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New univalence criteria for some integral operators

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Abstract. In this work we consider some integral operators for analytic functions in the open unit disk and we obtain new univalence criteria for these integral operators, using Mocanu's and Şerb's Lemma, Pascu's Lemma.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We denote $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$. We consider the integral operators

$$H_n(z) = \left\{\gamma \int_0^z u^{\gamma-1} \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} du\right\}^{\frac{1}{\gamma}}, \tag{1.1}$$

$$T_n(z) = \left\{\gamma \int_0^z u^{\gamma-1} \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} (f_1'(u))^{\beta_1} \cdots (f_n'(u))^{\beta_n} du\right\}^{\frac{1}{\gamma}}, \tag{1.2}$$

for the functions $f_j \in \mathcal{A}$ and the complex numbers $\gamma, \alpha_j, \beta_j, \gamma \neq 0, j = 1, n$.

In this work we define a new general integral operator V_n given by

$$V_n(z) = \left\{\delta \int_0^z u^{\delta-1} \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_j} \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du\right\}^{\frac{1}{\delta}} \tag{1.3}$$

for $f_j, g_j \in \mathcal{A}$ and the complex numbers $\alpha_j, \beta_j, \delta, \delta \neq 0, j = \overline{1, n}$.

The integral operator V_n is the most general integral operator.

Remarks. For different particular cases for parameters δ, α_j, β_j, $j = \overline{1,n}$, we obtain the integral operators which have been defined and studied by Kim-Merkes, Pfaltzgraff, Pascu, Pescar, Owa, D. Breaz and N. Breaz, Frasin, Ovesea.

- i1) For $n = 1, \delta = 1, \beta_1 = 0, g_1(z) = z$ we obtain the integral operator which was introduced and studied by Kim-Merkes [4].
- i2) For $n = 1, \delta = 1, \alpha_1 = 0$ we have the integral operator that was introduced and studied by Pfaltzgraff [10].
- i3) For $n = 1, \beta_1 = 0, g_1(z) = z$ we obtain the integral operator which was defined and studied by Pescar and Pascu [8].
- i4) For $n = 1, \alpha_1 = 0, g_1(z) = z$ we have the integral operator, which was defined and studied by Pescar and Owa [9].
- i5) For $g_i(z) = z, i = \overline{1,n}, \delta = \gamma$ and $\beta_1 = \cdots = \beta_n = 0$ we obtain the integral operator H_n, (1.1), which was defined and studied by D. Breaz and N. Breaz [1], and this integral operator is a generalization of the integral operator defined by Pescar and Pascu [8].
- i6) For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0, \delta = \gamma, g_i(z) = z, i = \overline{1,n}$ we have the integral operator which was defined and studied by D. Breaz, N. Breaz [2] and this integral operator is a generalization of the integral operator defined by Pescar and Owa [9].
- i7) For $n = 1, g_1(z) = z$ we obtain the integral operator which is defined and studied by Ovesea [6].
- i8) For $g_i(z) = z, i = \overline{1,n}, \delta = \gamma$ we obtain the integral operator T_n that was defined and studied by Frasin [3], and this integral operator is a generalization of the integral operator defined by Ovesea [6].

In this paper we derive certain sufficient conditions of univalence for the integral operators H_n, T_n, V_n, using Mocanu's and Şerb's Lemma, Pascu's Lemma.

2. Preliminary results

In order to prove main results we will use the lemmas.

Lemma 2.1. *Mocanu and Şerb* [5]. *Let $M_0 = 1,5936...$ the positive solution of equation*

$$(2 - M)e^M = 2. \tag{2.1}$$

If $f \in \mathcal{A}$ and

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0, \quad (z \in \mathcal{U}), \tag{2.2}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad (z \in \mathcal{U}). \tag{2.3}$$

The edge M_0 is sharp.

Lemma 2.2. *Pascu* [7]*. Let α be a complex number, $Re\ \alpha > 0$ and the function $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{2.4}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\ \beta \ge Re\ \alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{2.5}$$

is regular and univalent in $\mathcal{U}$.

