Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 4, 529–537

Integral operator defined by q-analogue of Liu-Srivastava operator

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Abstract. In this paper, we shall give an application of q-analogues theory in geometric function theory. We introduce an integral operator for meromorphic functions involving the q-analogue of differential operator. We also investigate several properties for this operator.

Mathematics Subject Classification (2010): 30C45.

Keywords: q-analogue, meromorphic function, Liu-Srivastava operator, integral operator.

1. Introduction

The theory of q-analogues or q-extensions of classical formulas and functions based on the observation that

$$\lim_{q\to 1}\frac{1-q^{\alpha}}{1-q}=\alpha, |q|<1,$$

therefore the number $(1-q^{\alpha})/(1-q)$ is sometimes called the basic number $[\alpha]_q$. In this work we derive q-analogue of Liu-Srivastava operator and employ this new differential operator to define an integral operator for meromorphic functions.

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k, \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^*=\{z:z\in\mathbb{C},0<|z|<1\}=\mathbb{U}\backslash\{0\}.$$

For complex parameters α_i,β_j $(i=1,\ldots,r,\ j=1,\ldots,s,\ \alpha_i\in\mathbb{C},\ \beta_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\})$ the basic hypergeometric function (or q- hypergeometric function) is the q-analogue of the familiar hypergeometric function and it is defined as follows:

$$\psi(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)=\sum_{k=0}^{\infty}\frac{(\alpha_1,q)_k\ldots(\alpha_r,q)_k}{(q,q)_k(\beta_1,q)_k\ldots(\beta_s,q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k, \tag{1.2}$$

with $\binom{k}{2}=k(k-1)/2$, where $q\neq 0$ when $r>s+1$, $(r,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$, and $(\alpha,q)_k$ is the q-analogue of the Pochhammer symbol $(\alpha)_k$ defined by

$$(\alpha,q)_k=\begin{cases}1, & k=0;\\ (1-\alpha)(1-\alpha q)(1-\alpha q^2)\ldots(1-\alpha q^{k-1}), & k\in\mathbb{N}.\end{cases}$$

It is clear that

$$\lim_{q\to 1}\frac{(q^\alpha;q)_k}{(1-q)^k}=(\alpha)_k.$$

The radius of convergence ρ of the basic hypergeometric series (1.2) for $|q|<1$ is given by

$$\rho=\begin{cases}\infty, & \text{if } r<s+1;\\ 1, & \text{if } r=s+1;\\ 0, & \text{if } r>s+1.\end{cases}$$

The basic hypergeometric series defined by (1.2) was first introduced by Heine in 1846. Therefore it is sometimes called Heine's series. For more details concerning the q-theory the reader may refer to (see [1],[2]).

Now for $z\in\mathbb{U}$, $|q|<1$, and $r=s+1$, the basic hypergeometric function defined in (1.2) takes the form

$${}_r\Phi_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)=\sum_{k=0}^{\infty}\frac{(\alpha_1;q)_k\ldots(\alpha_r;q)_k}{(q;q)_k(\beta_1;q)_k\ldots(\beta_s;q)_k}z^k$$

which converges absolutely in the open unit disk $\mathbb{U}$.

Corresponding to the function ${}_r\Phi_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)$, consider

$$\begin{aligned}{}_r\mathcal{G}_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)&=\frac{1}{z}\quad {}_r\Phi_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)\\ &=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{(\alpha_1,q)_{k+1}\ldots(\alpha_r,q)_{k+1}}{(q,q)_{k+1}(\beta_1,q)_{k+1}\ldots(\beta_s,q)_{k+1}}z^k.\end{aligned}$$

Next, we define the linear operator $\mathcal{L}_s^r(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;q):\Sigma\to\Sigma$ by

$$\begin{aligned}\mathcal{L}_s^r(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;q)f(z)&={}_r\mathcal{G}_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)*f(z)\\ &=\frac{1}{z}+\sum_{k=1}^{\infty}\nabla_s^r(\alpha_1,q,k)a_k z^k\end{aligned} \tag{1.3}$$

where

$$\nabla_s^r(\alpha_1,q,k)=\frac{(\alpha_1,q)_{k+1}\ldots(\alpha_r,q)_{k+1}}{(q,q)_{k+1}(\beta_1,q)_{k+1}\ldots(\beta_s,q)_{k+1}}.$$

