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Semi-φ_h and strongly $\log$-φ convexity

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Abstract. In this note, semi-φ_h-convexity as a generalization of h-convexity and semi φ-convexity, and strongly log-φ convex functions have been introduced and studied. Some properties of semi-φ_h-convex functions are proved. Also, some new results of Hemite-Hadamard type inequalities for semi-φ_h-convex functions, semi log- φ and strongly log-φ convex functions are obtained.

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1. Introduction

In 1883, Hermite proved an inequality, rediscovered by Hadamard in 1893, that for a convex function f on $[a,b] \in \mathbb{R}$, also continuous at the endpoints, one has that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This is known as Hermite-Hadamard inequality. In the literature, many modifications, generalizations and extensions of this inequality has been obtained for last few years.

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$, is said to be convex on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for all $x, y \in I$ and $t \in (0,1)$.

Let I be an interval in $\mathbb{R}$ and $h : (0,1) \to (0,\infty)$ be a given function. Then a function $f : I \to \mathbb{R}$ is said to be h-convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y),$$

for all $x, y \in I$ and $t \in (0,1)$.

If $h(t) = t^s; s \in (0,1)$, then f is said to be s-convex in second sense [2], if f is non-negative and $h(t) = \frac{1}{t}$ then f is said to be Godunova-Levin function [6] and if f is non-negative with $h(t) = 1$ then f is P-convex function [7].

In [14], Youness introduced a new class of functions called φ-convex functions and he established some results about these sets and functions. Later on, the result by Youness [14] were improved by Yang [13], Duca *et al.* [4] and Chen [3]. Throughout this paper, we assume that $\varphi : I \to I$, where I is a real interval and $h : (0,1) \to (0,\infty)$ are given maps.

Definition 1.1. *A function $f : I \to \mathbb{R}$ is said to be φ-convex on I if*

$$f(t\varphi(x) + (1-t)\varphi(y)) \le tf(\varphi(x)) + (1-t)f(\varphi(y)),$$

for all $x, y \in I$ and $t \in (0,1)$.

In [11], Sarikaya has studied φ_h- convexity and obtained some new inequalities.

Definition 1.2. *Let I be an interval in $\mathbb{R}$. We say that a function $f : I \to [0,\infty)$ is a φ_h-convex if*

$$f(t\varphi(x) + (1-t)\varphi(y)) \le h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)),$$

for all $t \in (0,1)$ and $x, y \in I$.

Theorem 1.3. (Th. 2, [11]) *Let $h : (0,1) \to (0,\infty)$ be a given function. If $f : I \to [0,\infty)$ is Lebesgue integrable on I and φ_h-convex for continuous function $\varphi : [a,b] \to [a,b]$, with $\varphi(a) \neq \varphi(b)$, then the following inequality holds:*

$$\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(b)+\varphi(a)-x)dx$$
$$\le \Big[f^2(\varphi(x)) + f^2(\varphi(y))\Big] \int_0^1 h(t)h(1-t)dt + 2f(\varphi(x))f(\varphi(y)) \int_0^1 h^2(t)dt.$$

Hu at al [8] studied firstly the notion of semi-φ-convexity. Chen in [3] modified their results and defined the following class of functions.

Definition 1.4. *The function $f : I \to \mathbb{R}$ is semi-φ-convex, if for every $x, y \in I$ and $t \in (0,1)$ we have*

$$f(t\varphi(x) + (1-t)\varphi(y)) \le tf(x) + (1-t)f(y).$$

Toader [12] defined the following function:

Definition 1.5. *Let be $b > 0$ and $m \in (0,1]$. A function $f : [0,b] \to [0,\infty)$ is said to be m-convex if*

$$f(tx + m(1-t)y) \le tf(x) + (1-t)f(y),$$

for all $x, y \in [0,b]$, $t \in [0,1]$.

In [5], Dragomir and Pečarić showed that the following result holds for m-convex functions.

