Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 2, 155–165

On vector variational-like inequalities and vector optimization problems in Asplund spaces

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Abstract. In this paper, we consider different kinds of generalized invexity for vector valued functions and a vector optimization problem. Some relations between some vector variational-like inequalities and a vector optimization problem are established using the properties of Mordukhovich limiting subdifferentials under $C-\eta-$strong pseudomonotonicity.

Mathematics Subject Classification (2010): 26A51, 26B25, 47N10, 65K10.

Keywords: Vector variational-like inequalities, Vector optimization problem, Mordukhovich limiting subdifferentials, Asplund space, $C-\eta-$strong pseudomonotonicity.

1. Introduction

In 1998, Giannessi [9] first used, so called, Minty type vector variational inequality (in short, $MVVI$) to establish the necessary and sufficient conditions for a point to be an efficient solution of a vector optimization problem (in short, (VOP)) for differentiable and convex functions. Since then, several researchers have studied (VOP) by using different kinds of $MVVI$ under different assumptions, see [1, 2, 10, 15, 19] and the references therein. Consequently, vector variational inequalities have been generalized in various directions, in particular, vector variational-like inequality problems, see [1, 13, 14, 20, 23, 28] and the references therein. The vector variational-like inequalities are closely related to the concept of the invex and preinvex functions which generalize the notion of the convexity of functions . The concept of the invexity was first introduced by Hanson [12]. More recently, the characterization and applications for generalized invexity were studied by many authors, see [11, 13, 19, 21, 24, 25, 27] and the references therein.

The relation between the vector variational inequality and the smooth vector optimization problem has been studied by many authors (see, for example, [9, 23, 26] and the references therein). Yang et al. [26] extended the result of Giannessi [9, 10] for differentiable but pseudoconvex functions. Yang and Yang [23] gave some relations between Minty variational-like inequalities and the vector optimization problems for differentiable but pseudo-invex vector-valued functions. Yang et al. [25, 26] and Garzon et al. [6, 7] studied the relations between generalized invexity of a differentiable function and generalized monotonicity of its gradient mapping. Very recently, Rezaie and Zafarani [20] showed some relations between the vector variational-like inequalities and vector optimization problems for nondifferential functions under generalized monotonicity. Al-Homidan and Ansari [1] studied the relation among the generalized Minty vector variational-like inequality, generalized Stampacchia vector variational-like inequality and vector optimization problems for nondifferential and nonconvex functions with Clarke's generalized directional derivative and then, Ansari and Lee [2] showed that for pseudoconvex functions with upper Dini directional derivative, similar results holds. Ansari, Rezaie and Zafarani [3] considered generalized Minty vector variational-like inequality problems, Stampacchia vector variational-like inequality problems and nonsmooth vector optimization problems under nonsmooth pseudo-invexity assumptions. They also considered the weak formulations of generalized Minty vector variational-like inequality problems and generalized Stampacchia vector variational-like inequality problems in a very general setting and established the existence results for their solutions. The main results in [1] and [20] were obtained in the setting of Clarke subdifferential. Since the class of Clarke subdifferential is larger than the class of Mordukhovich subdifferential, it is necessary to study the vector variational-like inequalities and vector optimization problems in the setting of Mordukhovich subdifferential (see [5, 16, 17]). Oveisiha and Zafarani [18] established some properties of pseudo-invex functions and Mordukhovich limiting subdifferential and relations between vector variational-like inequalities and vector optimization problems. Chen and Huang [4] considered the Minty vector variational-like inequality, Stampacchia vector variational-like inequality and the weak formulations of these inequalities, defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. They established some relations between the vector variational-like inequalities and vector optimization problems using the properties of Mordukhovich limiting subdifferential. Farajzadeh et al. [8] considered generalized variational-like inequalities with set-valued mappings in topological spaces, which include as a special case the strong vector variational-like inequalities. Motivated and inspired by the work mentioned above, in this paper we consider the Minty vector variational-like inequality, Stampacchia vector variational-like inequality and the weak formulations of these inequalities, defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. Some relations between vector variational-like inequalities and a vector optimization problem (respectively, between Minty vector variational-like inequality and Stampacchia vector variational-like inequality) are established using the properties of Mordukhovich limiting subdifferentials under different kinds of generalized invexity (respectively, $C-\eta-$strong pseudomonotonicity).

