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The minimum number of critical points of circular Morse functions

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. The minimum number of critical points for circular Morse functions on closed connected surfaces has been computed by the authors in [4]. Some bounds for the minimum characteristic number of closed connected orientable surfaces embedded in the first Heisenberg group with respect to its horizontal distribution are also given by [4]. In this paper we provide a more elementary proof for the minimum number of critical points of circular Morse functions and the details for the bounds on the mentioned minimum characteristic number.

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1. Introduction

In this paper we show that the circular Morse-Smale characteristic of a closed connected surface Σ is, except for the projective plane, the absolute value $|\chi(\Sigma)|$ of its Euler-Poincaré characteristic.

Definition 1.1. If M is a differential manifold, then the *circular Morse-Smale characteristic* of M is defined by

$$\gamma_{S^1}(M) := \min\{\text{card}(C(f)) : f \in \mathcal{F}(M, S^1)\}, \tag{1.1}$$

where $\mathcal{F}(M, S^1)$ stands for the set of all circular Morse functions $f : M \to S^1$.

Note that the *Morse-Smale characteristic* of a manifold M is defined by

$$\gamma(M) = \min\{\text{card}(C(f)) : f \in \mathfrak{F}(M)\},$$

where $\mathfrak{F}(M)$ denotes the set of all real-valued Morse functions defined on M, and it was studied by Andrica in [1, pp.106-129]. The *circular Morse-Smale characteristic* was defined by Andrica and Mangra [2, 3].

Proposition 1.1. ([4]) *If $\widetilde{M}$ is a k-fold cover of M, then $\gamma_{S^1}(\widetilde{M}) \le k \cdot \gamma_{S^1}(M)$.*

Constructing a circular Morse function on the closed connected orientable surface Σ_g, of genus g, with exactly $2(g-1)$ critical points is part of the strategy to compute the circular Morse-Smale characteristic of the surface Σ_g. We achieve this goal by producing a suitable embedding of Σ_g in $\mathbb{R}^3 \setminus Oz$, where Oz stands for the z-axis $\{(x,0,0) : x \in \mathbb{R}\}$, alongside a submersion $f : \mathbb{R}^3 \setminus Oz \longrightarrow S^1$, whose restriction $f\big|_{\Sigma_g}$ is a circular Morse function with exactly $2(g-1)$ critical points. In fact the suitable submersion is

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}(x,y,0). \tag{1.2}$$

In this respect we need to characterize somehow the critical points of such a restriction.

Proposition 1.2. *Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface and $f : \mathbb{R}^3 \longrightarrow N$ be a submersion, where N is either the real line or the circle S^1. The point $p = (x_0, y_0, z_0) \in \Sigma$ is critical for the restriction $f|_\Sigma$ if and only if the tangent plane of Σ at p is the tangent plane at p to the fiber $\mathcal{F}_p := f^{-1}(f(p))$ of the submersion (1.2) through p.*

Proposition 1.2 follows from the following more general statement.

Proposition 1.3. *Let M^m, N^n, P^p, $m \geq n > p$ be differential manifolds, let $f : M \to N$ be a differential map and $g : N \to P$ be a submersion. Then $x \in M$ is a regular point of $g \circ f$ if and only if $f \pitchfork_x \mathcal{F}_x$, where $\mathcal{F}_x$ stands for the fiber $g^{-1}(g(x))$ of g through x.*

Proof. Recall that we have the transversality property $f \pitchfork_x \mathcal{F}_x$ if and only if $\mathrm{Im}(df)_x + \ker(dg)_{f(x)} = T_{f(x)}(N)$, i.e. $\mathrm{Im}(df)_x + \ker(dg)_{f(x)} = T_{f(x)}(N)$, as $T_{f(x)}(\mathcal{F}_x) = \ker(dg)_{f(x)}$.

