Stud. Univ. Babeş-Bolyai Math. 59(2014), No. 1, 25–36

Fekete-Szegő problem for a new class of analytic functions with complex order defined by certain differential operator

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Abstract. In this paper, we obtain Fekete-Szegő inequalities for a new class of analytic functions $f \in \mathcal{A}$ for which $1+\frac{1}{b}[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z}+\gamma(D_\lambda^n(f*g)(z))'-1]$ $(\gamma, \lambda \geq 0; b \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}; n \in \mathbb{N}_0; z \in U)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic function, Fekete-Szegő problem, differential subordination.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} \text{ and } |z| < 1\}$. Further let S denote the family of functions of the form (1.1) which are univalent in U, and $g \in \mathcal{A}$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k. \tag{1.2}$$

A classical theorem of Fekete-Szegő [8] states that, for $f \in S$ given by (1.1), that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3-4\mu, & if\ \mu \leq 0, \\ 1+2\exp\left(\dfrac{-2\mu}{1-\mu}\right), & if\ 0 \leq \mu \leq 1, \\ 4\mu-3, & if\ \mu \geq 1. \end{cases} \tag{1.3}$$

The result is sharp.

Given two functions f and g, which are analytic in U with $f(0) = g(0)$, the function f is said to be subordinate to g if there exists a function w, analytic in U, such that $w(0) = 0$ and $|w(z)| < 1\,(z \in U)$ and $f(z) = g(w(z))$ $(z \in U)$. We denote this subordination by $f(z) \prec g(z)$ ([10]).
Let φ be an analytic function with positive real part on U, which satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$, and which maps the unit disc U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $S^*(\varphi)$ be the class of functions $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \tag{1.4}$$

and $C(\varphi)$ be the class of functions $f \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z). \tag{1.5}$$

The classes of $S^*(\varphi)$ and $C(\varphi)$ were introduced and studied by Ma and Minda [9]. The familier class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α $(0 \le \alpha < 1)$ are the special cases of $S^*(\varphi)$ and $C(\varphi)$, respectively, when $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \le \alpha < 1)$.
Ma and Minda [9] have obtained the Fekete-Szegő problem for the functions in the class $C(\varphi)$.

Definition 1.1. (Hadamard Product or Convolution) *Given two functions f and g in the class $\mathcal{A}$, where f is given by (1.1) and g is given by (1.2) the Hadamard product (or convolution) of f and g is defined (as usual) by*

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k g_k z^k = (g * f)(z). \tag{1.6}$$

For the functions f and g defined by (1.1) and (1.2) respectively, the linear operator $D_\lambda^n : \mathcal{A} \longrightarrow \mathcal{A}$ $(\lambda \ge 0; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, 3, ...\})$ is defined by(see [4], see also [7, with p = 1]):

$$\begin{aligned} D_\lambda^0(f * g)(z) &= (f * g)(z), \\ D_\lambda^n(f * g)(z) &= D_\lambda(D_\lambda^{n-1}(f * g)(z)) \\ &= z + \sum_{k=2}^{\infty}[1 + \lambda(k-1)]^n a_k g_k z^k \;\; (\lambda \ge 0; n \in \mathbb{N}_0). \end{aligned} \tag{1.7}$$

Remark 1.2. (i) Taking $g(z) = \dfrac{z}{1-z}$, then operator $D_\lambda^n(f * \dfrac{z}{1-z})(z) = D_\lambda^n f(z)$, was introduced and studied by Al-Oboudi [2];
(ii) Taking $g(z) = \dfrac{z}{1-z}$ and $\lambda = 1$, then operator $D_1^n(f * \dfrac{z}{1-z})(z) = D^n f(z)$, was introduced by Sălăgean [12].