Theorem 1.6. (Th. 197, [5]) *If $f : [0,\infty) \to [0,\infty)$ is a m-convex function with $m \in (0,1)$ and Lebesgue integrable on $[ma,b]$ where $0 \le a \le b$ and $mb \neq a$, then*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right] \le \frac{f(a)+f(b)}{2}.$$

The rest of the paper is organized as follows: In section 2, semi-φ_h-convexity has been defined and some properties are studied. In section 3, some new results of Hadamard type inequalities are proved. In the last section, semi log-φ and strongly log-φ convex functions are discussed and some inequalities are obtained.

2. Semi-φ_h-Convexity

In this section, we define the following function:

Definition 2.1. *Let $\varphi : [a, b] \to [a, b]$ and I be an interval in $\mathbb{R}$ such that $[a, b] \subseteq I$. Let $h : (0, 1) \to (0, \infty)$ be a given function. We say that a function $f : I \to [0, \infty)$ is a semi-φ_h-convex if for all $t \in (0, 1)$ and $x, y \in I$, we have*

$$f(t\varphi(x) + (1 - t)\varphi(y)) \leq h(t)f(x) + h(1 - t)f(y).$$

Remark 2.2. 1. If $h(t) = t$, f is a semi-φ-convex function on I.
2. If $h(t) = t^s$, f is a semi-φ_s-convex function on I.
3. If $h(t) = \frac{1}{t}$, f is a semi-φ Gudunova-Levin convex function on I.
4. If $h(t) = 1$, f is a semi-φP-convex function on I.
5. If $\varphi(x) = x$, f is a h-convex function on I.
6. If $\varphi(x) = x$ and $h(t) = t$, f is a convex function on I.

Example 2.3. [3] Let $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(x) = \begin{cases} 1, & 1 \leq x \leq 4 \\ 1 + \frac{2}{\pi}\arctan(1 - x), & x < 1 \\ 2 + \frac{\pi}{4}\arctan(x - 4), & x > 4. \end{cases}$$

and $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 7, & x < 1 \text{ or } x > 4 \\ x - 3, & 1 \leq x < 2 \\ 3 - x, & 2 \leq x \leq 3 \\ x - 3, & 3 < x \leq 4. \end{cases}$$

Here f is a semi-φ_h-convex function on $\mathbb{R}$ for $h(t) = t$.

Example 2.4. Let $h(t) = 1$ for all $t \in \mathbb{R}$, $\varphi(x) = -x^2$, for all $x \in \mathbb{R}$, and

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 2, & x \leq 0. \end{cases}$$

Then f is a semi-φ P-convex function on $\mathbb{R}$.

Now we prove some properties of semi-φ_h-convex functions.

Theorem 2.5. *If $f, g : I \to [0, \infty)$ are semi-φ_h-convex functions, where $h : (0, 1) \to (0, \infty)$ is a given function, and $\alpha > 0$ then $f + g$ and αf are semi-φ_h-convex functions.*

Proof. Since f, g are semi-φ_h convex functions then for $x, y \in I$ and $t \in (0, 1)$,

$$\begin{aligned}(f + g)(t\varphi(x) + (1 - t)\varphi(y)) &= f(t\varphi(x) + (1 - t)\varphi(y)) + g(t\varphi(x) + (1 - t)\varphi(y)) \\ &\leq h(t)(f + g)(x) + h(1 - t)(f + g)(y),\end{aligned}$$

and

$$\begin{aligned}(\alpha f)(t\varphi(x)+(1-t)\varphi(y)) &\le \alpha[h(t)f(x)+h(1-t)f(y))] \\ &= h(t)(\alpha f)(x)+h(1-t)(\alpha f)(y).\end{aligned}$$

□

Lemma 2.6. *If $f : I \to [0,\infty)$ is a semi-φ convex function and g is an increasing h-convex function, where range of f is contained in the domain of g and $h : (0,1) \to (0,\infty)$, then $g \circ f$ is a semi-φ_h- convex function.*

Proof. Since f is semi-φ-convex function then for $x, y \in I$ and $t \in (0,1)$,

$$f(t\varphi(x)+(1-t)\varphi(y)) \le tf(x)+(1-t)f(y).$$

Since g is increasing and h-convex we have

$$\begin{aligned}(g\circ f)(t\varphi(x)+(1-t)\varphi(y)) &\le g(tf(x)+(1-t)f(y)) \\ &\le h(t)(g\circ f)(x)+h(1-t)(g\circ f)(y).\end{aligned}$$