3. Main results

Theorem 3.1. *Let β, γ, α_j be complex numbers, $j = \overline{1,n}$, $Re\ \beta > 0, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \dots$, $j = \overline{1,n}$.*

If

$$\left|\frac{f''_j(z)}{f'_j(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.1}$$

$$Re\ \beta \ge |\alpha_1| + \dots + |\alpha_n|, \tag{3.2}$$

then for all γ be complex numbers, $Re\ \gamma \ge Re\ \beta$, the integral operator H_n given by (1.1) is in the class $\mathcal{S}$.

Proof. Let's consider the function

$$h_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} du, \quad (z \in \mathcal{U}), \tag{3.3}$$

which is regular in $\mathcal{U}$ and $h_n(0) = h'_n(0) - 1 = 0$.

We have

$$\frac{zh''_n(z)}{h'_n(z)} = \alpha_1\left(\frac{zf'_1(z)}{f_1(z)} - 1\right) + \dots + \alpha_n\left(\frac{zf'_n(z)}{f_n(z)} - 1\right)$$

and hence, we obtain

$$\frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zh''_n(z)}{h'_n(z)}\right| \le \frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\sum_{j=1}^{n}\left[|\alpha_j|\left|\frac{zf'_j(z)}{f_j(z)} - 1\right|\right], \tag{3.4}$$

for all $z \in \mathcal{U}$.

From (3.1) and Lemma Mocanu and Şerb we obtain

$$\left|\frac{zf'_j(z)}{f_j(z)} - 1\right| < 1, \quad (z \in \mathcal{U}; j = \overline{1,n}). \tag{3.5}$$

By (3.4) and (3.5) we get

$$\frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zh''_n(z)}{h'_n(z)}\right| \le \frac{|\alpha_1| + \dots + |\alpha_n|}{Re\ \beta}, \quad (z \in \mathcal{U}). \tag{3.6}$$

From (3.2) and (3.6) we have

$$\frac{1-|z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{3.7}$$

From (3.3) we get $h_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\gamma_1} \cdots \left(\frac{f_n(z)}{z}\right)^{\gamma_n}$ and by Lemma Pascu it results that $H_n \in \mathcal{S}$. □

Corollary 3.2. *Let α, β be complex numbers, $Re\ \beta > 0$, M_0 the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq M_0, \quad (z \in \mathcal{U}; j = \overline{1,n}), \tag{3.8}$$

$$Re\ [n(\alpha-1)+1] \geq Re\ \beta \geq n|\alpha-1|, \quad (n \in \mathbb{N} - \{0\}), \tag{3.9}$$

then the integral operator $G_{\alpha,n}$ defined by

$$G_{\alpha,n} = \left\{[n(\alpha-1)+1]\int_0^z (f_1(u))^{\alpha-1}\cdots(f_n(u))^{\alpha-1}\,du\right\}^{\frac{1}{n(\alpha-1)+1}} \tag{3.10}$$

is in the class $\mathcal{S}$.

Proof. From (3.10) we have

$$G_{\alpha,n}(z) = \left\{[n(\alpha-1)+1]\int_0^z u^{n(\alpha-1)}\left(\frac{f_1(u)}{u}\right)^{\alpha-1}\cdots\left(\frac{f_n(u)}{u}\right)^{\alpha-1}du\right\}^{\frac{1}{n(\alpha-1)+1}} \tag{3.11}$$

We take $\gamma = n(\alpha-1)+1$, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha - 1$ in Theorem 3.1 and we obtain the Corollary 3.2. □

Theorem 3.3. *Let $\delta, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $Re\ \delta > 0, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.12}$$

$$\left[\sum_{j=1}^n |\alpha_j|\right](2Re\ \delta+1)^{\frac{2Re\ \delta+1}{2Re\ \delta}} + 2M_0\left[\sum_{j=1}^n |\beta_j|\right]Re\ \delta \leq$$
$$\leq (2Re\ \delta+1)^{\frac{2Re\ \delta+1}{2Re\ \delta}}Re\ \delta, \tag{3.13}$$

then for all γ be complex numbers, $Re\ \gamma \geq Re\ \delta > 0$ the integral operator T_n given by (1.2) is in the class $\mathcal{S}$.

Proof. We consider the function

$$g_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} \cdot (f_1'(u))^{\beta_1} \cdots (f_n'(u))^{\beta_n}\, du. \tag{3.14}$$

The function g_n is regular in $\mathcal{U}$ and we have $g_n(0) = g_n'(0) - 1 = 0$.