Using the operator D_λ^n we introduce a new class of analytic functions with complex order as following:

Definition 1.3. *For $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ let the class $M_\lambda^n(f, g; \gamma, b; \varphi)$ denote the subclass of $\mathcal{A}$ consisting of functions f of the form (1.1) and g of the form (1.2) and satisfying the following subordination:*

$$1 + \frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z} + \gamma\left(D_\lambda^n(f*g)(z)\right)' - 1\right] \prec \varphi(z), \tag{1.8}$$
$$(\gamma, \lambda \geq 0;\ n \in \mathbb{N}_0).$$

Specializing the parameters γ, λ, b, n, g and φ, we obtain the following subclasses studied by various authors:

(i) $M_\lambda^0\left(f, z + \sum_{k=2}^{\infty} k^n z^k; \gamma, b; \frac{1+Az}{1+Bz}\right) = M_1^n\left(f, \frac{z}{1-z}; \gamma, b; \frac{1+Az}{1+Bz}\right)$ $= G_n(\gamma, b, A, B)$ $(\gamma, \lambda \geq 0, -1 \leq B < A \leq 1, b \in \mathbb{C}^*, n \in \mathbb{N}_0)$ (Sivasubramanian et al. [14]);

(ii) $M_\lambda^0\left(f, g; \gamma, b; \frac{1+(1-2\alpha)z}{1-z}\right) = S(f, g; \gamma, \alpha, b)$ $(0 \leq \alpha < 1, \gamma \geq 0, b \in \mathbb{C}^*)$ (Aouf et al. [5]);

(iii) $M_\lambda^0\left(f, z + \sum_{k=2}^{\infty} k^n z^k; \gamma, b; \frac{1+z}{1-z}\right) = M_1^n\left(f, \frac{z}{1-z}; \gamma, b; \frac{1+z}{1-z}\right) = G_n(\gamma, b)$ $(\gamma \geq 0, b \in \mathbb{C}^*, n \in \mathbb{N}_0)$ (Aouf [3]);

(iv) $M_\lambda^0\left(f, \frac{z}{1-z}; 1, b; (1-\ell)\frac{1+Az}{1+Bz} + \ell\right) = R_\ell^b(A, B)$ $(b \in \mathbb{C}^*, 0 \leq \ell < 1, -1 \leq B < A \leq 1)$ (Reddy and Reddy [11]);

(v) $M_\lambda^0\left(f, \frac{z}{1-z}; 1, b; \varphi\right) = R_b(\varphi)$ $(b \in \mathbb{C}^*)$ (Ali et al. [1]).

Also we note that:

(i) If $g(z) = z + \sum_{k=2}^{\infty} \Psi_k(\alpha_1) z^k$ (or $g_k = \Psi_k(\alpha_1)$), where

$$\Psi_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \cdots\cdots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \cdots\cdots (\beta_s)_{k-1}(k-1)!} \tag{1.9}$$

$(\alpha_i > 0, i = 1, ..., q; \beta_j > 0, j = 1, ..., s; q \leq s+1; q, s \in \mathbb{N} = \{1, 2, ...\})$, where $(\nu)_k$ is the Pochhammer symbol defined in terms to the Gamma function Γ, by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1, & \text{if } k = 0, \\ \nu(\nu+1)(\nu+2)...(\nu+k-1), & \text{if } k \in \mathbb{N}, \end{cases}$$

then the class $M_\lambda^n(f, z + \sum_{k=2}^{\infty} \Psi_k(\alpha_1) z^k; \gamma, b; \varphi)$ reduces to the class

$$\begin{aligned} & M_{\lambda,q,s}^n([\alpha_1]; \gamma, b; \varphi) \\ = & \left\{ f \in \mathcal{A} : 1 + \frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(\alpha_1, \beta_1) f(z)}{z} + \gamma(D_\lambda^n(\alpha_1, \beta_1) f(z))' - 1\right] \prec \varphi(z), \right. \\ & \left. \gamma, \lambda \geq 0; b \in \mathbb{C}^*; n \in \mathbb{N}_0 \right\}, \end{aligned}$$

where, the operator $D_\lambda^n(\alpha_1,\beta_1)$ was defined as (see Selvaraj and Karthikeyan [13], see also El-Ashwah and Aouf [6]):

$$D_\lambda^n(\alpha_1,\beta_1)f(z) = z + \sum_{k=2}^{\infty}[1+\lambda(k-1)]^n \frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}(1)_{k-1}} a_k z^k$$

(ii) $M_\lambda^n(f,g;1,b;\varphi) = G_\lambda^n(f,g;b;\varphi) = \{f(z)\in\mathcal{A} : 1+\frac{1}{b}[(D_\lambda^n(f*g)(z))' - 1] \prec \varphi(z)$ $(\lambda\geq 0; b\in\mathbb{C}^*; n\in\mathbb{N}_0)\}$;