For the sake of simplicity we write

$$\mathcal{L}_s^r(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q)f(z) = \mathcal{L}_s^r[\alpha_1, q]f(z).$$

Remark 1.1. **i.** For $\alpha_i = q^{\alpha_i}, \beta_j = q^{\beta_j}, \alpha_i > 0, \beta_j > 0, (i = 1, \ldots, r; j = 1, \ldots, s, r = s+1), q \to 1$ the operator $\mathcal{L}_s^r[\alpha_1, q]f(z) = \mathcal{H}_s^r[\alpha_1]f(z)$ which was investigated by Liu and Srivastava [3].

ii. For $r = 2, s = 1, \alpha_2 = q, q \to 1$, the operator $\mathcal{L}_1^2[\alpha_1, q, \beta_1, q]f(z) = \mathcal{L}[\alpha_1; \beta_1]f(z)$ was introduced and studied by Liu and srivastava [4]. Further, we note in passing that this operator $\mathcal{L}[\alpha_1; \beta_1]f(z)$ is closely related to the Carlson-Shaffer operator $\mathcal{L}[\alpha_1; \beta_1]f(z)$ defined on the space of analytic univalent functions in $\mathbb{U}$.

iii. For $r = 1, s = 0, \alpha_1 = \lambda + 1, q \to 1$, the operator $\mathcal{L}_0^1[\lambda + 1, q]f(z) = \mathcal{D}^\lambda f(z) = \frac{1}{z((1-z)^{\lambda+1}} * f(z)(\lambda > -1)$, where $\mathcal{D}^\lambda$ is the differential operator which was introduced by Ganigi and Uralegadi [5], and then it was generalized by Yang [6].

Analogue to the integral operator defined in [7] which involving q-hypergeometric functions on the normalized analytic functions, we now define the following integral operator on the space of meromorphic functions in the class Σ using the differential operator $\mathcal{L}_s^r[\alpha_1, q]$ defined in (1.3).

Definition 1.2. *Let $n \in \mathbb{N}, i \in \{1, 2, \ldots, n\}, \gamma_i > 0$. We define the integral operator $\mathcal{H}(f_1, f_2, \ldots, f_n)(z) : \Sigma^n \to \Sigma$ by*

$$\mathcal{H}(f_1, f_2, \ldots, f_n)(z) = \frac{1}{z^2} \int_0^z (u\, \mathcal{L}_s^r[\alpha_1, q]f_1(u))^{\gamma_1} \ldots (u\, \mathcal{L}_s^r[\alpha_1, q]f_n(u))^{\gamma_n} du. \quad (1.4)$$

For the sake of simplicity, we write $\mathcal{H}(z)$ instead of $\mathcal{H}(f_1, f_2, \ldots, f_n)(z)$.

We observe that in (1.4) for $r = 1, s = 0, a_1 = q$, we obtain the integral operator introduced and studied by Mohammed and Darus [8], see also ([9],[10],[11]).

The following definitions introduce subclasses of Σ which are of meromorphic starlike functions.

Definition 1.3. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_{r,s}^*(\alpha_1, q, \delta, b)$ if and only if, f satisfies*

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f(z)} + 1\right)\right\} > \delta,$$

where $\mathcal{L}_s^r[\alpha_1, q]f$ defined in (1.3) and $b \in \mathbb{C}\backslash\{0\}, 0 \le \delta < 1$.

Definition 1.4. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_{r,s}^*\mathcal{U}(\alpha_1, q, \alpha, \delta, b)$ if and only if, f satisfies*

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f(z)} + 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f(z)} + 1\right)\right| + \delta,$$

where $\mathcal{L}_s^r[\alpha_1, q]f$ defined in (1.3) and $\alpha \ge 0, -1 \le \delta < 1, \alpha + \delta \ge 0, b \in \mathbb{C}\backslash\{0\}$.