Assume that $x \in R(g \circ f)$, i.e. $\mathrm{Im}d(g \circ f)_x = T_{(g\circ f)(x)}(N)$. We only need to show that $T_{f(x)}(N) \subseteq \mathrm{Im}(df)_x + \ker(dg)_{f(x)}$, as the opposite inclusion is obvious. Consider $v \in T_{f(x)}(N)$ and observe that there exists $u \in T_x(M)$ such that $(dg)_{f(x)}(v) = d(g \circ f)_x(u)$, since $\mathrm{Im}[d(g \circ f)_x] = T_{(g\circ f)(x)}(N)$. Consequently we obtain successively:

$$\begin{aligned}
(dg)_{f(x)}(v) = d(g\circ f)_x(u) \ &\Leftrightarrow (dg)_{f(x)}(v) = (dg)_{f(x)}\left((df_x)(u)\right)\\
&\Leftrightarrow (dg)_{f(x)}(v) - (dg)_{f(x)}\left((df_x)(u)\right) = 0\\
&\Leftrightarrow (dg)_{f(x)}\left(v - (df_x)(u)\right) = 0\\
&\Leftrightarrow v - (df_x)(u) \in \ker(dg)_{f(x)}\\
&\Leftrightarrow v \in (df_x)(u) + \ker(dg)_{f(x)} \subseteq \mathrm{Im}(df)_x + \ker(dg)_{f(x)}.
\end{aligned}$$

In order to prove the opposite inclusion, we use the property of g to be a submersion and observe that we have successively:

$$\begin{aligned}
\mathrm{Im}(df)_x + \ker(dg)_{f(x)} = T_{f(x)}(N) \quad &\Rightarrow\\
(dg)_{f(x)}\left[\mathrm{Im}(df)_x + \ker(dg)_{f(x)}\right] = (dg)_{f(x)}\left[T_{f(x)}(N)\right] \quad &\Leftrightarrow\\
(dg)_{f(x)}\left[\mathrm{Im}(df)_x\right] + (dg)_{f(x)}\left[\ker(dg)_{f(x)}\right] = T_{g(f(x))}(N) \quad &\Leftrightarrow\\
(dg)_{f(x)}\left[\mathrm{Im}(df)_x\right] = T_{(g\circ f)(x)}(N) \quad &\Leftrightarrow\\
Im\left((dg)_{f(x)} \circ df)_x\right) = T_{(g\circ f)(x)}(N) \quad &\Leftrightarrow\\
Im\left(d(g\circ f)_x\right) = T_{(g\circ f)(x)}(N) \quad &\Leftrightarrow \quad x \in R(g\circ f).
\end{aligned}$$

$\square$

2. The circular Morse-Smale characteristic of closed surfaces

According to [4, Corollary 1.3], $\gamma_{S^1}(S^2) = \gamma(S^2) = 2$ and $\gamma_{S^1}(\mathbb{RP}^2) = \gamma(\mathbb{RP}^2) = 3$. Also $\gamma_{S^1}(\Sigma_1) = \gamma_{S^1}(T^2) = 0$, as the projection $T^2 = S^1 \times S^1 \to S^1$ is a submersion and it has no critical points. More generally, we shall prove the following:

Theorem 2.1. *The circular Morse-Smale characteristic of a closed surface* $\Sigma \neq \mathbb{RP}^2$ *is*

$$\gamma_{S^1}(\Sigma) = |\chi(\Sigma)| \tag{2.1}$$

2.1. The case of the closed orientable surfaces

In this case we only need to prove Theorem 2.1 for the compact orientable surface Σ_g of genus $g \geq 1$, as it is obvious for $\Sigma = S^2$ (see [4, Corollary 1.3]). In this respect we need:

1. to show that $\mu(F) := \mu_0(F) + \mu_1(F) + \mu_2(F) \geq 2(g-1)$ for every circular Morse function $F : \Sigma_g \longrightarrow S^1$, where $\mu_j(F)$ stands for the number of critical of index j of F and $\mu(F)$ for the total number $\operatorname{card}(C(F))$ of critical points of F;
2. to produce a circular Morse function on Σ_g with exactly $2(g-1)$ critical points.