From (3.14) we obtain

$$\frac{zg_n''(z)}{g_n'(z)} = \sum_{j=1}^{n} \left[\alpha_j \left(\frac{zf_j'(z)}{f_j(z)} - 1\right)\right] + \sum_{j=1}^{n} \left[\beta_j \frac{zf_j''(z)}{f_j'(z)}\right] \tag{3.15}$$

and hence, we get

$$\frac{1-|z|^{2Re\,\delta}}{Re\,\delta} \left|\frac{zg_n''(z)}{g_n'(z)}\right| \le$$

$$\le \frac{1-|z|^{2Re\,\delta}}{Re\,\delta} \sum_{j=1}^{n} \left[|\alpha_j| \left|\frac{zf_j'(z)}{f_j(z)} - 1\right| + |\beta_j||z| \left|\frac{f_j''(z)}{f_j'(z)}\right|\right], \tag{3.16}$$

for all $z \in \mathcal{U}$.

From (3.12), Lemma Mocanu and Şerb, by (3.16) we have

$$\frac{1-|z|^{2Re\,\delta}}{Re\,\delta} \left|\frac{zg_n''(z)}{g_n'(z)}\right| \le$$

$$\le \frac{1-|z|^{2Re\,\delta}}{Re\,\delta} \sum_{j=1}^{n} |\alpha_j| + \frac{1-|z|^{2Re\,\delta}}{Re\,\delta} |z| M_0 \sum_{j=1}^{n} |\beta_j|. \tag{3.17}$$

Since

$$\max_{|z|\le 1} \frac{1-|z|^{2Re\,\delta}}{Re\,\delta} |z| = \frac{2}{(2Re\,\delta + 1)^{\frac{2Re\,\delta+1}{2Re\,\delta}}},$$

from (3.17) we obtain

$$\frac{1-|z|^{2Re\,\delta}}{Re\,\delta} \left|\frac{zg_n''(z)}{g_n'(z)}\right| \le$$

$$\le \frac{1}{Re\,\delta} \sum_{j=1}^{n} |\alpha_j| + \frac{2M_0}{(2Re\,\delta + 1)^{\frac{2Re\,\delta+1}{2Re\,\delta}}} \sum_{j=1}^{n} |\beta_j|, \tag{3.18}$$

for all $z \in \mathcal{U}$.

From (3.13) and (3.18) we get

$$\frac{1-|z|^{2Re\,\delta}}{Re\,\delta} \left|\frac{zg_n''(z)}{g_n'(z)}\right| \le 1, \quad (z \in \mathcal{U}). \tag{3.19}$$

From (3.14) we have $g_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z}\right)^{\alpha_n} \cdot (f_1'(z))^{\beta_1} \cdots (f_n'(z))^{\beta_n}$ and by Lemma Pascu we obtain that $T_n \in \mathcal{S}$. □

Remark 3.4. For $\beta_1 = \beta_2 = \cdots = \beta_n = 0, \delta = \beta$, from Theorem 3.3 we obtain the Theorem 3.1.

Corollary 3.5. *Let $\delta, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $0 < Re\ \delta \le 1, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j \in \mathcal{A}$,*
$f_j(z) = z + a_{2j}z^2 + \dots$, $j = \overline{1,n}$.

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.20}$$

$$(2Re\ \delta + 1)^{\frac{2Re\ \delta+1}{2Re\ \delta}} \sum_{j=1}^{n} |\alpha_j| + 2(Re\ \delta)M_0 \left[\sum_{j=1}^{n} |\beta_j|\right] \le$$
$$\le (2Re\ \delta + 1)^{\frac{2Re\ \delta+1}{2Re\ \delta}} Re\ \delta, \tag{3.21}$$

then the integral operator I_n defined by

$$I_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} \cdot \left(f_1'(u)\right)^{\beta_1} \cdots \left(f_n'(u)\right)^{\beta_n} du. \tag{3.22}$$

belongs to the class $\mathcal{S}$.

Proof. We take $\gamma = 1$ in the Theorem 3.3. □

Theorem 3.6. *Let $\gamma, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $Re\ \gamma > 0, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j, g_j \in \mathcal{A}$,*
$f_j(z) = z + a_{2j}z^2 + \dots$, $g_j(z) = z + b_{2j}z^2 + \dots$, $j = \overline{1,n}$.

