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1,$$

shows that the coefficient bound given by (2.1) is sharp. □

Next, we will show that the sufficient condition (2.1) is also necessary for functions to be in the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$.

Theorem 2.2. *Let $f = h + \bar{g}$ be so that h and g are given by (1.3). Then $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} a_k + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} b_k \leq 1, \tag{2.3}$$

where $\beta \geq 0$, $0 \leq \gamma < 1$, $0 \leq t \leq 1$, $k^2(1-\gamma) \leq [(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k$ and $k^2(1-\gamma) \leq [(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k$ for $k \geq 2$.

Proof. Since $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t) \subset \mathcal{G}_{\mathcal{H}}(\Phi, \Psi; \beta, \gamma, b; t)$, we only need to prove the only if part of the theorem. To this end, for functions f of the form (1.3), we notice that the condition (1.7) is equivalent to

$$\operatorname{Re}\left\{ \frac{\begin{array}{l}(1+\beta e^{i\alpha})[h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}] \\ +[(1-\gamma)b - (1+\beta e^{i\alpha})][h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}]\end{array}}{b[h_t(z) \diamond \Phi(z) + \overline{g_t(z) \diamond \Psi(z)}]} \right\} \geq 0, \ z \in \mathcal{U}.$$

Upon choosing the values of z on the positive real axis where $0 \le |z| = r < 1$, the above inequality reduces to

$$\mathrm{Re}\left\{\frac{\begin{array}{c}(1-\gamma)b - \sum_{k=2}^{\infty}[(k-t)+(1-\gamma)bt]\frac{\lambda_k}{k}|a_k|r^{k-1} \\ -\sum_{k=1}^{\infty}[(k+t)-(1-\gamma)bt]\frac{\mu_k}{k}|b_k|r^{k-1}\end{array}}{b\left[1-\sum_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}\right]}\right\}$$

$$-\mathrm{Re}\left\{\beta e^{i\alpha}\frac{\sum_{k=2}^{\infty}(k-t)\frac{\lambda_k}{k}|a_k|r^{k-1}+\sum_{k=1}^{\infty}(k+t)\frac{\mu_k}{k}|b_k|r^{k-1}}{b\left[1-\sum_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}\right]}\right\} \ge 0.$$

Since $Re(-e^{i\alpha}) \ge -|e^{i\alpha}| = -1$, the above inequality reduces to

$$\left\{\frac{\begin{array}{c}(1-\gamma)|b| - \sum_{k=2}^{\infty}[(k-t)(1+\beta)+(1-\gamma)t|b|]\frac{\lambda_k}{k}|a_k|r^{k-1} \\ -\sum_{k=1}^{\infty}[(k+t)(1+\beta)-(1-\gamma)t|b|]\frac{\mu_k}{k}|b_k|r^{k-1}\end{array}}{|b|\left[1-\sum_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}\right]}\right\} \ge 0. \tag{2.4}$$

If the condition (2.3) does not hold then the numerator in (2.4) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.4) is negative. This contradicts the condition for $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$, hence the proof is complete. □

3. Extreme points and distortion bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$.

Theorem 3.1. *Let f be given by (1.3). Then $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k(z) + Y_k g_k(z)), \tag{3.1}$$

where

$$h_1(z) = z,\ \ h_k(z) = z - \frac{k(1-\gamma)|b|}{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}z^k\ \ (k=2,3,...),$$

$$g_k(z) = z + \frac{k(1-\gamma)|b|}{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}\overline{z}^k \ (k=1,2,3,...),$$

$$\sum_{k=1}^{\infty}(X_k+Y_k)=1,\ X_k\geq 0,\ Y_k\geq 0.$$

In particular, the extreme points of $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. For functions f of the form (3.1), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\left(X_k h_k(z)+Y_k g_k(z)\right) \\ &= \sum_{k=1}^{\infty}\left(X_k+Y_k\right)z - \sum_{k=2}^{\infty}\frac{k(1-\gamma)|b|}{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}X_k z^k \\ &+ \sum_{k=1}^{\infty}\frac{k(1-\gamma)|b|}{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}Y_k\overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}\left(\frac{k(1-\gamma)|b|}{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}\right)X_k \\ &+\sum_{k=1}^{\infty}\frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}\left(\frac{k(1-\gamma)|b|}{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}\right)Y_k \\ &=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k = 1-X_1\leq 1 \end{aligned}$$

and so $f \in clco\ \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$.