2. Preliminaries

Let X be a Banach space endowed with a norm $\|.\|$ and X^* its dual space with a norm $\|.\|_*$. Denote $\langle .,. \rangle$, $[x,y]$, $]x,y[$ the dual pair between X and X^*, the line segment for $x,y \in X$ and $[x,y]\backslash\{x,y\}$, respectively. Let Ω be a nonempty open subset of X.

When functions are not differentiable, we use the concept of subdifferential: Fréchet subdifferential, Limiting subdifferential and Clarke-Rockafellar subdifferential.

Definition 2.1. *Let X be a Banach space and $f : X \to \mathbf{R} \cup \{\infty\}$ a proper l.s.c. function. We say that f is Fréchet-subdifferentiable and ξ^* is Fréchet-subderivative of f at x ($\xi^* \in \partial_F f(x)$) if $x \in dom\ f$ and*

$$\liminf_{\|h\|\to 0} \frac{f(x+h) - f(x) - \langle \xi^*, h\rangle}{\|h\|} \geq 0.$$

Definition 2.2. [16] *Let $x \in \Omega$ and $\varepsilon \geq 0$. The set of $\varepsilon-$normals to Ω at x is defined by*

$$\widehat{N}_\varepsilon(x,\Omega) = \{x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \leq \varepsilon\}.$$

If $x \notin \Omega$, we put $\widehat{N}_\varepsilon(x,\Omega) = \varnothing$ for all $\varepsilon \geq 0$.

Definition 2.3. [16] *Let $\overline{x} \in \Omega$. Then $x^* \in X^*$ is a limiting normal to Ω at $\overline{x}$ if there are sequences $\varepsilon_k \searrow 0$, $x_k \xrightarrow{\Omega} \overline{x}$ and $x_k^* \xrightarrow{w^*} \overline{x}^*$ such that $x_k^* \in \widehat{N}_\varepsilon(x_k,\Omega)$, for all $k \in \mathbf{N}$. The set of such normals*

$$N(\overline{x},\Omega) = \limsup_{\substack{x\to\bar{x}\\ \varepsilon\searrow 0}} \widehat{N}_\varepsilon(x,\Omega)$$

is the limiting normal cone to Ω at $\overline{x}$. If $\overline{x} \notin \Omega$, we put $N(\overline{x},\Omega) = \varnothing$.

Remark 2.4. Note that the symbol $u \xrightarrow{\Omega} x$ means that $u \to x$ with $u \in \Omega$. The symbol $\xrightarrow{w^*}$ stands for convergence in weak* topology.

Definition 2.5. [16] *Considering the extended-real-valued function $\varphi : X \to \overline{\mathbf{R}} = [-\infty, +\infty]$ we say that φ is proper if $\varphi(x) > -\infty$ for all $x \in X$ and its domain, $dom\varphi = \{x \in X : \varphi(x) < \infty\}$, is nonempty. The epigraph of φ is defined as*

$$epi\varphi = \{(x,a) \in X \times \mathbf{R} / \varphi(x) \leq a\}.$$

Definition 2.6. [16] *Considering a point $\overline{x} \in X$ with $\mid \varphi(\overline{x}) \mid < \infty$, the set*

$$\partial_L \varphi(\overline{x}) = \{x^* \in X^* \mid (x^*, -1) \in N((\overline{x}, \varphi(\overline{x})), epi\ \varphi)\}$$

is the limiting subdifferential of φ at $\bar{x}$ and its elements are limiting subdifferentials of φ at this point. If $\mid \varphi(\overline{x}) \mid = \infty$, we put $\partial_L \varphi(\overline{x}) = \varnothing$.

Remark 2.7. [16] It is well known that

$$\partial_F f(x) \subseteq \partial_L f(x) \subseteq \partial_C f(x),$$

where $\partial_C f$ is the Clarke subdifferential.