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.23}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \le M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.24}$$

$$(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}} \sum_{j=1}^{n} |\alpha_j| + 2(Re\ \gamma)M_0 \sum_{j=1}^{n} |\beta_j| \le$$
$$\le \frac{(2Re\ \gamma + 1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}} Re\ \gamma}{2}, \tag{3.25}$$

then for every complex number δ , $Re\ \delta \ge Re\ \gamma$ the integral operator V_n defined by (1.3) is in the class $\mathcal{S}$.

Proof. We consider the function

$$p_n(z) = \int_0^z \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} \cdot \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du \tag{3.26}$$

The function p_n is regular in $\mathcal{U}$ and $p_n(0) = p_n'(0) - 1 = 0$.

We have

$$\frac{zp_n''(z)}{p_n'(z)} = \sum_{j=1}^{n} \left[\alpha_j \left(\frac{zf_j'(z)}{f_j(z)} - \frac{zg_j'(z)}{g_j(z)} \right) + \beta_j \left(\frac{zf_j''(z)}{f_j'(z)} - \frac{zg_j''(z)}{g_j'(z)} \right) \right],$$

and hence, we get

$$\frac{zp_n''(z)}{p_n'(z)} =$$

$$\sum_{j=1}^{n} \left\{ \alpha_j \left[\left(\frac{zf_j'(z)}{f_j(z)} - 1 \right) - \left(\frac{zg_j'(z)}{g_j(z)} - 1 \right) \right] + \beta_j \left(\frac{zf_j''(z)}{f_j'(z)} - \frac{zg_j''(z)}{g_j'(z)} \right) \right\} \tag{3.27}$$

for all $z \in \mathcal{U}$.

From (3.27) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \left| \frac{zp_n''(z)}{p_n'(z)} \right| \leq \tag{3.28}$$

$$\leq \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \left\{ \sum_{j=1}^{n} \left[|\alpha_j| \left(\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| + \left| \frac{zg_j'(z)}{g_j(z)} - 1 \right| \right) + \right. \right.$$

$$\left. \left. + \ |\beta_j| \left(\left| \frac{zf_j''(z)}{f_j'(z)} \right| + \left| \frac{zg_j''(z)}{g_j'(z)} \right| \right) \right] \right\}$$

for all $z \in \mathcal{U}$.

From (3.23), (3.24) and Lemma Mocanu and Şerb we have

$$\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| < 1, \tag{3.29}$$

$$\left| \frac{zg_j'(z)}{g_j(z)} - 1 \right| < 1, \tag{3.30}$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$ and hence, we get

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \left| \frac{zp_n''(z)}{p_n'(z)} \right| \leq$$

$$\leq \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \cdot 2 \sum_{j=1}^{n} |\alpha_j| + \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} |z| \cdot 2M_0 \sum_{j=1}^{n} |\beta_j|. \tag{3.31}$$

Since

$$\max_{|z|\leq 1} \frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} |z| = \frac{2}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}, \tag{3.32}$$

from (3.31) we obtain

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} \left| \frac{zp_n''(z)}{p_n'(z)} \right| \leq \frac{2}{Re\,\gamma} \sum_{j=1}^{n} |\alpha_j| + \frac{4M_0}{(2Re\,\gamma+1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}} \sum_{j=1}^{n} |\beta_j|, \tag{3.33}$$

for all $z \in \mathcal{U}$.

From (3.25) and (3.33) we get

$$\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zp_n''(z)}{p_n'(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{3.34}$$

From (3.26) we obtain

$$p_n'(z) = \left(\frac{f_1(z)}{g_1(z)}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{g_n(z)}\right)^{\alpha_n} \cdot \left(\frac{f_1'(z)}{g_1'(z)}\right)^{\beta_1} \cdots \left(\frac{f_n'(z)}{g_n'(z)}\right)^{\beta_n}$$

and by Lemma Pascu it results that $V_n \in \mathcal{S}$. □

Corollary 3.7. *Let γ, α_j be complex numbers, $j = \overline{1,n}$, $Re\,\gamma > 0, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j, g_j \in \mathcal{A}$,*

$$f_j(z) = z + a_{2j}z^2 + \dots,\ g_j(z) = z + b_{2j}z^2 + \dots,\ j = \overline{1,n}.$$

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.35}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \leq M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.36}$$

and

$$\sum_{i=1}^{n} |\alpha_j| \leq \frac{Re\,\gamma}{2} \tag{3.37}$$

then for all complex numbers δ , $Re\,\delta \geq Re\,\gamma$ the integral operator K_n defined by

$$K_n(z) = \left\{\delta \int_0^z u^{\delta-1}\left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} du\right\}^{\frac{1}{\delta}}, \tag{3.38}$$

is in the class $\mathcal{S}$.