(iii) $M_\lambda^n(f,g;0,b;\varphi) = R_\lambda^n(f,g;b;\varphi) = \{f(z)\in\mathcal{A} : 1+\frac{1}{b}[\frac{D_\lambda^n(f*g)(z)}{z} - 1] \prec \varphi(z)$ $(\lambda\geq 0; b\in\mathbb{C}^*; n\in\mathbb{N}_0)\}$;

(iv) $M_\lambda^n\left(f,g;\gamma,(1-\rho)\cos\eta e^{-i\eta};\varphi\right) = E_{\lambda,\rho}^{n,\eta}(f,g;\gamma;\varphi) = \{f(z\in\mathcal{A} : e^{i\eta}[(1-\gamma)$ $\cdot\frac{D_\lambda^n(f*g)(z)}{z} + \gamma\left(D_\lambda^n(f*g)(z)\right)'] \prec (1-\rho)\cos\eta\varphi(z) + i\sin\eta + \rho\cos\eta$ $(|\eta|\leq\frac{\pi}{2};$ $\gamma,\lambda\geq 0;\ 0\leq\rho<1; b\in\mathbb{C}^*; n\in\mathbb{N}_0)\}$.

In this paper, we obtain the Fekete-Szegő inequalities for functions in the class $M_\lambda^n(f,g;\gamma,b;\varphi)$.

2. Fekete-Szegő problem

Unless otherwise mentioned, we assume in the reminder of this paper that $\lambda\geq 0$, $b\in\mathbb{C}^*$ and $z\in U$.

To prove our results, we shall need the following lemmas:

Lemma 2.1. [9] *If* $p(z) = 1 + c_1 z + c_2 z^2 + \ldots..$ $(z\in U)$ *is a function with positive real part in* U *and* μ *is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \leq 2\max\{1; |2\mu - 1|\}. \tag{2.1}$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z} \quad (z\in U). \tag{2.2}$$

Lemma 2.2. [9] *If* $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots..$ *is a function with positive real part in* U, *then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2, & if \quad \nu\leq 0, \\ 2, & if \quad 0\leq\nu\leq 1, \\ 4\nu - 2, & if \quad \nu\geq 1. \end{cases}$$

When $\nu<0$ *or* $\nu>1$, *the equality holds if and only if* $p_1(z) = \frac{1+z}{1-z}$ *or one of its rotations. If* $0<\nu<1$, *then the equality holds if and only if* $p_1(z) = \frac{1+z^2}{1-z^2}$ *or one of its rotations. If* $\nu=0$, *the equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2}+\frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2}-\frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0\leq\gamma\leq 1),$$

or one of its rotations. If $\nu = 1$, *the equality holds if and only if*

$$\frac{1}{p_1(z)} = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0 \le \gamma \le 1).$$

Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \le 2 \ (0 < \nu < \frac{1}{2}),$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \le 2 \ (\frac{1}{2} < \nu < 1).$$

Using Lemma 2.1, we have the following theorem:

Theorem 2.3. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...$, *where* $\varphi(z) \in \mathcal{A}$ *and* $\varphi^{'}(0) > 0$. *If* $f(z)$ *given by (1.1) belongs to the class* $M_\lambda^n(f, g; \gamma, b; \varphi)$ *and if* μ *is a complex order, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1 |b|}{(1+2\lambda)^n (1+2\gamma) g_3} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{(1+2\lambda)^n (1+2\gamma) g_3}{(1+\lambda)^{2n} (1+\gamma)^2 g_2^2}\mu b B_1\right|\right\}. \tag{2.3}$$

The result is sharp.