Definition 2.8. *A Banach space X is Asplund, or it has the Asplund property, if every convex continuous function $\varphi : U \to \mathbf{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.*

Remark 2.9. One of the most popular Asplund spaces is any reflexive Banach space [16].

Theorem 2.10. [16] *Let X be an Asplund space and $\varphi : X \to \overline{\mathbf{R}}$ be proper and l.s.c. around $\overline{x} \in dom\varphi$, then*

$$\partial_L \varphi(\overline{x}) = \limsup_{\substack{x \to \overline{x} \\ \varepsilon \searrow 0}} \partial_F \varphi(x).$$

For more details and applications, see [16].

Definition 2.11. *Let $\eta : X \times X \to X$. A subset Ω of X is said to be invex with respect to η if for any $x, y \in \Omega$ and $\lambda \in [0, 1]$, we have $y + \lambda\eta(x, y) \in \Omega$.*

Hereafter, unless otherwise specified, we assume that X is an Asplund space and $\Omega \subseteq X$ is a nonempty open invex set with respect to the mapping $\eta : \Omega \times \Omega \to X$.

Definition 2.12. *A mapping $\eta : \Omega \times \Omega \to X$ is said to be skew if for any $x, y \in \Omega$,*

$$\eta(x, y) + \eta(y, x) = 0.$$

Definition 2.13. *Let $x_0 \in \Omega$. A mapping $\eta : \Omega \times \Omega \to X$ is said to be skew at x_0 if for any $x \in \Omega$, $x \neq x_0$,*

$$\eta(x, x_0) + \eta(x_0, x) = 0.$$

Definition 2.14. [21] *Let $f : \Omega \to \mathbf{R}$ be a function. f is said to be*

1. *$weakly - quasi - invex$ with respect to η on Ω if for any $x, y \in \Omega$,*

$$f(x) \le f(y) \Rightarrow \exists\ \xi^* \in \partial_L f(y)\ \langle \xi^*, \eta(x, y)\rangle \le 0;$$

2. *$quasi - invex$ with respect to η on Ω if for any $x, y \in \Omega$,*

$$f(x) \le f(y) \Rightarrow \forall\ \xi^* \in \partial_L f(y)\ \langle \xi^*, \eta(x, y)\rangle \le 0;$$

3. *$pseudo - invex$ with respect to η on Ω if for any $x, y \in \Omega$,*

$$\langle \xi^*, \eta(x, y)\rangle \ge 0,\ \exists\ \xi^* \in \partial_L f(y) \Rightarrow f(x) \ge f(y).$$

In some results of the paper we need to consider some further assumptions on η. These assumptions are known in invexity literature (Jabarootian and Zafarani (2006) [13]).
Condition C. Let $\eta : \Omega \times \Omega \to X$. Then for any $x, y \in \Omega$, $\lambda \in [0, 1]$,

$$\begin{cases} C_1 : \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y); \\ C_2 : \eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y). \end{cases}$$

Remark 2.15. Yang et al. [27] have shown that if $\eta : \Omega \times \Omega \to X$ satisfies **condition C**, then for all $x, y \in \Omega$, $\lambda \in [0, 1]$,

$$\eta(y + \lambda\eta(x, y), y) = \lambda\eta(x, y).$$

Definition 2.16. *Let $\eta : \Omega \times \Omega \to X$, $x_0 \in \Omega$. We say that $\eta : \Omega \times \Omega \to X$ satisfies* **condition C** *at x_0 if for all $x \in \Omega$, $\lambda \in [0,1]$,*

$$\eta(x_0 + \lambda\eta(x, x_0), x_0) = \lambda\eta(x, x_0).$$

Definition 2.17. *Let $f = (f_1, ..., f_n) : \Omega \to \mathbf{R}^n$ be a vector-valued function and $x_0 \in \Omega$. f is said to be*