In order to do so, we first observe that

$$2 - 2g = \mu_0(F) - \mu_1(F) + \mu_2(F). \tag{2.2}$$

Indeed, by using the Poincaré-Hopf Theorem one obtains

$$2 - 2g = \chi(\Sigma_g) = \sum_{p \in C(F)} \operatorname{ind}_p(\nabla F),$$

where ∇F is the gradient vector field of F with respect to some Riemann metric on Σ_g. To finish the proof of relation 2.2, we just need to observe that the index of the gradient vector field ∇F at a critical point of index one is -1 and the index of ∇F at the critical points of index zero and two is 1. Indeed the local behavior of F around the critical points of index one is $F = x^2 - y^2$ and its gradient behaves locally around such a point like the vector field $(x, -y)$. The degree of its normalized restriction to the circle S^1 is -1 as the normalized restriction is a diffeomorphism which reverses the orientation. Similarly, the index of ∇F at a critical point of index zero or two is one as the local behavior of F around such a critical point is $F = x^2 + y^2$ or $F = -x^2 - y^2$ and its gradient behaves locally around such a point like the vector field (x, y) or $(-x, -y)$ respectively. The normalized restrictions of these vector fields to the circle S^1 are diffeomorphisms preserving the orientation and their degree is therefore one. Thus, the relation (2.2) is now completely proved via the Poincaré-Hopf Theorem.

For the second item of the above observation we prove the following

Lemma 2.2. *The surface* Σ_g *can be suitably embedded into the three dimensional space* $\mathbb{R}^3 \setminus Oz$ *such that the restriction* $f\big|_{\Sigma_g} : \Sigma_g \longrightarrow S^1$ *is a circular Morse function with exactly* $2(g-1)$ *critical points, where* $f : \mathbb{R}^3 \setminus Oz \longrightarrow S^1$ *is the submersion given by*

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0).$$

2.1.1. The embedding of Σ_g into $\mathbb{R}^3 \setminus Oz$. Recall that $\Sigma_1 = T^2 = S^1 \times S^1$ is being usually identified with the surface of revolution in $\mathbb{R}^3$ obtained by rotating a circle in the plane xOz centered at a point on the x-axis around the z-axis. The radius of the circle is supposed to be strictly smaller than the distance from the origin to its center. A certain embedding of the surface Σ_g in $\mathbb{R}^3$, obtained from the one of Σ_1 on which we perform some surgery, will be useful in our approach. However the above mentioned embedding of Σ_1 in $\mathbb{R}^3$ has one circle on '*its top*' and one circle on '*its bottom*', where the Gauss curvature vanishes. The two circles form the critical set of the height function $f_{\vec{k}}$ in the direction of the z-axis, on the embedded copy of T^2 in $\mathbb{R}^3$. Thus, this height function is not a Morse function.

In order to construct our suitable embedding of Σ_g we need to rotate around the z-axis a closed convex curve of nonconstant curvature with a unique center of symmetry, on the x-axis, which lies in the plane xOz and has no overlaps with the z-axis, rather than a circle with the same properties except the requirement on the curvature. This curve is also required to contain two segments mutually symmetric with respect to the x-axis, one on '*its top*' and the other on '*its bottom*'. These two segments form the critical set of the height function $f_{\vec{k}}$ restricted to the curve itself.

Instead of rotating a circle within the plane xOz, we consider the embedding of Σ_1 obtained by rotating, around the z-axis a closed convex curve described above. The