Proof. If $f \in M_\lambda^n(f, g; \gamma, b; \varphi)$, then there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$1 + \frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z} + \gamma\left(D_\lambda^n(f*g)(z)\right)' - 1\right] = \varphi(w(z)). \tag{2.4}$$

Define the function p_1 by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + ... \ . \tag{2.5}$$

Since w is a Schwarz function, we see that $\mathrm{Re} p_1(z) > 0$ and $p_1(0) = 1$.
Let define the function p by:

$$p(z) = 1 + \frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z} + \gamma\left(D_\lambda^n(f*g)(z)\right)' - 1\right] = 1 + b_1 z + b_2 z^2 + ... \tag{2.6}$$

In view of the equations (2.4), (2.5) and (2.6), we have

$$\begin{aligned} p(z) &= \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \varphi\left(\frac{c_1 z + c_2 z^2 + ...}{2 + c_1 z + c_2 z^2 + ...}\right) \\ &= \varphi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + ...\right) \\ &= 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + ... \end{aligned} \tag{2.7}$$

Thus

$$b_1 = \frac{1}{2}B_1 c_1 \quad \text{and} \quad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2. \tag{2.8}$$

Since

$$1+\frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z}+\gamma\left(D_\lambda^n(f*g)(z)\right)'-1\right]$$

$$=1+\left(\frac{1}{b}(1+\lambda)^n(1+\gamma)a_2g_2\right)z+\left(\frac{1}{b}(1+2\lambda)^n(1+2\gamma)a_3g_3\right)z^2+...,$$

from (2.6) and (2.8), we obtain

$$a_2=\frac{B_1c_1b}{2(1+\lambda)^n(1+\gamma)g_2}, \tag{2.9}$$

and

$$a_3=\frac{B_1c_2b}{2(1+2\lambda)^n(1+2\gamma)g_3}+\frac{c_1^2}{4(1+2\lambda)^n(1+2\gamma)g_3}\left[(B_2-B_1)b\right]. \tag{2.10}$$

Therefore, we have

$$a_3-\mu a_2^2=\frac{B_1b}{2(1+2\lambda)^n(1+2\gamma)g_3}\left[c_2-\nu c_1^2\right], \tag{2.11}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(1+2\lambda)^n(1+2\gamma)g_3\mu}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}B_1b\right]. \tag{2.12}$$

Our result now follows by an application of Lemma 2.1. The result is sharp for the functions f satisfying

$$1+\frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z}+\gamma\left(D_\lambda^n(f*g)(z)\right)'-1\right]=\varphi(z^2), \tag{2.13}$$

and

$$1+\frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z}+\gamma\left(D_\lambda^n(f*g)(z)\right)'-1\right]=\varphi(z). \tag{2.14}$$

This completes the proof of Theorem 2.3.

Remark 2.4. (i) Taking $\gamma=1$, $n=0$ and $g(z)=\dfrac{z}{1-z}$ in Theorem 2.3, we obtain the result obtained by Ali et al. [1, Theorem 2.3, with $k=1$];
(ii) Taking $\gamma=1$, $n=0$, $g(z)=\dfrac{z}{1-z}$ and $\varphi(z)=(1-\ell)\dfrac{1+Az}{1+Bz}+\ell$ $(0\le\ell<1,$ $-1\le B<A\le1)$ in Theorem 2.3, we obtain the result obtained by Reddy and Reddy [11, Theorem 4].

Also by specializing the parameters in Theorem 2.3, we obtain the following new sharp results.

Putting $n=0$, $g(z)=z+\sum_{k=2}^{\infty}k^nz^k$ $(n\in\mathbb{N}_0)$ and $\varphi(z)=\frac{1+Az}{1-Bz}$ $(-1\le B<A\le1)$ (or equivalently, $B_1=A-B$ and $B_2=-B(A-B)$) in Theorem 2.3, we obtain the corollary:

Corollary 2.5. *If f given by (1.1) belongs to the class $G_n(\gamma, b; A, B)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)|b|}{(1+2\gamma)3^n} \max\left\{1, \left|\frac{(1+2\gamma)3^n}{(1+\gamma)^2 2^{2n}}\mu(A-B)b + B\right|\right\}. \tag{2.15}$$

The result is sharp.

Putting $n = 0$ and $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.6. *If f given by (1.1) belongs to the class $S(f, g; \gamma, \alpha, b)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2(1-\alpha)|b|}{(1+2\gamma)g_3} \max\left\{1, \left|1 - \frac{2(1+2\gamma)g_3}{(1+\gamma)^2 g_2^2}\mu(1-\alpha)b\right|\right\}. \tag{2.16}$$

The result is sharp.