1. *pseudo − invex with respect to η on Ω if for any $x, y \in \Omega$,*
$$f(x) - f(y) \in -\mathbf{R}^n_+\backslash\{0\} \implies \langle \partial_L f(y), \eta(x, y)\rangle \subseteq -\mathbf{R}^n_+\backslash\{0\};$$
2. *quasi − invex with respect to η on Ω if for any $x, y \in \Omega$,*
$$\langle \xi^*, \eta(x, y)\rangle \in \mathbf{R}^n_+\backslash\{0\},\ \exists\ \xi^* \in\ \partial_L f(y) \implies f(x) - f(y) \in \mathbf{R}^n_+\backslash\{0\};$$
3. *weakly − quasi − invex with respect to η on Ω if for any $x, y \in \Omega$,*
$$\langle \partial_L f(y), \eta(x, y)\rangle \subseteq \mathbf{R}^n_+\backslash\{0\} \implies f(x) - f(y) \in \mathbf{R}^n_+\backslash\{0\};$$
4. *weakly − quasi − invex at x_0 with respect to η if for any $x \in \Omega$,*
$$\langle \partial_L f(x_0), \eta(x, x_0)\rangle \subseteq \mathbf{R}^n_+\backslash\{0\} \implies f(x) - f(x_0) \in \mathbf{R}^n_+\backslash\{0\}.$$

Remark 2.18. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be pseudo-invex with respect to η on Ω and there exists $k, 1 \le k \le n$, such that f_k is not pseudo-invex with respect to η on Ω.

Example 2.19. Let us consider $X = \mathbf{R}$, $\Omega = [-1, 1]$, $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ defined as

$$f_1(x) = \begin{cases} \sqrt{x}, & x \ge 0, \\ x, & x < 0, \end{cases}$$

$$f_2(x) = x$$

and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L f(x) = \begin{cases} (\frac{1}{2\sqrt{x}}, 1) & x > 0, \\ [0, \infty[\times\{1\}, & x = 0, \\ (1, 1), & x < 0. \end{cases}$$

It is not difficult to see that f is pseudo-invex with respect to η. Function f_1 is not pseudo-invex with respect to η on Ω because for $x = -1, y = 0$ there exists $\xi^* = 0 \in \partial_L f(y)$ such that $\langle \xi^*, \eta(x, y)\rangle = 0$ and $f(x) < f(y)$.

Definition 2.20. [8] *A set valued mapping $F : \Omega \to 2^{X^*}$ is said to be $C - \eta-$strong pseudomonotone if for any $x, y \in \Omega$,*

$$\langle Fx, \eta(x, y)\rangle \nsubseteq -C(x)\backslash\{0\} \Longrightarrow \langle Fy, \eta(y, x)\rangle \subseteq -C(y).$$

Definition 2.21. *A set valued mapping $F : \Omega \to 2^{X^*}$ is said to be $(C, K) - \eta-$strong pseudomonotone if for any $x, y \in \Omega$,*

$$\langle Fx, \eta(x, y)\rangle \nsubseteq C \Longrightarrow \langle Fy, \eta(y, x)\rangle \subseteq K.$$

Definition 2.22. *A set valued mapping* $F : \Omega \to 2^{X^*}$ *is said to be strictly* $(C, K)-\eta-$*strong pseudomonotone if for any* $x, y \in \Omega, x \neq y$,

$$\langle Fx, \eta(x, y)\rangle \nsubseteq C \Longrightarrow \langle Fy, \eta(y, x)\rangle \subseteq K.$$

Let $f = (f_1, ..., f_n) : \Omega \to \mathbf{R}^n$ be a vector-valued function, where $f_i : \Omega \to \mathbf{R}$ $(i = 1, ..., n)$ is non-differentiable locally Lipschitz function.

In this paper, we consider the following vector optimization problem:

$$(VOP) \quad \text{Minimize } f(x) = (f_1(x), ..., f_n(x)) \quad \text{subject to } x \in \Omega.$$

Definition 2.23. *A point* $x_0 \in \Omega$ *is said to be an efficient (or Pareto) solution (respectively, weak efficient solution) of* (VOP) *if for all* $x \in \Omega$,

$$\begin{aligned} f(x) - f(x_0) &= (f_1(x) - f_1(x_0), ..., f_n(x) - f_n(x_0)) \notin -\mathbf{R}^n_+ \backslash \{0\}, \\ (\textit{respectively, } f(x) - f(x_0) &= (f_1(x) - f_1(x_0), ..., f_n(x) - f_n(x_0)) \notin -int\mathbf{R}^n_+), \end{aligned}$$

where $\mathbf{R}^n_+$ *is the nonnegative orthant of* $\mathbf{R}^n$ *and* 0 *is the origin of the nonnegative orthant.*

3. Characterization

We consider the following Minty vector variational-like inequality problems and Stampacchia vector variational-like inequality problems.