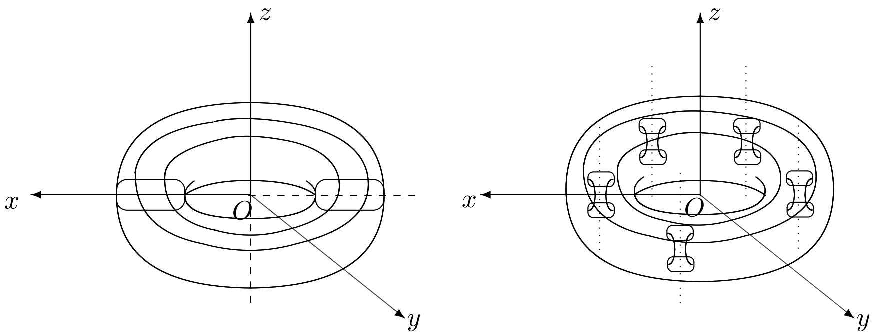

FIGURE 1. An embedded copy of Σ_6 constructed out of an embedded copy of Σ_1

obtained copy of Σ_1 is flat on the two annuli $\mathcal{A}$ and $\mathcal{A}'$ generated by the two symmetric segments of the generating curve, which lie in two horizontal parallel planes. Consider the points $p_1, \dots p_{g-1} \in \mathcal{A}$ and $q_1, \dots, q_{g-1} \in \mathcal{A}'$ such that the lines p_iq_i, $i = 1, \dots, g-1$ are vertical, i.e. parallel to the z-axis. In order to obtain a topological copy of the surface Σ_g we next remove some small open discs $D_1, \dots, D_{g-1} \subseteq \mathcal{A}$ centered at $p_1, \dots p_{g-1}$ and $D'_1, \dots, D'_{g-1} \subseteq \mathcal{A}'$ centered at $q_1, \dots, q_{g-1}$ respectively. The radii of the disks D_i and D'_i are supposed to be the same. We next consider suitable planar curves

$$\gamma_i : [0,1] \longrightarrow \mathrm{cl}\,(\mathcal{B}) \cap \pi_i,\ i = 1, \dots, g-1$$

such that $\gamma_i(0) \in \partial D_i$ and $\gamma_i(1) \in \partial D_i'$, where$p_i q_i \cap xOy = \{(x_i, y_i, 0)\}$, π_i is the plane parallel to xOz through the point $(x_i, y_i, 0)$ (i.e. $\pi_i : y = y_i$) and $\mathcal{B}$ is the bounded component of the complement of the embedded copy of Σ_1. The curves γ_i are chosen in such a way to complete, by their rotation around the axes $p_i q_i$, the embedded copy of $\Sigma_1 \setminus [D_1 \cup \ldots \cup D_{g-1} \cup D_1' \cup \ldots \cup D_{g-1}']$ up to a smooth embedded copy of Σ_g.

2.1.2. The cardinality of the set $C\big(f\big|_{\Sigma_g}\big)$ and the nondegeneracy of its points. Since our embedded copy of Σ_g is constructed out of several surfaces of revolutions, we are going to investigate the critical set of the restriction of the submersion (1.2) to such a surface, by using the geometric interpretation coming from Proposition 1.2.

Proposition 2.3. card $\big(C(f|_{\Sigma_g})\big) = 2(g-1)$.

Proof. Every surface of revolution Σ around a vertical line of equations $x = x_0$, $y = y_0$ can be parametrized as follows:

$$\begin{cases} x = x_0 + \alpha(v) \cos u \\ y = y_0 + \alpha(v) \sin u \quad u \in (0, 2\pi), v \in [0, 1]. \\ z = \beta(v) \end{cases}$$

In our considerations the function α is supposed to be strictly positive. Recall that a point $p(u,v) = (x(u,v), y(u,v), z(u,v))$ is, according to Proposition 1.3, critical for the restriction $f|_\Sigma$ if and only if the tangent plane of Σ at $p(u,v)$ contains the fiber of f through $p(u,v)$, i.e. its equation is $y(u,v)x = x(u,v)y$. On the other hand the equation of the tangent plane of Σ at $p(u,v)$ is

$$\begin{aligned} (x - x(u,v))\alpha(v)\beta'(v) \cos u + (y - y(u,v))\alpha(v)\beta'(v) \sin u& \\ - \alpha(v)\alpha'(v)(z - z(u,v)) = 0& \end{aligned} \tag{2.3}$$