Putting $n = 0$, $g(z) = z + \sum_{k=2}^{\infty} k^n z^k$ $(n \in \mathbb{N}_0)$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.7. *If f given by (1.1) belongs to the class $G_n(\gamma, b)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2|b|}{(1+2\gamma)3^n} \max\left\{1, \left|1 - \frac{(1+2\gamma)3^n}{(1+\gamma)^2 2^{2n-1}}\mu b\right|\right\}. \tag{2.17}$$

The result is sharp.

Putting $\gamma = 1$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.8. *If f given by (1.1) belongs to the class $G_\lambda^n(f, g; b; \varphi)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1|b|}{3(1+2\lambda)^n g_3} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3(1+2\lambda)^n g_3}{4(1+\lambda)^{2n} g_2^2}\mu B_1 b\right|\right\}. \tag{2.18}$$

The result is sharp.

Putting $\gamma = 0$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.9. *If f given by (1.1) belongs to the class $R_\lambda^n(f, g; b; \varphi)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1|b|}{(1+2\lambda)^n g_3} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{(1+2\lambda)^n g_3}{(1+\lambda)^{2n} g_2^2}\mu B_1 b\right|\right\}. \tag{2.19}$$

The result is sharp.

Putting $(1-\rho)\cos\eta e^{-i\eta}$ $\left(0 \leq \rho < 1; |\eta| \leq \frac{\pi}{2}\right)$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.10. *If f given by (1.1) belongs to the class $E_{\lambda,\rho}^{n,\eta}(f,g;\gamma;\varphi)$, then for any complex number μ, we have*

$$\left|a_3-\mu a_2^2\right| \leq \frac{(1-\rho)B_1\cos\eta}{(1+2\lambda)^n(1+2\gamma)g_3}\max\left\{1,\left|\frac{B_2}{B_1}e^{i\eta}-\frac{(1+2\lambda)^n(1+2\gamma)(1-\rho)\cos\eta}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}g_3\mu B_1\right|\right\}. \tag{2.20}$$

The result is sharp.

Using Lemma 2.2, we have the following theorem:

Theorem 2.11. *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+...,\ (b>0;B_i>0;i\in\mathbb{N})$. Also let*

$$\sigma_1=\frac{(1+\lambda)^{2n}(1+\gamma)^2g_2^2(B_2-B_1)}{(1+2\lambda)^n(1+2\gamma)g_3bB_1^2},$$

and

$$\sigma_2=\frac{(1+\lambda)^{2n}(1+\gamma)^2g_2^2(B_2+B_1)}{(1+2\lambda)^n(1+2\gamma)g_3bB_1^2}.$$

If f is given by (1.1) belongs to the class $M_\lambda^n(f,g;\gamma,b;\varphi)$, then we have the following sharp results:

(i) If $\mu\leq\sigma_1$, then

$$\left|a_3-\mu a_2^2\right| \leq \frac{b}{(1+2\lambda)^n(1+2\gamma)g_3}\left[B_2-\frac{(1+2\lambda)^n(1+2\gamma)g_3b}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}\mu B_1^2\right]; \tag{2.21}$$

(ii) If $\sigma_1\leq\mu\leq\sigma_2$, then

$$\left|a_3-\mu a_2^2\right| \leq \frac{bB_1}{(1+2\lambda)^n(1+2\gamma)g_3}; \tag{2.22}$$

(iii) If $\mu\geq\sigma_2$, then

$$\left|a_3-\mu a_2^2\right| \leq \frac{b}{(1+2\lambda)^n(1+2\gamma)g_3}\left[-B_2+\frac{(1+2\lambda)^n(1+2\gamma)g_3b}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}\mu B_1^2\right]. \tag{2.23}$$

Proof. For $f\in M_\lambda^n(f,g;\gamma,b;\varphi)$, $p(z)$ given by (2.6) and p_1 given by (2.5), then a_2 and a_3 are given as in Theorem 2.3. Also

$$a_3-\mu a_2^2=\frac{B_1b}{2(1+2\lambda)^n(1+2\gamma)g_3}\left[c_2-\nu c_1^2\right], \tag{2.24}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(1+2\lambda)^n(1+2\gamma)g_3\mu}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}B_1b\right]. \tag{2.25}$$

First, if $\mu\leq\sigma_1$, then we have $\nu\leq0$, and by applying Lemma 2.2 to equality (2.24), we have

$$\left|a_3-\mu a_2^2\right| \leq \frac{b}{(1+2\lambda)^n(1+2\gamma)g_3}\left[B_2-\frac{(1+2\lambda)^n(1+2\gamma)g_3b}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}\mu B_1^2\right],$$

which is evidently inequality (2.21) of Theorem 2.11.