This completes the proof. □

Lemma 2.7. *If f is semi-φ-convex and $h(t) \ge t$ then f is semi-φ_h-convex.*

Proof.

$$f(t\varphi(x)+(1-t)\varphi(y)) \le tf(x)+(1-t)f(y) \le h(t)f(x)+h(1-t)f(y).$$

This completes the proof. □

Lemma 2.8. *If f is semi-φ_h convex and $h(t) \le t$ then f is semi-φ-convex.*

Proof.

$$f(t\varphi(x)+(1-t)\varphi(y)) \le h(t)f(x)+h(1-t)f(y) \le tf(x)+(1-t)f(y).$$

This completes the proof. □

Lemma 2.9. *Let $h_1, h_2 : (0,1) \to (0,\infty)$ such that $h_2(t) \le h_1(t)$. If f is semi-φ_{h_2} convex then f is semi-φ_{h_1} convex.*

Proof. Since f is semi-φ_{h_2} convex then for $x, y \in I$ and $t \in (0,1)$ we have

$$f(t\varphi(x)+(1-t)\varphi(y)) \le h_2(t)f(x)+h_2(1-t)f(y) \le h_1(t)f(x)+h_1(1-t)f(y).$$

This completes the proof. □

3. Hermite-Hadamard Type Inequalities

Theorem 3.1. *If $[a,b] \subseteq I$, $\varphi : [a,b] \to [a,b]$ is a continuous function such that $\varphi(a) \ne \varphi(b)$ and the function $f : I \to [0,\infty)$ is Lebesgue integrable on I and semi-φ_h convex, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx \le \Big(f(a)+f(b)\Big)\int_0^1 h(t)dt.$$

Proof. Since f is semi-φ_h convex, we have for $t \in (0,1)$,

$$f(t\varphi(a) + (1-t)\varphi(b)) \leq h(t)f(a) + h(1-t)f(b).$$

Integrating the above inequality over the interval $(0,1)$,

$$\int_0^1 f(t\varphi(a) + (1-t)\varphi(b))dt \leq (f(a) + f(b)) \int_0^1 h(t)dt.$$

Substituting $x = t\varphi(a) + (1-t)\varphi(b)$ we get the required inequality. □

Corollary 3.2. *Under the assumptions of Theorem* 3.1 *with* $h(t) = t$ *for all* $t \in (0,1)$, *we have*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Corollary 3.3. *Under the assumptions of Theorem* 3.1 *with* $s \in (0,1)$ *and* $h(t) = t^s$ *for all* $t \in (0,1)$, *we have*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq \frac{f(a) + f(b)}{s+1}.$$

Corollary 3.4. *Under the assumptions of Theorem* 3.1 *with* $h(t) = 1$ *for* $t \in (0,1)$, *we have*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq f(a) + f(b).$$

Remark 3.5. If $h(t) = t$ for $t \in (0,1)$ and $\varphi(x) = x$ we have

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Theorem 3.6. *If* $[a,b] \subseteq I$, $\varphi : [a,b] \to [a,b]$ *is a continuous function such that* $\varphi(a) \neq \varphi(b)$ *and the function* $f : I \to [0,\infty)$ *is Lebesgue integrable on* I *and semi-*φ_h *convex, then*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(a) + \varphi(b) - x)dx$$
$$\leq (f^2(a) + f^2(b))(\int_0^1 h(t)h(1-t)dt + 2f(a)f(b) \int_0^1 h^2(t)dt).$$

Proof. Since f is semi-φ_h convex we have for $t \in (0,1)$

$$f(t\varphi(a) + (1-t)\varphi(b)) \leq h(t)f(a) + h(1-t)f(b),$$

and

$$f((1-t)\varphi(a) + (t\varphi(b)) \leq h(1-t)f(a) + h(t)f(b).$$

By multiplying both inequalities, we get

$$f(t\varphi(a) + (1-t)\varphi(b))f((1-t)\varphi(a) + t\varphi(b))$$
$$\leq h(1-t)h(t)(f^2(a) + f^2(b)) + f(a)f(b)(h^2(t) + h^2(1-t)).$$

