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Multivariate weighted fractional representation formulae and Ostrowski type inequalities

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Abstract. Here we derive multivariate weighted fractional representation formulae involving ordinary partial derivatives of first order. Then we present related multivariate weighted fractional Ostrowski type inequalities with respect to uniform norm.

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1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Suppose now that $w : [a, b] \to [0, \infty)$ is some probability density function, i.e. it is a nonnegative integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a, b]$, $W(t) = 0$ for $t \le a$ and $W(t) = 1$ for $t \ge b$. Then, the following identity (Pecarić, [5]) is the weighted generalization of the Montgomery identity ([4])

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x,t) f'(t)\,dt, \tag{1.1}$$

where the weighted Peano Kernel is

$$P_w(x,t) := \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases} \tag{1.2}$$

In [1] we proved

Theorem 1.1. *Let $w : [a, b] \to [0, \infty)$ be a probability density function, i.e. $\int_a^b w(t)\,dt = 1$, and set $W(t) = \int_a^t w(x)\,dx$ for $a \le t \le b$, $W(t) = 0$ for $t \le a$ and $W(t) = 1$ for $t \ge b$, $\alpha \ge 1$, and f is as in (1.1). Then the generalization of the weighted Montgomery* *identity for fractional integrals is the following*

$$f(x) = (b-x)^{1-\alpha} \Gamma(\alpha) J_a^{\alpha} (w(b) f(b)) - J_a^{\alpha-1} (Q_w(x,b) f(b)) + J_a^{\alpha} (Q_w(x,b) f'(b)), \tag{1.3}$$

where the weighted fractional Peano Kernel is

$$Q_w(x,t) := \begin{cases} (b-x)^{1-\alpha} \Gamma(\alpha) W(t), & a \le t \le x, \\ (b-x)^{1-\alpha} \Gamma(\alpha) (W(t) - 1), & x < t \le b, \end{cases} \tag{1.4}$$

i.e. $Q_w(x,t) = (b-x)^{1-\alpha} \Gamma(\alpha) P_w(x,t)$.

When $\alpha = 1$ then the weighted generalization of the Montgomery identity for fractional integrals in (1.3) reduces to the weighted generalization of the Montgomery identity for integrals in (1.1).

So for $\alpha \ge 1$ and $x \in [a,b)$ we can rewrite (1.3) as follows

$$\begin{aligned} f(x) &= (b-x)^{1-\alpha} \int_a^b (b-t)^{\alpha-1} w(t) f(t) \, dt \\ &- (b-x)^{1-\alpha} (\alpha-1) \int_a^b (b-t)^{\alpha-2} P_w(x,t) f(t) \, dt \\ &+ (b-x)^{1-\alpha} \int_a^b (b-t)^{\alpha-1} P_w(x,t) f'(t) \, dt. \end{aligned} \tag{1.5}$$

In this article based on (1.5), we establish a multivariate weighted general fractional representation formula for $f(x)$, $x \in \prod_{i=1}^m [a_i, b_i] \subset \mathbb{R}^m$, and from there we derive an interesting multivariate weighted fractional Ostrowski type inequality. We finish with an application.

2. Main Results

We make

Assumption 2.1. Let $f \in C^1 \left(\prod_{i=1}^m [a_i, b_i] \right)$.

Assumption 2.2. Let $f : \prod_{i=1}^m [a_i, b_i] \to \mathbb{R}$ be measurable and bounded, such that there exist $\frac{\partial f}{\partial x_j} : \prod_{i=1}^m [a_i, b_i] \to \mathbb{R}$, and it is x_j-integrable for all $j = 1, ..., m$. Furthermore $\frac{\partial f}{\partial x_i} (t_1, ..., t_i, x_{i+1}, ..., x_m)$ it is integrable on $\prod_{j=1}^i [a_j, b_j]$, for all $i = 1, ..., m$, for any $(x_{i+1}, ..., x_m) \in \prod_{j=i+1}^m [a_j, b_j]$.