Definition 1.5. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma^*_{r,s}\mathcal{UH}(\alpha_1, q, \alpha, b)$ if and only if, f satisfies*

$$\left|1 - \frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f(z)} + 1\right) - 2\alpha(\sqrt{2} - 1)\right| < \Re\left\{\sqrt{2}\left(1 - \frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f(z)} + 1\right)\right)\right\} + 2\alpha(\sqrt{2} - 1),$$

where $\mathcal{L}^r_s[\alpha_1, q]f$ defined in (1.3) and $\alpha > 0, b \in \mathbb{C}\backslash\{0\}$.

For $r = 1, s = 0$ and $\alpha_1 = q$ in Definitions 1.3, 1.4 and 1.5, we obtain $\Sigma^*_b(\delta), \Sigma^*\mathcal{U}(\alpha, \delta, b)$ and $\Sigma^*\mathcal{UH}(\alpha, b)$ the classes of meromorphic functions, introduced and studied by Mohammed and Darus [12].

Now, let us introduce the following families of subclasses of meromorphic functions $\Sigma\mathcal{F}_1(\delta, b), \Sigma\mathcal{F}_2(\alpha, \delta, b)$ and $\Sigma\mathcal{F}_3(\alpha, b)$ as follows.

Definition 1.6. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_1(\delta, b)$ if and only if, f satisfies*

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right)\right\} > \delta, \tag{1.5}$$

where $b \in \mathbb{C}\backslash\{0\}, 0 \le \delta < 1$.

Definition 1.7. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_2(\alpha, \delta, b)$ if and only if, f satisfies*

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right)\right| + \delta, \tag{1.6}$$

where $\alpha \ge 0, -1 \le \delta < 1, \alpha + \delta \ge 0, b \in \mathbb{C}\backslash\{0\}$.

Definition 1.8. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_3(\alpha, b)$ if and only if, f satisfies*

$$\left|1 - \frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right) - 2\alpha(\sqrt{2} - 1)\right| < \Re\left\{\sqrt{2}\left(1 - \frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right)\right)\right\} + 2\alpha(\sqrt{2} - 1), \tag{1.7}$$

where $\alpha > 0, b \in \mathbb{C}\backslash\{0\}$.

2. Main results

In this section, we investigate some properties for the integral operator $\mathcal{H}(z)$ defined by (1.4)of the subclasses given by Definitions 1.3, 1.4 and 1.5

Theorem 2.1. *For $i \in \{1, 2, \ldots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^*_{r,s}(\alpha_1, q, \delta_i, b)(0 \le \delta < 1)$ and $b \in \mathbb{C}\backslash\{0\}$. If*

$$0 < \sum_{i=1}^{n} \gamma_i(1 - \delta_i) \le 1,$$

then $\mathcal{H}(z)$ *is in the class* $\Sigma\mathcal{F}_1(\mu, b)$, $\mu = 1 - \sum_{i=1}^{n} \gamma_i(1-\delta_i)$

Proof. A differentiation of $\mathcal{H}(z)$ which is defined by (1.4), we obtain

$$z^2\mathcal{H}'(z) + 2z\mathcal{H}(z) = (z\ \mathcal{L}_s^r[\alpha_1, q]f_1(z))^{\gamma_1} \ldots (z\ \mathcal{L}_s^r[\alpha_1, q]f_n(z))^{\gamma_n}, \tag{2.1}$$

$$\begin{aligned} z^2\mathcal{H}''(z) + 4z\mathcal{H}'(z) + 2\mathcal{H}(z) \\ = \sum_{i=1}^{n} \gamma_i \Big(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f)_i'(z) + \mathcal{L}_s^r[\alpha_1, q]f_i(z)}{z\mathcal{L}_s^r[\alpha_1, q]f_i(z)} \Big) \\ [(z\ \mathcal{L}_s^r[\alpha_1, q]f_1(z))^{\gamma_1} \ldots (z\ \mathcal{L}_s^r[\alpha_1, q]f_n(z))^{\gamma_n}] \end{aligned} \tag{2.2}$$

Then from (2.1) and (2.2),we obtain

$$\frac{z^2\mathcal{H}''(z) + 4z\mathcal{H}'(z) + 2\mathcal{H}(z)}{z^2\mathcal{H}'(z) + 2z\mathcal{F}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + \frac{1}{z} \right). \tag{2.3}$$