We obtain

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))f((1-t)\varphi(a)+(t\varphi(b))dt$$
$$\leq (f^2(a)+f^2(b))\int_0^1 h(1-t)h(t)dt+2f(a)f(b)\int_0^1 h^2(t)dt.$$

Substituting $x=t\varphi(a)+(1-t)\varphi(b)$, we get the required inequality. □

Corollary 3.7. *Under the assumptions of Theorem* 3.6 *with* $h(t)=t$ *for all* $t\in(0,1)$, *we have*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(b)+\varphi(a)-x)dx$$
$$\leq \frac{f^2(a)+f^2(b)}{6}+\frac{2f(a)f(b)}{3}.$$

Theorem 3.8. *If* $[a,b]\subseteq I$, $\varphi:[a,b]\to[a,b]$ *is a continuous function such that* $\varphi(a)\neq\varphi(b)$ *and the functions* $f,g:I\to[0,\infty)$ *is Lebesgue integrable on* I *and semi-*φ_h *convex, then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)g(x)dx$$
$$\leq M(a,b)\int_0^1 h^2(t)dt+N(a,b)\int_0^1 h(t)h(1-t)dt.$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b),$$
$$N(a,b)=f(a)g(b)+f(b)g(a).$$

Proof. Since f,g are semi-φ_h-convex we have for $t\in(0,1)$

$$f(t\varphi(a)+(1-t)\varphi(b))\leq h(t)f(a)+h(1-t)f(b),$$

and

$$g(t\varphi(a)+(1-t)\varphi(b))\leq h(t)g(a)+h(1-t)g(b).$$

By multiplying both sides, we get

$$f(t\varphi(a)+(1-t)\varphi(b))g(t\varphi(a)+(1-t)\varphi(b))$$
$$\leq h^2(t)f(a)g(a)+h^2(1-t)f(b)g(b)+h(t)h(1-t)f(a)g(b)+h(t)h(1-t)f(b)g(a).$$

Integrating over the interval $(0,1)$, we obtain

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))g(t\varphi(a)+(1-t)\varphi(b))dt$$
$$\leq (f(a)g(a)+f(b)g(b))\int_0^1 h^2(t)dt+(f(a)g(b)+f(b)g(a))\int_0^1 h(t)h(1-t)dt.$$

Replacing $x=t\varphi(a)+(1-t)\varphi(b)$, we get the required inequality. □

Definition 3.9. *Let be* $m\in(0,1]$. *A function* $f:[0,b]\to[0,\infty)$ *is said to be semi-*φ_m*-convex if*

$$f(t\varphi(x)+m(1-t)\varphi(y))\leq tf(x)+m(1-t)f(y),$$

for all $x,y\in[0,b]$, $t\in[0,1]$.

Remark 3.10. If $m = 1$, then f is semi-φ-convex, and if $m = 1, \varphi(x) = x$ for all $x \in [0, b]$, then f is convex on $[0, b]$.

Theorem 3.11. *If $f : [0,\infty) \to [0,\infty)$ is a semi- φ_m-convex function, with $m \in (0, 1)$ such that $m\varphi(b) \neq \varphi(a)$ and $m\varphi(a) \neq \varphi(b)$ and f is Lebesgue integrable on $[ma, b]$ then*

$$\frac{1}{m+1}\Big[\frac{1}{m\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{m\varphi(b)} f(x)dx + \frac{1}{\varphi(b)-m\varphi(a)}\int_{m\varphi(a)}^{\varphi(b)} f(x)dx\Big] \leq \frac{f(a)+f(b)}{2}.$$

Proof. Since f is semi- φ_m-convex we have following inequalities

$$f(t\varphi(a) + m(1-t)\varphi(b)) \leq tf(a) + m(1-t)f(b),$$

$$f((1-t)\varphi(a) + mt\varphi(b)) \leq (1-t)f(a) + mtf(b),$$

$$f(mt\varphi(a) + (1-t)\varphi(b)) \leq mtf(a) + (1-t)f(b),$$

$$f(m(1-t)\varphi(a) + t\varphi(b)) \leq m(1-t)f(a) + tf(b).$$

Adding the above four inequalities, we get

$$f(t\varphi(a) + m(1-t)\varphi(b)) + f((1-t)\varphi(a) + mt\varphi(b))$$
$$+f(mt\varphi(a) + (1-t)\varphi(b)) + f(m(1-t)\varphi(a) + t\varphi(b))$$
$$\leq (m+1)(f(a)+f(b)).$$