Convention 2.3. We set

$$\prod_{j=1}^{0} \cdot = 1. \tag{2.1}$$

Notation 2.4. Here $x = \overrightarrow{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$. Likewise $t = \overrightarrow{t} = (t_1, ..., t_m)$, and $d\overrightarrow{t} = dt_1 dt_2 ... dt_m$. Here w_i, W_i correspond to $[a_i, b_i]$, $i = 1, ..., m$, and are as w, W of Theorem 1.1.

We need

Definition 2.5. *(see* [2] *and* [3]*) Let* $\prod_{i=1}^{m}[a_i,b_i]\subset\mathbb{R}^m$, $m\in\mathbb{N}-\{1\}$, $a_i<b_i$, $a_i,b_i\in\mathbb{R}$. *Let* $\alpha>0$, $f\in L_1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$. *We define the left mixed Riemann-Liouville fractional multiple integral of order* α*:*

$$\left(I_{a+}^{\alpha}f\right)(x):=\frac{1}{(\Gamma(\alpha))^m}\int_{a_1}^{x_1}...\int_{a_m}^{x_m}\left(\prod_{i=1}^{m}(x_i-t_i)\right)^{\alpha-1}f(t_1,...,t_m)\,dt_1...dt_m, \quad (2.2)$$

where $x_i\in[a_i,b_i]$, $i=1,...,m$, *and* $x=(x_1,...,x_m)$, $a=(a_1,...,a_m)$, $b=(b_1,...,b_m)$.

We present the following multivariate weighted fractional representation formula

Theorem 2.6. *Let* f *as in Assumption* 2.1 *or Assumption* 2.2, $\alpha\geq1$, $x_i\in[a_i,b_i)$, $i=1,...,m$. *Here* P_{w_i} *corresponds to* $[a_i,b_i]$, $i=1,...,m$, *and it is as in (*1.2*). The probability density function* w_j *is assumed to be bounded for all* $j=1,...,m$. *Then*

$$f(x_1,...,x_m)=\left(\prod_{j=1}^{m}(b_j-x_j)\right)^{1-\alpha}(\Gamma(\alpha))^m\left(I_{a+}^{\alpha}\left(\prod_{j=1}^{m}w_j\right)f\right)(b)$$
$$+\sum_{i=1}^{m}A_i(x_1,...,x_m)+\sum_{i=1}^{m}B_i(x_1,...,x_m), \quad (2.3)$$

where for $i=1,...,m$*:*

$$A_i(x_1,...,x_m):=-(\alpha-1)\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i-1}(b_j-t_j)\right)^{\alpha-1} \quad (2.4)$$
$$\cdot(b_i-t_i)^{\alpha-2}\left(\prod_{j=1}^{i-1}w_j(t_j)\right)P_{w_i}(x_i,t_i)\,f(t_1,...,t_i,x_{i+1},...,x_m)\,dt_1...dt_i,$$

and

$$B_i(x_1,...,x_m):=\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i}(b_j-t_j)\right)^{\alpha-1} \quad (2.5)$$
$$\cdot\left(\prod_{j=1}^{i-1}w_j(t_j)\right)P_{w_i}(x_i,t_i)\frac{\partial f}{\partial x_i}(t_1,...,t_i,x_{i+1},...,x_m)\,dt_1...dt_i.$$

Proof. We have that

$$f(x_1,x_2,...,x_m)\overset{(1.5)}{=}(b_1-x_1)^{1-\alpha}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}w_1(t_1)\,f(t_1,x_2,...,x_m)\,dt_1$$
$$+A_1(x_1,...,x_m)+B_1(x_1,...,x_m). \quad (2.6)$$

Similarly it holds

$$f(t_1,x_2,...,x_m)\overset{(1.5)}{=}(b_2-x_2)^{1-\alpha}\int_{a2}^{b_2}(b_2-t_2)^{\alpha-1}w_2(t_2)\,f(t_1,t_2,x_3,...,x_m)\,dt_2$$

$$-\left(\alpha-1\right)\left(b_2-x_2\right)^{1-\alpha}\int_{a_2}^{b_2}\left(b_2-t_2\right)^{\alpha-2}P_{w_2}\left(x_2,t_2\right)f\left(t_1,t_2,x_3,...,x_m\right)dt_2$$

$$+\left(b_2-x_2\right)^{1-\alpha}\int_{a_2}^{b_2}\left(b_2-t_2\right)^{\alpha-1}P_{w_2}\left(x_2,t_2\right)\frac{\partial f}{\partial x_2}\left(t_1,t_2,x_3,...,x_m\right)dt_2. \tag{2.7}$$

Next we plug (2.7) into (2.6).