$(GGMVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ and all $\xi_i \in \partial_L f_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi_1^*, \eta(x, x_0)\rangle, ..., \langle \xi_n^*, \eta(x, x_0)\rangle) \notin -\mathbf{R}^n_+ \backslash \{0\}.$$

$(GMVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ there exists $\xi_i \in \partial_L f_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi_1^*, \eta(x, x_0)\rangle, ..., \langle \xi_n^*, \eta(x, x_0)\rangle) \notin -\mathbf{R}^n_+ \backslash \{0\}.$$

$(WGGMVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ and all $\xi_i \in \partial_L f_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi_1^*, \eta(x, x_0)\rangle, ..., \langle \xi_n^*, \eta(x, x_0)\rangle) \notin -int\mathbf{R}^n_+.$$

$(WGMVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ there exists $\xi_i \in \partial_L f_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi_1^*, \eta(x, x_0)\rangle, ..., \langle \xi_n^*, \eta(x, x_0)\rangle) \notin -int\mathbf{R}^n_+.$$

$(SVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ there exists $\xi_i \in \partial_L f_i(x_0)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi_1^*, \eta(x, x_0)\rangle, ..., \langle \xi_n^*, \eta(x, x_0)\rangle) \notin -\mathbf{R}^n_+ \backslash \{0\}.$$

$(WSVVLIP)$ Find $x_0 \in \Omega$ such that, for all $x \in \Omega$ there exists $\xi_i \in \partial_L f_i(x_0)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi_1^*, \eta(x, x_0)\rangle, ..., \langle \xi_n^*, \eta(x, x_0)\rangle) \notin -int\mathbf{R}^n_+.$$

Theorem 3.1. *If x_0 is a solution of $(SVVLIP)$, $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone, η is skew at x_0 and $\langle\ \partial_L f(x_0), \eta(x_0, x_0)\rangle \subseteq \mathbf{R}^n_+$, then x_0 is a solution of $(GGMVVLIP)$.*

Proof. Suppose that x_0 is not a solution of $(GGMVVLIP)$.
Since $\langle \partial_L f(x_0), \eta(x_0, x_0)\rangle \subseteq \mathbf{R}^n_+$ it follows that there exist $\overline{x} \in \Omega, \overline{x} \neq x_0$, $\overline{\zeta} \in \partial_L f(\overline{x})$ such that

$$\langle \overline{\zeta}, \eta(\overline{x}, x_0)\rangle \in -\mathbf{R}^n_+ \backslash \{0\}.$$

Therefore,

$$\langle \overline{\zeta}, \eta(\overline{x}, x_0)\rangle \notin int\mathbf{R}^n_+. \tag{3.1}$$

Since $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone, by (3.1) we obtain

$$\langle \partial_L f(x_0), \eta(x_0, \overline{x})\rangle \subseteq int\mathbf{R}^n_+. \tag{3.2}$$

Since η is skew at x_0, by (3.2) it follows that

$$\langle \partial_L f(x_0), \eta(\overline{x}, x_0)\rangle \subseteq -int\mathbf{R}^n_+,$$

which contradicts the fact that x_0 is a solution of $(SVVLIP)$. Therefore, it follows that x_0 is a solution of $(GGMVVLIP)$. □

Example 3.2. Let us consider $X = \mathbf{R}$, $\Omega = [-1, 1]$, $f : \Omega \to \mathbf{R}$ defined as

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0, \\ [0, \infty[\cup\{-1\}, & x = 0, \\ -1, & x < 0. \end{cases}$$

and $\partial_L f$ is strictly $(int\mathbf{R}_+, int\mathbf{R}_+) - \eta-$strong pseudomonotone. It is not difficult to see that $x_0 = 0$ is a solution of $(SVVLIP)$ and η is skew at x_0. Therefore, x_0 is a solution of $(GGMVVLIP)$.