Now, integrating over the interval $(0, 1)$, we have

$$\int_0^1 f(t\varphi(a) + m(1-t)\varphi(b))dt + \int_0^1 f((1-t)\varphi(a) + mt\varphi(b))dt +$$
$$\int_0^1 f(mt\varphi(a) + (1-t)\varphi(b))dt + \int_0^1 f(m(1-t)\varphi(a) + t\varphi(b))dt$$
$$\leq (m+1)(f(a)+f(b)).$$

Using the substitution $x = t\varphi(a) + (1-t)\varphi(b)$, we have

$$\int_0^1 f(t\varphi(a) + m(1-t)\varphi(b))dt = \int_0^1 f((1-t)\varphi(a) + mt\varphi(b))dt$$
$$= \frac{1}{m\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{m\varphi(b)} f(x)dx,$$

and using the substitution $x = t\varphi(a) + (1-t)\varphi(b)$, we have

$$f(mt\varphi(a) + (1-t)\varphi(b))dt = f(m(1-t)\varphi(a) + t\varphi(b))dt$$
$$= \frac{1}{\varphi(b)-m\varphi(a)}\int_{m\varphi(a)}^{\varphi(b)} f(x)dx.$$

Using the above equations, we get the required inequality. ☐

4. Semi-φ and strongly log-φ convexity

Definition 4.1. [3] *A function $f : I \to [0, \infty)$ is a semi log-φ convex if, for all $t \in (0, 1)$ and $x, y \in I$, one has*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq f(x)^t f(y)^{1-t}.$$

Polyak [9] introduced strongly convex functions which plays an important role in optimization theory and mathematical economics.

A function $f : I \to \mathbb{R}$ is said to be strongly convex with modulus $c > 0$ on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + ct(1-t)(x-y)^2,$$

for all $x, y \in I$ and $t \in (0, 1)$.

Sarikaya [11] defined strongly log-convex functions as:

Definition 4.2. *A positive function $f : I \to (0, \infty)$ is said to be strongly log-convex with respect to $c > 0$ if*

$$f(tx + (1-t)y) \leq f(x)^t f(y)^{1-t} - ct(1-t)(x-y)^2,$$

for all $x, y \in I$ and $t \in (0, 1)$.

In this section we relate Hermite Hadamard type inequalities to some special means. Firstly, let us recall the following means for positive $a, b \in \mathbb{R}$:
Arithmetic mean:

$$A(a, b) = \frac{a+b}{2},$$

Geometric mean:

$$G(a, b) = \sqrt{ab},$$

Logarithmic mean:

$$L(a, b) = \frac{b-a}{\log(b) - \log(a)}.$$

Theorem 4.3. *If the positive function $f : I \to (0, \infty)$ is semi log-φ convex function and Lebesgue integrable on I, then*

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a) + \varphi(b) - x))dx \leq G(f(a), f(b)),$$

for all $a, b \in I$, $a < b$.

Proof. Since f is semi log-φ convex, we have

$$f(t\varphi(a) + (1-t)\varphi(b)) \leq f(a)^t f(b)^{1-t}, \ \forall\, t \in (0, 1)$$

and

$$f((1-t)\varphi(a) + t\varphi(b)) \leq f(a)^{1-t} f(b)^t, \ \forall\, t \in (0, 1).$$

By multiplying both inequalities, we get

$$f(t\varphi(a) + (1-t)\varphi(b)) f((1-t)\varphi(a) + t\varphi(b)) \leq f(a)f(b).$$

Now, taking square root, we get

$$G(f(t\varphi(a) + (1-t)\varphi(b)), f((1-t)\varphi(a) + t\varphi(b))) \leq G(f(a), f(b)).$$

By integrating over the interval $(0,1)$ and replacing $x = t\varphi(a) + (1-t)\varphi(b)$, we get the required inequality. □

Theorem 4.4. *If the positive function $f : I \to (0,\infty)$ is semi log-φ convex function and Lebesgue integrable on I, then*

$$\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \le L(f(b), f(a)) \le \frac{f(a)+f(b)}{2},$$

for all $a, b \in I,\ a < b$.