The two planes are equal, i.e. $p(u,v) \in C\left(f|_\Sigma\right)$, if and only if

$$\begin{cases} \alpha(v)\beta'(v) \cos u \cdot x(u,v) + \alpha(v)\beta'(v) \sin u \cdot y(u,v) = 0 \\ \alpha(v)\alpha'(v) = 0, \end{cases}$$

or equivalently

$$\begin{cases} x_0 \cos u + y_0 \sin u + \alpha(v) = 0 \\ \alpha'(v) = 0. \end{cases} \tag{2.4}$$

The equation $x_0 \cos u + y_0 \sin u + \alpha(v) = 0$ is equivalent to $\cos(u - \alpha) = -\frac{\alpha(v)}{x_0} \cos x$, and has two solutions on the interval $(-x, 2\pi - x)$, where $\tan x = \frac{y_0}{x_0}$ and x_0 is assumed to be nonzero. Since $\cos x = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}$, the condition $|\frac{\alpha(v)}{x_0} \cos x| < 1$ is equivalent to $\alpha(v) < \sqrt{x_0^2 + y_0^2}$. Up to now we use the fact that $x_0^2 + y_0^2 > 0$ several times. Note that for $x_0 = y_0 = 0$ the restriction $f|_\Sigma$ has no critical points at all, as the first equation of the system (2.4) has no solutions in such a case. In particular the restriction $f|_{\Sigma_1}$ has no critical points at all as the embedded copy of Σ_1 is a surface of revolution around the z-axis, i.e. $x_0 = y_0 = 0$.

We now recall that $p_i q_i \cap xOy = (x_i, y_i, 0)$ and choose

$$\gamma_i : [0, 1] \longrightarrow \mathrm{cl}\,(\mathcal{B}) \cap \pi_i, \ \gamma_i(t) = (\alpha_i(t), y_i, \beta_i(t)).$$

such that $\alpha_i(0) = \alpha_i(1)$, the equations $\alpha_i'(v) = 0$ has one solution in $(0,1)$, $\alpha_i'' > 0$ and $\lim_{v\to 0} \beta_i'(v) = -\infty$, $\lim_{v\to 1} g_i'(v) = +\infty$. With such choices of the functions α_i and β_i, the revolution surfaces of the curves γ_i around the axes p_iq_i completes the surface $\Sigma_1 \setminus [D_1 \cup \ldots \cup D_{g-1} \cup D_1' \cup \ldots \cup D_{g-1}']$ up to a smooth embedded copy of Σ_g. Moreover the restriction of f to each of these revolution surfaces has exactly two critical points. Thus, the restriction $f|_{\Sigma_g}$ has precisely $2(g-1)$ critical points. □

Proposition 2.4. *The restriction $f|_{\Sigma_g}$ is a circular Morse function, i.e. its critical points are nondegenerated. Moreover the critical points of $f|_{\Sigma_g}$ have all index* 1.

Proof. The local representations of the restriction $f|_{\Sigma_g}$ have one of the following form:

$$\varphi(u,v) = x_0 + \alpha(v)\cos u \text{ or } \psi(u,v) = y_0 + \alpha(v)\sin u.$$

The nodegeneracy of a critical point (u_0, v_0), via the local representations φ or ψ, is quite obvious as $\det\left(\mathrm{Hess}_{(u_0,v_0)}\varphi\right)$ or $\det\left(\mathrm{Hess}_{(u_0,v_0)}\psi\right)$ is either

$$-\alpha(v_0)\alpha''(v_0)\cos^2 u_0 - (\alpha'(v_0))^2 \sin^2 u_0 < 0$$

or

$$-\alpha(v_0)\alpha''(v_0)\sin^2 u_0 - (\alpha'(v_0))^2 \cos^2 u_0 < 0$$

respectively.