Corollary 3.3. *If x_0 is a solution of $(SVVLIP)$, $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone and η is skew at x_0, then x_0 is a solution of $(GMVVLIP)$.*

Corollary 3.4. *If x_0 is a solution of $(WSVVLIP)$, $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone, η is skew at x_0 and $\langle\ \partial_L f(x_0), \eta(x_0, x_0)\rangle \subseteq \mathbf{R}^n_+\backslash\{0\}$, then x_0 is a solution of $(GGMVVLIP)$.*

Corollary 3.5. *If x_0 is a solution of $(WSVVLIP)$, $\partial_L f$ is strictly $(int\mathbf{R}^n_+, int\mathbf{R}^n_+) - \eta-$strong pseudomonotone and η is skew at x_0, then x_0 is a solution of $(WGMVVLIP)$.*

Theorem 3.6. *If x_0 is a solution of (VOP), f is quasi-invex with respect to η on Ω and η is skew, then x_0 is a solution of $(GGMVVLIP)$.*

Proof. Suppose that x_0 is not a solution of $(GGMVVLIP)$. It follows that there exist $\overline{x} \in \Omega$, $\overline{\zeta} \in \partial_L f(\overline{x})$ such that we have

$$\langle \overline{\zeta}, \eta(\overline{x}, x_0) \rangle \in -\mathbf{R}^n_+ \backslash \{0\}. \tag{3.3}$$

Since η is skew, by (3.3) we obtain

$$\langle \overline{\zeta}, \eta(x_0, \overline{x}) \rangle \in \mathbf{R}^n_+ \backslash \{0\}.$$

Since f is quasi-invex, it follows that

$$f(x_0) - f(\overline{x}) \in \mathbf{R}^n_+ \backslash \{0\},$$

which contradicts the fact that x_0 is a solution of (VOP). Therefore, x_0 is a solution of $(GGMVVLIP)$. □

Remark 3.7. In [4] (Theorem 3.1) the authors obtained this result by assuming that $f_i (i = 1, ..., n)$ are invex with respect to η on Ω. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be quasi-invex with respect to η on Ω and there exists $k, 1 \leq k \leq n$, such that f_k is not invex with respect to η on Ω.

Example 3.8. Let us consider $X = \mathbf{R}$, $\Omega = [-\frac{1}{5}, \frac{1}{5}]$, $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ defined as

$$f_1(x) = \begin{cases} x^2 + 2x, & x > 0, \\ -x, & x \leq 0, \end{cases}$$

$$f_2(x) = \begin{cases} x^3 - 2x^2 + x, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L f(x) = \begin{cases} (2x + 2, 3x^2 - 4x + 1), & x > 0, \\ (k, t), & k \in \{2, -1\}, t \in \{1, -1\}, x = 0. \\ (-1, -1), & x < 0. \end{cases}$$

It is easy to observe that $x_0 = 0$ is a solution of (VOP), η is skew and function f is quasi-invex with respect to η on Ω. Function f_2 is not invex with respect to η on Ω because for $x = 1, y = 0$ we obtain

$$f_2(1) - f_2(0) < \langle \xi^*, \eta(1, 0) \rangle,$$

for $\xi^* = 1$.

Corollary 3.9. *If x_0 is a solution of (VOP), f is quasi-invex with respect to η on Ω and η is skew, then x_0 is a solution of $(GMVVLIP)$.*

Theorem 3.10. *If x_0 is a solution of (VOP), f is weakly quasi-invex at x_0 with respect to η on Ω and η is skew at x_0, then x_0 is a solution of $(GMVVLIP)$.*

Proof. Suppose that x_0 is not a solution of $(GMVVLIP)$. Therefore, there exists $\overline{x} \in \Omega$ such that for all $\xi^* \in \partial_L f(\overline{x})$ we have

$$\langle \xi^*, \eta(\overline{x}, x_0) \rangle \in -\mathbf{R}^n_+ \backslash \{0\}. \tag{3.4}$$

Hence,

$$\langle \partial_L f(\overline{x}), \eta(\overline{x}, x_0)\rangle \subseteq -\mathbf{R}^n_+\backslash\{0\}. \tag{3.5}$$