If $\mu = \sigma_1$, then we have $\nu = 0$, therefore equality holds if and only if

$$p_1(z) = (\tfrac{1+\gamma}{2})\frac{1+z}{1-z} + (\tfrac{1-\gamma}{2})\frac{1-z}{1+z} \ (0 \le \gamma \le 1; z \in U).$$

Next, if $\sigma_1 \le \mu \le \sigma_2$, we note that

$$\max\left\{\frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{(1+2\lambda)^n (1+2\gamma)\, g_3 \mu}{(1+\lambda)^{2n} (1+\gamma)^2 g_2^2} B_1 b\right]\right\} \le 1, \tag{2.26}$$

then applying Lemma 2.2 to equality (2.24), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{b}{(1+2\lambda)^n (1+2\gamma)\, g_3},$$

which is evidently inequality (2.22) of Theorem 2.11.
If $\sigma_1 < \mu < \sigma_2$, then we have

$$p_1(z) = \frac{1+z^2}{1-z^2}.$$

Finally, If $\mu \ge \sigma_2$, then we have $\nu \ge 1$, therefore by applying Lemma 2.2 to (2.24), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{b}{(1+2\lambda)^n (1+2\gamma)\, g_3}\left[\frac{(1+2\lambda)^n (1+2\gamma)\, g_3 b}{(1+\lambda)^{2n} (1+\gamma)^2 g_2^2}\mu B_1^2 - B_2\right],$$

which is evidently inequality (2.23) of Theorem 2.11.
If $\mu = \sigma_2$, then we have $\nu = 1$, therefore equality holds if and only if

$$\frac{1}{p_1(z)} = \frac{1+\gamma}{2}\frac{1+z}{1-z} + \frac{1-\gamma}{2}\frac{1-z}{1+z} \ (0 \le \gamma \le 1; z \in U).$$

To show that the bounds are sharp, we define the functions $K_\varphi^s (s \ge 2)$ by

$$1 + \frac{1}{b}\left[(1-\gamma)\tfrac{D_\lambda^n (K_\varphi^s * g)(z)}{z} + \gamma\left(D_\lambda^n (K_\varphi^s * g)(z)\right)' - 1\right] = \varphi(z^{s-1}), \tag{2.27}$$

$$K_\varphi^s(0) = 0 = K_\varphi^{'s}(0) - 1,$$

and the functions F_t and G_t $(0 \le t \le 1)$ by

$$1 + \frac{1}{b}\left[(1-\gamma)\tfrac{D_\lambda^n (F_t * g)(z)}{z} + \gamma\left(D_\lambda^n (F_t * g)(z)\right)' - 1\right] = \varphi\left(\tfrac{z(z+t)}{1+tz}\right), \tag{2.28}$$

$$F_t(0) = 0 = F_t'(0) - 1,$$

and

$$1 + \frac{1}{b}\left[(1-\gamma)\tfrac{D_\lambda^n (G_t * g)(z)}{z} + \gamma\left(D_\lambda^n (G_t * g)(z)\right)' - 1\right] = \varphi\left(-\tfrac{z(z+t)}{1+tz}\right), \tag{2.29}$$

$$G_t(0) = 0 = G_t'(0) - 1.$$

Cleary the functions K_φ^s, F_t and $G_t \in M_\lambda^n(f, g; \gamma, b; \varphi)$. Also we write $K_\varphi = K_\varphi^2$.
If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if f is K_φ^3 or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_t or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_t or one of its rotations.

Remark 2.12. Taking $\gamma = 1$, $b = 1$, $n = 0$ and $g(z) = \dfrac{z}{1-z}$ in Theorem 2.11, we obtain the result obtained by Ali et al. [1, Corollary 2.5, with $k = 1$].