Proof. Since f is semi log-φ convex, we have

$$f(t\varphi(a) + (1-t)\varphi(b)) \le f(a)^t f(b)^{1-t},\ \forall\, t \in (0,1).$$

Integrating over the interval $(0,1)$, we get

$$\int_0^1 f(t\varphi(a) + (1-t)\varphi(b))dt \le \int_0^1 f(a)^t f(b)^{1-t} dt$$
$$= \frac{f(b)-f(a)}{\log f(b) - \log f(a)} = L(f(b), f(a)) \le \frac{f(a)+f(b)}{2}.$$

Substituting $x = t\varphi(a) + (1-t)\varphi(b)$, we get the required result. □

Theorem 4.5. *If the functions $f, g : I \to (0,+\infty)$ are semi log-φ convex and Lebesgue integrable on I, then*

$$\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)g(x)dx \le L(f(b)g(b), f(a)g(a))$$
$$\le \frac{1}{4}\{(f(b)+f(a))L(f(b), f(a)) + (g(a)+g(b))L(g(b), g(a))\},$$

for all $a, b \in I,\ a < b$.

Proof. Since f, g are semi log-φ convex, we have

$$f(t\varphi(a) + (1-t)\varphi(b)) \le f(a)^t f(b)^{1-t},\ \forall\, t \in (0,1)$$

and

$$g(t\varphi(a) + (1-t)\varphi(b)) \le g(a)^t g(b)^{1-t},\ \forall\, t \in (0,1).$$

Multiplying both inequalities and integrating over the interval $(0,1)$, we get

$$\int_0^1 f(t\varphi(a) + (1-t)\varphi(b))g(t\varphi(a) + (1-t)\varphi(b))dt$$
$$\le \int_0^1 f(a)^t f(b)^{1-t} g(a)^t g(b)^{1-t} dt$$
$$= \frac{f(b)g(b) - f(a)g(a)}{\log(f(b)g(b)) - \log(f(a)g(a))}$$
$$= L(f(b)g(b), f(a)g(b)). \tag{4.1}$$

By Young's inequality, we have

$$\int_0^1 f(a)^t f(b)^{1-t} g(a)^t g(b)^{1-t} dt$$

$$\leq \frac{1}{2}\int_0^1 \{[f(a)^t f(b)^{1-t}]^2 + [g(a)^t g(b)^{1-t}]^2\}dt$$
$$= \frac{1}{4}\left[\frac{(f(b))^2 - (f(a))^2}{\log(f(b)) - \log(f(a))} + \frac{(g(b))^2 - (g(a))^2}{\log(g(b)) - \log(g(a))}\right]$$
$$= \frac{1}{4}\left\{(f(a)+f(b))L(f(b),f(a)) + (g(a)+g(b))L(g(b),g(a))\right\}. \qquad (4.2)$$

Using (4.1) and (4.2) and substituting $x = t\varphi(a)+(1-t)\varphi(b)$, we get the required result. □

Definition 4.6. *Let $f : I \to (0,\infty)$ be a positive function. We say that f is strongly log-φ convex with respect to $c > 0$ if*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq f(\varphi(x))^t f(\varphi(y))^{1-t} - ct(1-t)(\varphi(x) - \varphi(y))^2,$$

for all $x, y \in I$ and $t \in (0,1)$.

Remark 4.7. From the above inequality, using arithmetic mean- geometric mean, we have

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq f(\varphi(x))^t f(\varphi(y))^{1-t} - ct(1-t)(\varphi(x) - \varphi(y))^2$$

$$\leq tf(\varphi(x)) + (1-t)f(\varphi(y)) - ct(1-t)(\varphi(x) - \varphi(y))^2$$

$$\leq \max\{f(\varphi(x)), f(\varphi(y))\} - ct(1-t)(\varphi(x) - \varphi(y))^2.$$

Example 4.8. Let

$$\varphi(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0. \end{cases}$$

Then for $c = \frac{1}{4}$ the function

$$f(x) = \begin{cases} 0, & -1 < x < 1 \\ 1, & \text{otherwise} \end{cases}$$

is strongly log-φ convex function with respect to c on $\mathbb{R}$.