Since η is skew at x_0 we obtain

$$\langle \partial_L f(\overline{x}), \eta(x_0, \overline{x})\rangle \subseteq \mathbf{R}^n_+\backslash\{0\}.$$

Since f is weakly quasi-invex at x_0 with respect to η on Ω it follows that

$$f(x_0) - f(\overline{x}) \in \mathbf{R}^n_+\backslash\{0\},$$

which contradicts the fact that x_0 is a solution of (VOP). Therefore, x_0 is a solution of $(GMVVLIP)$. □

Remark 3.11. In [18] (Theorem 13) the authors obtained this result by assuming that $f_i (i = 1, ..., n)$ are pseudo-invex with respect to η on Ω. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be weakly quasi-invex with respect to η on Ω and there exists $k, 1 \le k \le n$, such that f_k is not pseudo-invex with respect to η on Ω.

Example 3.12. Let us consider $X = \mathbf{R}$, $\Omega = [-1, 1]$, $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ defined as

$$f_1(x) = \begin{cases} \sqrt{x}, & x \ge 0, \\ x, & x < 0, \end{cases}$$

$$f_2(x) = \begin{cases} \frac{1}{2}\sqrt{x}, & x \ge 0, \\ -x, & x < 0, \end{cases}$$

$x_0 = 0$ and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We obtain that

$$\partial_L f_1(x) = \begin{cases} (\frac{1}{2\sqrt{x}}, \frac{1}{4\sqrt{x}}), & x > 0, \\ [0, \infty[\times([0, \infty[\cup\{-1\}), & x = 0, \\ (1, -1), & x < 0. \end{cases}$$

It is not difficult to verify that f is weakly quasi-invex at x_0 with respect to η, $x_0 = 0$ is solution of (VOP), η is skew at x_0 and f_1 is not pseudo-invex with respect to η on Ω because for $x = -1, y = 0$ there exists $\xi^* = 0 \in \partial_L f(y)$ such that $\langle \xi^*, \eta(x, y)\rangle = 0$ and $f(x) < f(y)$.

Theorem 3.13. *Suppose that x_0 is a solution of $(SVVLIP)$ and f is pseudo-invex with respect to η on Ω. Then, x_0 is a solution of (VOP).*

Proof. Suppose that x_0 is not a solution of (VOP). Therefore, there exists $\overline{x} \in \Omega$ such that

$$f(\overline{x}) - f(x_0) \in -\mathbf{R}^n_+\backslash\{0\}.$$

Since f is pseudo-invex with respect to η on Ω, it follows that

$$\langle \partial_L f(x_0), \eta(\overline{x}, x_0)\rangle \subseteq -\mathbf{R}^n_+\backslash\{0\},$$

which contradicts the fact that x_0 is a solution of $(SVVLIP)$. Therefore, x_0 is a solution of (VOP).

Remark 3.14. In [4] (Theorem 3.2) the authors obtained this result by assuming that $f_i (i = 1, ..., n)$ are invex with respect to η on Ω. Next, we provide an example which shows that a function $f = (f_1, ..., f_n)$ it can be pseudo-invex with respect to η on Ω and there exists $k, 1 \leq k \leq n$, such that f_k is not invex with respect to η on Ω.

Example 3.15. Let us consider $X = \mathbf{R}$, $\Omega = [-1, 1]$, $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ defined as

$$f_1(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

$$f_2(x) = x$$

and $\eta : \Omega \times \Omega \to \mathbf{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L f_1(x) = \begin{cases} (\frac{1}{2\sqrt{x}}, 1), & x > 0, \\ ([0, \infty[\cup\{-1\}) \times \{1\}, & x = 0, \\ (-1, 1), & x < 0. \end{cases}$$

It is not difficult to see that $x_0 = 0$ is solution of $(SVVLIP)$, f is pseudo-invex with respect to η. Function f_1 is not invex with respect to η on Ω because for $x = 1, y = 0$ we obtain

$$f(1) - f(0) < \langle \xi^*, \eta(1, 0) \rangle,$$

for $\xi^* = 2$.

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