We get

$$f\left(x_1,...,x_m\right)=\left(\left(b_1-x_1\right)\left(b_2-x_2\right)\right)^{1-\alpha}$$

$$\cdot\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left(\left(b_1-t_1\right)\left(b_2-t_2\right)\right)^{\alpha-1}w_1\left(t_1\right)w_2\left(t_2\right)f\left(t_1,t_2,x_3,...,x_m\right)dt_1dt_2 \tag{2.8}$$

$$+A_2\left(x_1,...,x_m\right)+B_2\left(x_1,...,x_m\right)+A_1\left(x_1,...,x_m\right)+B_1\left(x_1,...,x_m\right).$$

We continue as above.

We also have

$$f\left(t_1,t_2,x_3,...,x_m\right)\overset{(1.5)}{=}\left(b_3-x_3\right)^{1-\alpha}$$

$$\cdot\int_{a_3}^{b_3}\left(b_3-t_3\right)^{\alpha-1}w_3\left(t_3\right)f\left(t_1,t_2,t_3,x_4,...,x_m\right)dt_3$$

$$-\left(\alpha-1\right)\left(b_3-x_3\right)^{1-\alpha}\int_{a_3}^{b_3}\left(b_3-t_3\right)^{\alpha-2}P_{w_3}\left(x_3,t_3\right)f\left(t_1,t_2,t_3,x_4,...,x_m\right)dt_3 \tag{2.9}$$

$$+\left(b_3-x_3\right)^{1-\alpha}\int_{a_3}^{b_3}\left(b_3-t_3\right)^{\alpha-1}P_{w_3}\left(x_3,t_3\right)\frac{\partial f}{\partial x_3}\left(t_1,t_2,t_3,x_4,...,x_m\right)dt_3.$$

We plug (2.9) into (2.8). Therefore it holds

$$f\left(x_1,...,x_m\right)=\left(\prod_{j=1}^{3}\left(b_j-x_j\right)\right)^{1-\alpha}\int_{\prod_{j=1}^{3}[a_j,b_j]}\left(\prod_{j=1}^{3}\left(b_j-t_j\right)\right)^{\alpha-1}$$

$$\cdot\left(\prod_{j=1}^{3}w_j\left(t_j\right)\right)f\left(t_1,t_2,t_3,x_4,...,x_m\right)dt_1dt_2dt_3$$

$$+\sum_{j=1}^{3}A_j\left(x_1,...,x_m\right)+\sum_{j=1}^{3}B_j\left(x_1,...,x_m\right). \tag{2.10}$$

Continuing similarly we finally obtain

$$f\left(x_1,...,x_m\right)=\left(\prod_{j=1}^{m}\left(b_j-x_j\right)\right)^{1-\alpha}$$

$$\cdot\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{j=1}^{m}\left(b_j-t_j\right)\right)^{\alpha-1}\left(\prod_{j=1}^{m}w_j\left(t_j\right)\right)f\left(\overrightarrow{t}\right)d\overrightarrow{t}$$

$$+\sum_{i=1}^{m}A_i\left(x_1,...,x_m\right)+\sum_{i=1}^{m}B_i\left(x_1,...,x_m\right), \tag{2.11}$$

that is proving the claim. □

We make

Remark 2.7. Let $f \in C^1\left(\prod_{i=1}^m [a_i, b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i, b_i)$, $i = 1, ..., m$. Denote by

$$\|f\|_\infty^{\sup} := \sup_{x \in \prod_{i=1}^m [a_i, b_i]} |f(x)|. \tag{2.12}$$