Theorem 4.9. *Let $\varphi : [a,b] \to [a,b]$ be a continuous function and $f : I \to (0,\infty)$ be a positive strongly log-φ convex function with respect to $c > 0$, where $a, b \in I$. If f is Lebesgue integrable on I then*

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{2}(\varphi(a)-\varphi(b))^2 \leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a)+\varphi(b)-x))dx$$
$$\leq \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx$$
$$\leq L(f(\varphi(b)), f(\varphi(a))) - \frac{c}{6}(\varphi(a) - \varphi(b))^2$$
$$\leq \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{c}{6}(\varphi(a) - \varphi(b))^2.$$

Proof. Since f is strongly log-φ convex, we have for $t \in (0,1)$

$$f(\frac{\varphi(a)+\varphi(b)}{2})$$

$$\leq \sqrt{f(t\varphi(a)+(1-t)\varphi(b))f((1-t)\varphi(a)+t\varphi(b))} - \frac{c}{4}(\varphi(a)-\varphi(b))^2(1-2t)^2$$

$$\leq \frac{f(t\varphi(a)+(1-t)\varphi(b))}{2} + \frac{f((1-t)\varphi(a)+t\varphi(b))}{2} - \frac{c}{4}(\varphi(a)-\varphi(b))^2(1-2t)^2.$$

Integrating the above inequality over $(0,1)$ and substituting $x = t\varphi(a)+(1-t)\varphi(b)$ we get

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{12}(\varphi(a)-\varphi(b))^2$$

$$\leq \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a)+\varphi(b)-x))dx \tag{4.3}$$

$$\leq \int_{\varphi(a)}^{\varphi(b)} A(f(x), f(\varphi(a)+\varphi(b)-x))dx. \tag{4.4}$$

Using $\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_{\varphi(a)}^{\varphi(b)} f(\varphi(a)+\varphi(b)-x)dx$, (4.3) becomes

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + \frac{c}{12}(\varphi(a)-\varphi(b))^2$$

$$\leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a)+\varphi(b)-x))dx$$

$$\leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx.$$

Again, using strongly log-φ convexity of f, we get

$$\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_0^1 f(t\varphi(a)+(1-t)\varphi(b))dt$$

$$\leq \int_0^1 [f(\varphi(a)]^t[f(\varphi(b)]^{1-t}dt - \int_0^1 ct(1-t)(\varphi(a)-\varphi(b))^2dt$$

$$= \frac{f(\varphi(b))-f(\varphi(a))}{\log(f(\varphi(b)))-\log(f(\varphi(a)))} - \frac{c}{6}(\varphi(a)-\varphi(b))^2$$

$$= L(f(\varphi(b)), f(\varphi(a))) - \frac{c}{6}(\varphi(a)-\varphi(b))^2$$

$$\leq A(f(\varphi(b)), f(\varphi(a))) - \frac{c}{6}(\varphi(a)-\varphi(b))^2$$

$$= \frac{f(\varphi(b))+f(\varphi(a))}{2} - \frac{c}{6}(\varphi(a)-\varphi(b))^2.$$

□

Theorem 4.10. *Let $\varphi : [a,b] \to [a,b]$ be a continuous function, where $a, b \in I$, and let $f : I \to (0,\infty)$ be a positive strongly log-φ convex function with respect to $c > 0$. If f is Lebesgue integrable on I then*

$$\frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)f(\varphi(b)+\varphi(a)-x)dx$$

$$\leq f(\varphi(a))f(\varphi(b)) + \frac{c^2}{30}(\varphi(b)-\varphi(a))^4$$

$$-4c\frac{(\varphi(b)-\varphi(a))^2}{(\log(f(\varphi(b)))-\log(f(\varphi(a))))^2}[A(f(\varphi(b)),f(\varphi(a)))-L(f(\varphi(b)),f(\varphi(a)))].$$