By multiplying (2.3) with z we have

$$\frac{z^2\mathcal{H}''(z) + 4z\mathcal{H}'(z) + 2\mathcal{H}_{\gamma_i}(z)}{z\mathcal{H}'_{\gamma_i}(z) + 2\mathcal{H}_{\gamma_i}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + 1 \right).$$

That is equivalent to

$$\frac{z\left(z\mathcal{H}''(z) + 3\mathcal{H}'(z)\right)}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 = \sum_{i=1}^{n} \gamma_i \left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + 1 \right). \tag{2.4}$$

Equivalently, (2.4) can be written as

$$1 - \frac{1}{b}\left\{ \frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 \right\} = \sum_{i=1}^{n} \gamma_i \left\{ 1 - \frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i.$$

Taking the real part of both sides of the last expression, we have

$$\begin{aligned} \Re\left\{ 1 - \frac{1}{b}\left(\frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 \right) \right\} \\ = \sum_{i=1}^{n} \gamma_i \Re\left\{ 1 - \frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i. \end{aligned}$$

Since $f_i \in \Sigma_{r,s}^*(\alpha_1, q, \delta_i, b)$, hence

$$\Re\left\{ 1 - \frac{1}{b}\left(\frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 \right) \right\} > \sum_{i=1}^{n} \gamma_i\delta_i + 1 - \sum_{i=1}^{n} \gamma_i.$$

Therefore

$$\Re\left\{ 1 - \frac{1}{b}\left(\frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 \right) \right\} > 1 - \sum_{i=1}^{n} \gamma_i(1-\delta_i).$$

Then $\mathcal{H}(z) \in \Sigma\mathcal{F}_1(\mu, b)$, $\quad \mu = 1 - \sum_{i=1}^{n} \gamma_i(1-\delta_i)$ □

Theorem 2.2. *For $i \in \{1,2,\ldots,n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^*_{r,s}\mathcal{U}(\alpha,\delta,b)(\alpha \geq 0, -1 \leq \delta < 1, \alpha+\delta \geq 0)$ and $b \in \mathbb{C}\backslash\{0\}$. If*

$$\sum_{i=1}^{n} \gamma_i \leq 1,$$

then $\mathcal{H}(z)$ is in the class $\Sigma\mathcal{F}_2(\alpha,\delta,b)$.

Proof. Since $f_i \in \Sigma^*_{r,s}\mathcal{U}(\alpha_1,q,\alpha,\delta,b)$, it follows from Definition 1.3 that

$$\Re\left\{1-\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1,q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1,q]f_i(z)}+1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1,q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1,q]f_i(z)}+1\right)\right|+\delta. \quad (2.5)$$

Considering (2.2) and (2.5) we obtain

$$\Re\left\{1-\frac{1}{b}\left(\frac{z(z\mathcal{H}''(z)+3\mathcal{H}'(z))}{z\mathcal{H}'(z)+2\mathcal{H}(z)}+1\right)\right\}-\alpha\left|\frac{1}{b}\left(\frac{z(z\mathcal{H}''(z)+3\mathcal{H}'(z))}{z\mathcal{H}'(z)+2\mathcal{H}(z)}+1\right)\right|-\delta.$$

$$=1-\sum_{i=1}^{n}\gamma_i+\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1,q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1,q]f_i(z)}+1\right)\right\}$$
$$-\alpha\left|\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1,q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1,q]f_i(z)}+1\right)\right|-\delta$$

$$>1-\sum_{i=1}^{n}\gamma_i+\sum_{i=1}^{n}\gamma_i\left\{\alpha\left|\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1,q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1,q]f_i(z)}+1\right)\right|+\delta\right\}$$
$$-\alpha\sum_{i=1}^{n}\gamma_i\left|\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1,q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1,q]f_i(z)}+1\right)\right|-\delta$$

$$=(1-\delta)(1-\sum_{i=1}^{n}\gamma_i)\geq 0.$$

This completes the proof. □

Theorem 2.3. *For $i \in \{1,2,\ldots,n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^*\mathcal{U}\mathcal{H}(\alpha,b)$ $(\alpha > 0$ and $b \in \mathbb{C}\backslash\{0\})$. If*

$$\sum_{i=1}^{n} \gamma_i \leq 1,$$

then $\mathcal{H}(z)$ is in the class $\Sigma\mathcal{F}_3(\alpha,b)$.