Also, using Lemma 2.2 we have the following theorem:

Theorem 2.13. *For $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, $(b > 0; B_i > 0; i \in \mathbb{N})$ and $f(z)$ given by (1.1) belongs to the class $M_\lambda^n(f, g; \gamma, b; \varphi)$ and $\sigma_1 \leq \mu \leq \sigma_2$, then in view of Lemma 2.2, Theorem 2.11 can be improved. Let*

$$\sigma_3 = \frac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2 B_2}{(1+2\lambda)^n (1+2\gamma) g_3 b B_1^2},$$

(i) If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \tfrac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2}{(1+2\lambda)^n (1+2\gamma) g_3 b B_1}\left[1 - \frac{B_2}{B_1} + \tfrac{(1+2\lambda)^n(1+2\gamma)g_3}{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2}\mu b B_1\right]|a_2|^2$$
$$\leq \frac{B_1 b}{(1+2\lambda)^n (1+2\gamma) g_3}; \tag{2.30}$$

(ii) If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \tfrac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2}{(1+2\lambda)^n (1+2\gamma) g_3 b B_1}\left[1 + \frac{B_2}{B_1} - \tfrac{(1+2\lambda)^n(1+2\gamma)g_3}{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2}\mu b B_1\right]|a_2|^2$$
$$\leq \frac{B_1 b}{(1+2\lambda)^n (1+2\gamma) g_3}. \tag{2.31}$$

Proof. For the values of $\sigma_1 \leq \mu \leq \sigma_3$, we have

$$\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)|a_2|^2$$
$$= \tfrac{B_1 b}{2(1+2\lambda)^n(1+2\gamma)g_3}\left|c_2 - \nu c_1^2\right| + \left(\mu - \tfrac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2 (B_2 - B_1)}{(1+2\lambda)^n(1+2\gamma)g_3 b B_1^2}\right)\tfrac{B_1^2 b^2}{4(1+2\lambda)^{2n}(1+\gamma)^2 g_2^2}|c_1|^2$$
$$= \frac{B_1 b}{(1+2\lambda)^n (1+2\gamma) g_3}\left\{\frac{1}{2}\left(\left|c_2 - \nu c_1^2\right| + \nu |c_1|^2\right)\right\}. \tag{2.32}$$

Now applying Lemma 2.2 to equality (2.32), we have

$$\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)|a_2|^2 \leq \frac{B_1 b}{(1+2\lambda)^n (1+2\gamma) g_3},$$

which is the inequality (2.30) of Theorem 2.13.

Next, for the values of $\sigma_3 \leq \mu \leq \sigma_2$, we have

$$\left|a_3 - \mu a_2^2\right| + (\sigma_2 - \mu)|a_2|^2$$
$$= \tfrac{b B_1}{2(1+2\lambda)^n(1+2\gamma)g_3}\left|c_2 - \nu c_1^2\right| + \left(\tfrac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2 (B_2 + B_1)}{(1+2\lambda)^n(1+2\gamma)g_3 b B_1^2} - \mu\right)$$
$$\cdot \tfrac{B_1^2 b^2}{4(1+2\lambda)^{2n}(1+\gamma)^2 g_2^2}|c_1|^2$$
$$= \frac{B_1 b}{(1+2\lambda)^n (1+2\gamma) g_3}\left\{\frac{1}{2}\left(\left|c_2 - \nu c_1^2\right| + (1-\nu)|c_1|^2\right)\right\}. \tag{2.33}$$

Now applying Lemma 2.2 to equality (2.33), we have

$$\left|a_3 - \mu a_2^2\right| + (\sigma_2 - \mu)|a_2|^2 \leq \frac{B_1 b}{(1+2\lambda)^n (1+2\gamma) g_3},$$

which is the inequality (2.31). This completes the proof of Theorem 2.13.

Remark 2.14. (i) Specializing the parameters γ, λ, b, n, g and φ in Theorem 2.11 and Theorem 2.13, we obtain the corresponding results of the classes $G_n(\gamma, b, A, B)$, $S(f, g; \gamma, \alpha, b)$, $G_n(\gamma, b)$, $R_\ell^b(A, B)$, $M_{\lambda,q,s}^n([\alpha_1]; \gamma, b; \varphi)$, $G_\lambda^n(f, g; b; \varphi)$, $R_\lambda^n(f, g; b; \varphi)$ and $E_{\lambda,\rho}^{n,\eta}(f, g; \gamma; \varphi)$, as special cases as defined before.

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