Proof. Since f is strongly log-φ convex, we have for $t \in (0,1)$

$$f(t\varphi(a)+(1-t)\varphi(b)) \leq f(\varphi(a))^t f(\varphi(b))^{1-t} - ct(1-t)(\varphi(a)-\varphi(b))^2,$$

and

$$f((1-t)\varphi(a)+t\varphi(b)) \leq f(\varphi(a))^{1-t} f(\varphi(b))^{t} - ct(1-t)(\varphi(a)-\varphi(b))^2.$$

Multiplying both inequalities and integrating over $(0,1)$, we get

$$\int_0^1 f(t\varphi(a)+(1-t)\varphi(b))f((1-t)\varphi(a)+t\varphi(b))dt$$

$$\leq f(\varphi(a))f(\varphi(b)) - (\varphi(a)-\varphi(b))^2\int_0^1 ct(1-t)\left\{f(\varphi(b))\left[\frac{f(\varphi(a))}{f(\varphi(b))}\right]^t\right.$$

$$\left.+f(\varphi(a))\left[\frac{f(\varphi(b))}{f(\varphi(a))}\right]^t\right\}dt + c^2(\varphi(a)-\varphi(b))^4\int_0^1 t^2(1-t)^2dt. \tag{4.5}$$

Since

$$\int_0^1 t(1-t)\left[\frac{f(\varphi(a))}{f(\varphi(b))}\right]^t dt$$

$$= \frac{2}{f(\varphi(b))(\log(f(\varphi(a)))-\log(f(\varphi(b))))^2}[A(f(\varphi(b)),f(\varphi(a)))-L(f(\varphi(b)),f(\varphi(a)))]. \tag{4.6}$$

Similarly,

$$\int_0^1 t(1-t)\left[\frac{f(\varphi(a))}{f(\varphi(b))}\right]^t dt$$

$$= \frac{2}{f(\varphi(a))(\log(\varphi(b))-\log(\varphi(a)))^2}\Big[A(f(\varphi(b)),f(\varphi(a)))-L(f(\varphi(b)),f(\varphi(a)))\Big]. \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.5) and replacing $x = t\varphi(a)+(1-t)\varphi(b)$, we get the required inequality. □

Theorem 4.11. *Let $\varphi : [a,b] \to [a,b]$ be a continuous function, where $a, b \in I$, and let $f, g : I \to (0,\infty)$ be a positive strongly log-φ convex functions with respect to $c > 0$. If f and g are Lebesgue integrable, then*

$$\frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)g(x)dx$$

$$\leq L(f(\varphi(b))g(\varphi(b)), f(\varphi(a))g(\varphi(a))) + \frac{c^2}{30}(\varphi(a)-\varphi(b))^4 - 2c(\varphi(b)-\varphi(a))^2$$

$$\times \left[\frac{A(f(\varphi(b)), f(\varphi(a))) - L(f(\varphi(b)), f(\varphi(a)))}{(\log(f(\varphi(b))) - \log(f(\varphi(a))))^2} + \frac{A(g(\varphi(b)), g(\varphi(a))) - L(g(\varphi(b)), g(\varphi(a)))}{(\log(g(\varphi(b))) - \log(g(\varphi(a))))^2} \right]$$

$$\leq \frac{1}{4} \left[\{f(\varphi(a)) + f(\varphi(b))\} L(f(\varphi(b)), f(\varphi(a))) + \{g(\varphi(a)) + g(\varphi(b))\} L(g(\varphi(b)), g(\varphi(a))) \right]$$

$$+ \frac{c^2}{30}(\varphi(a)-\varphi(b))^4 - 2c(\varphi(b)-\varphi(a))^2$$

$$\times \left[\frac{A(f(\varphi(b)), f(\varphi(a))) - L(f(\varphi(b)), f(\varphi(a)))}{(\log(f(\varphi(b))) - \log(f(\varphi(a))))^2} + \frac{A(g(\varphi(b)), g(\varphi(a))) - L(g(\varphi(b)), g(\varphi(a)))}{(\log(g(\varphi(b))) - \log(g(\varphi(a))))^2} \right].$$

Proof. The proof is similar to Theorem 4.10 □

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