Proof. Since $f_i \in \Sigma^*_{r,s}\mathcal{U}\mathcal{H}(\alpha_1,q,\alpha,b)$, it follows from Definition 1.4 that

$$\Re\left\{\sqrt{2}\left(1-\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1,q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1,q]f_i(z)}+1\right)\right)\right\}+2\alpha(\sqrt{2}-1)$$
$$-\left|1-\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1,q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right|>0. \quad (2.6)$$

Considering (2.2) and (2.6), we obtain

$$\Re\left\{\sqrt{2}\left(1-\frac{1}{b}\left(\frac{z(z\mathcal{H}''(z)+3\mathcal{H}'(z))}{z\mathcal{H}'(z)+2\mathcal{H}(z)}+1\right)\right)\right\}+2\alpha(\sqrt{2}-1)$$
$$-\left|1-\frac{1}{b}\left(\frac{z(z\mathcal{H}''(z)+3\mathcal{H}'(z))}{z\mathcal{H}'(z)+2\mathcal{H}(z)}+1\right)-2\alpha(\sqrt{2}-1)\right| \quad (2.7)$$

$$=\Re\left\{\sqrt{2}\left[1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right]\right\}+2\alpha(\sqrt{2}-1)$$
$$-\left|1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right|$$

$$=\sqrt{2}-\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)+2\alpha(\sqrt{2}-1)$$
$$-\left|1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right|$$

$$=\sqrt{2}+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right\}-\sqrt{2}\sum_{i=1}^{n}\gamma_i+2\alpha(\sqrt{2}-1)$$
$$-\left|1+\sum_{i=1}^{n}\gamma_i\left[1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right]-\sum_{i=1}^{n}\gamma_i\right.$$
$$\left.+2\alpha(\sqrt{2}-1)\sum_{i=1}^{n}\gamma_i-2\alpha(\sqrt{2}-1)\right|$$

$$=\sqrt{2}\left(1-\sum_{i=1}^{n}\gamma_i\right)+2\alpha(\sqrt{2}-1)+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right\}$$
$$-\left|[1-2\alpha(\sqrt{2}-1)]\left(1-\sum_{i=1}^{n}\gamma_i\right)+\sum_{i=1}^{n}\gamma_i\left[1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right.\right.$$
$$\left.\left.-2\alpha(\sqrt{2}-1)\right]\right|$$

$$\geq\sqrt{2}\left(1-\sum_{i=1}^{n}\gamma_i\right)+2\alpha(\sqrt{2}-1)+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right\}$$
$$-\sum_{i=1}^{n}\gamma_i\left|1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right|-|1-2\alpha(\sqrt{2}-1)|\left(1-\sum_{i=1}^{n}\gamma_i\right)$$

$$= \sum_{i=1}^{n} \gamma_i \Bigg\{ \Re\sqrt{2} \left[1 - \frac{1}{b} \left(\frac{z(\mathcal{L}_s^r[\alpha_1, q] f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q] f_i(z)} + 1 \right) \right] + 2\alpha(\sqrt{2} - 1)$$

$$- \left| 1 - \frac{1}{b} \left(\frac{z(\mathcal{L}_s^r[\alpha_1, q] f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q] f_i(z)} + 1 \right) - 2\alpha(\sqrt{2} - 1) \right| \Bigg\} + \sqrt{2} \left(1 - \sum_{i=1}^{n} \gamma_i \right)$$

$$+ 2\alpha(\sqrt{2} - 1) - 2\alpha(\sqrt{2} - 1) \sum_{i=1}^{n} \gamma_i - |1 - 2\alpha(\sqrt{2} - 1)| \left(1 - \sum_{i=1}^{n} \gamma_i \right)$$

$$> \left[\sqrt{2} + 2\alpha(\sqrt{2} - 1) - |1 - 2\alpha(\sqrt{2} - 1)| \right] \left(1 - \sum_{i=1}^{n} \gamma_i \right)$$

$$> \left(1 - \sum_{i=1}^{n} \gamma_i \right) \min\left\{ (\sqrt{2} - 1)(1 + 4\alpha), \sqrt{2} + 1 \right\} \geq 0.$$

This completes the proof. □

Acknowledgements. The work is supported by GUP-2013-004.

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