From (1.2) we get that

$$|P_w(x,t)| \leq \left\{ \begin{array}{ll} W(x), & a \leq t \leq x, \\ 1 - W(x), & x < t \leq b \end{array} \right\}$$

$$\leq \max\{W(x), 1 - W(x)\} = \frac{1 + |2W(x) - 1|}{2}. \tag{2.13}$$

That is

$$|P_w(x,t)| \leq \frac{1 + |2W(x) - 1|}{2}, \tag{2.14}$$

for all $t \in [a,b]$, where $x \in [a,b]$ is fixed.

Consequently it holds

$$|P_{w_i}(x_i, t_i)| \leq \frac{1 + |2W_i(x_i) - 1|}{2}, \quad i = 1, ..., m. \tag{2.15}$$

Assume here that

$$w_j(t_j) \leq K_j, \tag{2.16}$$

for all $t_j \in [a_j, b_j]$, where $K_j > 0$, $j = 1, ..., m$.

Therefore we derive

$$|B_i(x_1, ..., x_m)| \leq \left(\prod_{j=1}^i (b_j - x_j)\right)^{1-\alpha} \left(\prod_{j=1}^{i-1} K_j\right)$$

$$\cdot \left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup} \prod_{j=1}^i \left(\int_{a_j}^{b_j} (b_j - t_j)^{\alpha - 1} dt_j\right). \tag{2.17}$$

That is

$$|B_i(x_1, ..., x_m)| \leq \left(\prod_{j=1}^i (b_j - x_j)\right)^{1-\alpha} \left(\frac{\prod_{j=1}^i (b_j - a_j)^\alpha}{\alpha^i}\right) \left(\prod_{j=1}^{i-1} K_j\right)$$

$$\cdot \left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup}, \tag{2.18}$$

for all $i = 1, ..., m$.

Based on the above and Theorem 2.6 we have established the following multivariate weighted fractional Ostrowski type inequality.

Theorem 2.8. *Let $f \in C^1\left(\prod_{i=1}^m [a_i, b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i, b_i)$, $i = 1, ..., m$. Here P_{w_i} corresponds to $[a_i, b_i]$, $i = 1, ..., m$, and it is as in (1.2). Assume that $w_j(t_j) \leq K_j$, for all $t_j \in [a_j, b_j]$, where $K_j > 0$, $j = 1, ..., m$. And $A_i(x_1, ..., x_m)$ is as in (2.4), $i = 1, ..., m$. Then*

$$\left| f(x_1, ..., x_m) - \left(\prod_{j=1}^m (b_j - x_j)\right)^{1-\alpha} (\Gamma(\alpha))^m \left(I_{a+}^\alpha \left(\prod_{j=1}^m w_j\right) f\right)(b) \right.$$

$$\left. - \sum_{i=1}^m A_i(x_1, ..., x_m) \right| \leq \sum_{i=1}^m \left\{ \left(\prod_{j=1}^i (b_j - x_j)\right)^{1-\alpha} \left(\frac{\prod_{j=1}^i (b_j - a_j)^\alpha}{\alpha^i}\right) \right.$$

$$\left. \cdot \left(\prod_{j=1}^{i-1} K_j\right) \left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\| \frac{\partial f}{\partial x_i} \right\|_\infty^{\sup} \right\}. \tag{2.19}$$

3. Application

Here we operate on $[0,1]^m$, $m \in \mathbb{N} - \{1\}$. We notice that

$$\int_0^1 \left(\frac{e^{-x}}{1 - e^{-1}}\right) dx = 1, \tag{3.1}$$

and

$$\frac{e^{-x}}{1 - e^{-1}} \leq \frac{1}{1 - e^{-1}}, \text{ for all } x \in [0,1]. \tag{3.2}$$

So here we choose as w_j the probability density function

$$w_j^*(t_j) := \frac{e^{-t_j}}{1 - e^{-1}}, \tag{3.3}$$

$j = 1, ..., m$, $t_j \in [0,1]$.