Proof. We take $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ in Theorem 3.6. □

Corollary 3.8. *Let $\gamma, \alpha_j, \beta_j$ be complex numbers, $j = \overline{1,n}$, $0 < Re\,\gamma \leq 1, M_0$ the positive solution of the equation (2.1), $M_0 = 1,5936...$ and $f_j, g_j \in \mathcal{A}$,*

$$f_j(z) = z + a_{2j}z^2 + \dots,\ g_j(z) = z + b_{2j}z^2 + \dots,\ j = \overline{1,n}.$$

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.39}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \leq M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.40}$$

and

$$(2Re\ \gamma+1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}}\sum_{j=1}^{n}|\alpha_j|+2M_0(Re\ \gamma)\sum_{j=1}^{n}|\beta_j|\leq$$

$$\leq\frac{(2Re\ \gamma+1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}}\cdot Re\ \gamma}{2}, \tag{3.41}$$

then the integral operator J_n *defined by*

$$J_n(z)=\int_0^z\left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1}\cdots\left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n}\cdot\left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1}\cdots\left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n}du \tag{3.42}$$

is in the class $\mathcal{S}$.

Proof. For $\delta=1$ in Theorem 3.6, we obtain Corollary 3.8. □

Corollary 3.9. *Let* γ,β_j *be complex numbers,* $j=\overline{1,n}$, $Re\ \gamma>0, M_0$ *the positive solution of the equation* (2.1), $M_0=1,5936...$ *and* $f_j,g_j\in\mathcal{A}$,
$f_j(z)=z+a_{2j}z^2+\ldots$, $g_j(z)=z+b_{2j}z^2+\ldots$, $j=\overline{1,n}$.

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right|\leq M_0,\quad (z\in\mathcal{U},\ j=\overline{1,n}), \tag{3.43}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right|\leq M_0,\quad (z\in\mathcal{U},\ j=\overline{1,n}), \tag{3.44}$$

and

$$\sum_{j=1}^{n}|\beta_j|\leq\frac{(2Re\ \gamma+1)^{\frac{2Re\ \gamma+1}{2Re\ \gamma}}}{4M_0}, \tag{3.45}$$

then for all complex number δ, $Re\ \delta\geq Re\ \gamma$, *the integral operator* Q_n *defined by*

$$Q_n(z)=\left\{\delta\int_0^z u^{\delta-1}\left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1}\cdots\left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n}du\right\}^{\frac{1}{\delta}} \tag{3.46}$$

is in the class $\mathcal{S}$.

Proof. For $\alpha_1=\alpha_2=\cdots=\alpha_n=0$ in Theorem 3.6, we obtain Corollary 3.9. □

Corollary 3.10. *Let* γ,β_j *be complex numbers,* $j=\overline{1,n}$, $0<Re\ \gamma\leq 1, M_0$ *the positive solution of the equation* (2.1), $M_0=1,5936...$ *and* $f_j,g_j\in\mathcal{A}$,
$f_j(z)=z+a_{2j}z^2+\ldots$, $g_j(z)=z+b_{2j}z^2+\ldots$, $j=\overline{1,n}$.

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right|\leq M_0,\quad (z\in\mathcal{U},\ j=\overline{1,n}), \tag{3.47}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \leq M_0, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{3.48}$$

and

$$\sum_{j=1}^{n} |\beta_j| \leq \frac{(2Re\,\gamma + 1)^{\frac{2Re\,\gamma+1}{2Re\,\gamma}}}{4M_0}, \tag{3.49}$$

then the integral operator L_n defined by

$$L_n(z) = \int_0^z \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du \tag{3.50}$$

is in the class $\mathcal{S}$.

Proof. We take $\delta = 1$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ in Theorem 3.6. □

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