Conversely, suppose that $f \in clco\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ and set

$$X_k = \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}|a_k|,\ k=2,3,\ldots,$$

and

$$Y_k = \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}|b_k|,\ k=1,2,\ldots,$$

where

$$\sum_{k=1}^{\infty}(X_k+Y_k)=1.$$

Then, by Theorem 2.2, we have $0\leq X_k\leq 1$ $(k=2,3,\ldots)$ and $0\leq Y_k\leq 1$ $(k=1,2,3,\ldots)$. We define

$$X_1 = 1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k$$

and use Theorem 2.2 again to get $X_1\geq 0$. Consequently, we obtain

$$f(z)=\sum_{k=1}^{\infty}\left(X_k h_k(z)+Y_k g_k(z)\right).$$

Another application of Theorem 2.2 shows that $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ is convex and closed, so *clco* $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t) = \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. In other words, the statement of Theorem 3.1 holds. □

The following theorem gives the distortion bounds for functions in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ which yields a covering result for this class.

Theorem 3.2. *Let* $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ *and*

$$A \leq [(k-t)(1+\beta) + (1-\gamma)t|b|]\frac{\lambda_k}{k},$$
$$A \leq [(k+t)(1+\beta) - (1-\gamma)t|b|]\frac{\mu_k}{k}$$

for $k \geq 2$, *where*

$$A = \min\left\{[(2-t)(1+\beta) + (1-\gamma)t|b|]\frac{\lambda_2}{2},\ [(2+t)(1+\beta) - (1-\gamma)t|b|]\frac{\mu_2}{2}\right\}$$

then

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\gamma}{A} - \frac{(1+t)(1+\beta) - (1-\gamma)t|b|}{A}|b_1|\right)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\gamma}{A} - \frac{(1+t)(1+\beta) - (1-\gamma)t|b|}{A}|b_1|\right)r^2.$$

Proof. Let $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. Taking the absolute value of f, we obtain

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\
&\leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k| + |b_k|) \\
&= (1+|b_1|)r + \frac{1-\gamma}{A}r^2\sum_{k=2}^{\infty}\left(\frac{A}{1-\gamma}|a_k| + \frac{A}{1-\gamma}|b_k|\right) \\
&\leq (1+|b_1|)r + \frac{1-\gamma}{A}r^2\sum_{k=2}^{\infty}\left(\frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}|a_k|\right. \\
&\qquad \left. + \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}|b_k|\right) \\
&\leq (1+|b_1|)r + \frac{1-\gamma}{A}\left(1 - \frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{(1-\gamma)}|b_1|\right)r^2 \\
&= (1+|b_1|)r + \left(\frac{1-\gamma}{A} - \frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)r^2
\end{aligned}$$

and similarly,

$$|f(z)| \geq (1-|b_1|)r - \left(\frac{1-\gamma}{A} - \frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)r^2.$$

The upper and lower bounds given in Theorem 3.2 are respectively attained by the following functions:

$$f(z)=z+|b_1|\overline{z}+\left(\frac{1-\gamma}{A}-\frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)\overline{z}^2$$

and

$$f(z)=(1-|b_1|)z-\left(\frac{1-\gamma}{A}-\frac{(1+t)(1+\beta)-(1-\gamma)t|b|}{A}|b_1|\right)z^2. \qquad \square$$

The next covering result follows from the left hand inequality in Theorem 3.2.

Corollary 3.3. *Let f of the form (1.3) be so that $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ and*

$$A\leq[(k-t)(1+\beta)+(1-\gamma)t|b|]\frac{\lambda_k}{k},$$

$$A\leq[(k+t)(1+\beta)-(1-\gamma)t|b|]\frac{\mu_k}{k}$$

for $k\geq 2$, where

$$A=\min\left\{[(2-t)(1+\beta)+(1-\gamma)t|b|]\frac{\lambda_2}{2},\ [(2+t)(1+\beta)-(1-\gamma)t|b|]\frac{\mu_2}{2}\right\}.$$

Then

$$\left\{\omega\in\mathbb{C}:|\omega|<\frac{A+1-\gamma}{A}+\frac{A-1+\gamma}{A}|b_1|\right\}\subset f(\mathcal{U}).$$

4. Convolution and convex combinations

In this section we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For

$$f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|\overline{z}^k$$

and

$$F(z)=z-\sum_{k=2}^{\infty}|A_k|z^k+\sum_{k=1}^{\infty}|B_k|\overline{z}^k,$$

we define the convolution of two harmonic functions f and F as

$$(f*F)(z)=f(z)*F(z)=z-\sum_{k=2}^{\infty}|a_k||A_k|z^k+\sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k. \qquad (4.1)$$

Using the definition, we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ is closed under convolution.