So we have the corrsponding W_j as

$$W_j^*(t_j) = \frac{1 - e^{-t_j}}{1 - e^{-1}}, \quad t_j \in [0,1], \tag{3.4}$$

and the corresponding P_{w_j} as

$$P_{w_j}^*(x_j, t_j) = \begin{cases} \frac{1 - e^{-t_j}}{1 - e^{-1}}, & 0 \leq t_j \leq x_j, \\ \frac{e^{-1} - e^{-t_j}}{1 - e^{-1}}, & x_j < t_j \leq 1, \end{cases} \tag{3.5}$$

$j = 1, ..., m$.

Set $\overrightarrow{0} = (0, ..., 0)$ and $\overrightarrow{1} = (1, ..., 1)$.

First we apply Theorem 2.6.

We have

Theorem 3.1. *Let* $f \in C^1([0,1]^m)$, $\alpha \geq 1$, $x_i \in [0,1)$, $i = 1,...,m$. *Then*

$$f(x_1,...,x_m) = \left(\prod_{j=1}^{m}(1-x_j)\right)^{1-\alpha}\left(\frac{\Gamma(\alpha)}{1-e^{-1}}\right)^m\left(I^{\alpha}_{\overrightarrow{0}+}\left(e^{-\sum_{j=1}^{m}t_j}f(\cdot)\right)\right)\left(\overrightarrow{1}\right)$$
$$+\sum_{i=1}^{m}A_i^*(x_1,...,x_m)+\sum_{i=1}^{m}B_i^*(x_1,...,x_m), \tag{3.6}$$

where for $i = 1,...,m$:

$$A_i^*(x_1,...,x_m) := \frac{-(\alpha-1)}{(1-e^{-1})^{i-1}}\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}\int_{[0,1]^i}\left(\prod_{j=1}^{i-1}(1-t_j)\right)^{\alpha-1} \tag{3.7}$$
$$\cdot(1-t_i)^{\alpha-2}e^{-\sum_{j=1}^{i-1}t_j}P^*_{w_i}(x_i,t_i)f(t_1,...,t_i,x_{i+1},...,x_m)\,dt_1...dt_i,$$

and

$$B_i^*(x_1,...,x_m) := \frac{\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}}{(1-e^{-1})^{i-1}}\int_{[0,1]^i}\left(\prod_{j=1}^{i}(1-t_j)\right)^{\alpha-1} \tag{3.8}$$
$$\cdot e^{-\sum_{j=1}^{i-1}t_j}P^*_{w_i}(x_i,t_i)\frac{\partial f}{\partial x_i}(t_1,...,t_i,x_{i+1},...,x_m)\,dt_1...dt_i.$$

Above we set $\sum_{i=1}^{0}\cdot = 0$.

Finally we apply Theorem 2.8.

Theorem 3.2. *Let* $f \in C^1([0,1]^m)$, $\alpha \geq 1$, $x_i \in [0,1)$, $i = 1,...,m$. *Here* $P^*_{w_i}$ *is as in (*3.5*) and* $A_i^*(x_1,...,x_m)$ *as in (*3.7*),* $i = 1,...,m$. *Then*

$$\left|f(x_1,...,x_m) - \left(\prod_{j=1}^{m}(1-x_j)\right)^{1-\alpha}\left(\frac{\Gamma(\alpha)}{1-e^{-1}}\right)^m\left(I^{\alpha}_{\overrightarrow{0}+}\left(e^{-\sum_{j=1}^{m}t_j}f(\cdot)\right)\right)\left(\overrightarrow{1}\right)\right.$$
$$\left.-\sum_{i=1}^{m}A_i^*(x_1,...,x_m)\right| \leq \sum_{i=1}^{m}\left\{\frac{\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}}{\alpha^i(1-e^{-1})^{i-1}}\right.$$
$$\left.\cdot\left(\frac{1+|2W_i^*(x_i)-1|}{2}\right)\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\right\}. \tag{3.9}$$

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