Theorem 4.1. *For $0\leq\gamma<1$, let $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ and $F\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$. Then $f*F\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$.*

Proof. Let

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| \overline{z}^k$$

be in $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. Then the convolution $f * F$ is given by (4.1). We wish to show that the coefficient of $f * F$ satisfy the required condition given in Theorem 2.2. For $F \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now for the convolution function $f * F$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} |a_k||A_k| \\
&\qquad + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} |b_k||B_k| \\
&\leq \sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} |a_k| \\
&\qquad + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} |b_k| \\
&\leq 1,
\end{aligned}$$

since $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. Therefore $f * F \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$. □

Next, we show that the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ is closed under convex combination of its members.

Theorem 4.2. *The class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$ let $f_i(z) \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, \gamma, b; t)$, where f_i is given by

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{ik}| z^k + \sum_{k=1}^{\infty} |b_{ik}| \overline{z}^k.$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{ik}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{ik}| \right) \overline{z}^k.$$

Since,

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta) + (1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|} |a_{ik}| \\
&\qquad + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta) - (1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|} |b_{ik}| \leq 1,
\end{aligned}$$

from the above equation we obtain

$$\sum_{k=2}^{\infty}\frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}\sum_{i=1}^{\infty}t_i|a_{ik}| + \sum_{k=1}^{\infty}\frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}\sum_{i=1}^{\infty}t_i|b_{ik}|$$

$$= \sum_{i=1}^{\infty}t_i\left\{\sum_{k=2}^{\infty}\frac{[(k-t)(1+\beta)+k(1-\gamma)|b|t]}{k(1-\gamma)|b|}|a_{ik}| + \sum_{k=1}^{\infty}\frac{[(k+t)(1+\beta)-k(1-\gamma)|b|t]}{k(1-\gamma)|b|}|b_{ik}|\right\}$$

$$\leq \sum_{i=1}^{\infty}t_i = 1.$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty}t_i f_i(z)\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$. □

5. Class preserving integral operator

Finally, we consider the closure property of the class $\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$ under the generalized Bernardi-Libera-Livingston integral operator$\mathcal{L}_c$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c}\int_0^z \xi^{c-1}f(\xi)d\xi \quad (c>-1).$$

Theorem 5.1. *If $f\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$, then $\mathcal{L}_c[f(z)]\in\mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$.*

Proof. From the representation of $\mathcal{L}_c[f(z)]$, it follows that

$$\begin{aligned}\mathcal{L}_c[f(z)] &= \frac{c+1}{z^c}\int_0^z \xi^{c-1}h(\xi)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}g(\xi)d\xi}\\ &= \frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\xi-\sum_{k=2}^{\infty}|a_k|\xi^k\right)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\sum_{k=1}^{\infty}|b_k|\xi^k\right)d\xi}\\ &= z-\sum_{k=2}^{\infty}A_k z^k + \sum_{k=1}^{\infty}B_k z^k,\end{aligned}$$

where $A_k = \frac{c+1}{c+k}|a_k|$ and $B_k = \frac{c+1}{c+k}|b_k|$. Hence

$$\sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}\left(\frac{c+1}{c+k}|a_k|\right)$$
$$+\sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}\left(\frac{c+1}{c+k}|b_k|\right)$$
$$\leq \sum_{k=2}^{\infty} \frac{[(k-t)(1+\beta)+(1-\gamma)t|b|]\lambda_k}{k(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty} \frac{[(k+t)(1+\beta)-(1-\gamma)t|b|]\mu_k}{k(1-\gamma)|b|}|b_k|$$
$$\leq 1,$$

since $f \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$, therefore by Theorem 2.2, $\mathcal{L}_c(f(z)) \in \mathcal{G}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,\gamma,b;t)$. □

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