Thus the critical point (u_0, v_0) of the local representation φ or ψ of the restriction $f|_{\Sigma_g}$ is, indeed, non-degenerate of index one. □

Proof of Theorem 2.1 *in the orientable case.* We only need to treat the case $g \geq 2$ as for $g \in \{0,1\}$ we obviously have $\gamma_{S^1}(\Sigma_0) = \gamma_{S^1}(S^2) = \gamma(S^2) = 2$ and $\gamma_{S^1}(\Sigma_1) = \gamma_{S^1}(T^2) = 0$. For the inequality $\gamma_{S^1}(\Sigma_g) \geq 2(g-1)$ we just need to use relation (2.2) that is $2 - 2g = \mu_0(F) - \mu_1(F) + \mu_2(F) \geq -\mu_1(F)$, for every circular Morse function $F : \Sigma_g \longrightarrow S^1$. This shows that $2(g-1) \leq \mu_1(F) \leq \mu_0(F) + \mu_1(F) + \mu_2(F) = \mu(F)$, for every circular Morse function $F : \Sigma_g \longrightarrow S^1$, and the inequality $2(g-1) \leq \gamma_{S^1}(\Sigma_g)$ therefore. The opposite inequality is proved by the existence of the circular Morse function $f\big|_{\Sigma_g}$ which has exactly $2(g-1)$ critical points.

Remark 2.5. No real valued Morse function defined on a compact manifold M^m $(m \geq 2)$ can merely have critical points of index one, as the global minimum of such a function has index zero and its global maximum has index m. Thus the restriction $f|_{\Sigma_g}$ cannot be lifted to any map $\widetilde{f} : \Sigma_g \longrightarrow \mathbb{R}$, i.e. $\exp \circ \widetilde{f} = f$ and the induced group homomorphism $f_* : \pi(\Sigma_g) \longrightarrow \mathbb{Z} = \pi(S^1)$ is nontrivial therefore.

Proof of Theorem 2.1 *in the non-orientable case.* In this case we rely on Proposition 1.1 in order to prove the inequality

$$\gamma_{S^1}\left(g\mathbb{RP}^2\right) \geq |\chi\left(g\mathbb{RP}^2\right)|,$$

for $g \geq 2$, where $k\mathbb{RP}^2$ stands for the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ of k copies of the projective plane. Indeed, by applying Proposition 1.1 to the orientable double cover

$$\Sigma_{g-1} \to g\mathbb{RP}^2$$

we obtain successively:

$$\begin{aligned}\gamma_{S^1}(g\mathbb{RP}^2) &\geq \tfrac{1}{2}\gamma_{S^1}(\Sigma_{g-1}) = \tfrac{1}{2}|\chi(\Sigma_{g-1})| \\ &= \tfrac{1}{2}|2-2(g-1)| = |2-g| = |\chi(g\mathbb{RP}^2)|.\end{aligned}$$

For the opposite inequality we first recall that

$$f:\mathbb{RP}^2 \longrightarrow \mathbb{R},\ f([x_1,x_2,x_3]) = \frac{x_1^2+2x_2^2+3x_3^2}{x_1^2+x_2^2+x_3^2},$$

is a perfect Morse function with exactly three critical points of indices $0,1,2$, i.e a minimum point p, a maximum point q and a saddle point s. If $\varepsilon > 0$ is small enough, then the inverse images $D := f^{-1}(-\infty, f_2(p)+\varepsilon)$ and $D' := f^{-1}(f(q)-\varepsilon, \infty)$ are open disks and the inverse image $f^{-1}[f(p)+\varepsilon, f(q)-\varepsilon] = \mathbb{RP}^2 \setminus (D_1 \cup D_2)$ is a compact surface with two circular boundary components $f^{-1}(f(p)+\varepsilon)$ and $f^{-1}(f(q)-\varepsilon)$. Observe that the restriction

$$f\big|_{\mathbb{RP}^2\setminus(D\cup D')} : \mathbb{RP}^2 \setminus (D_1 \cup D_2) \longrightarrow [f(p)+\varepsilon, f(q)-\varepsilon]$$

has one critical point of index one, i.e the saddle point s. We next glue successively g copies of $\mathbb{RP}^2 \setminus (D \cup D')$, say

$$M_1 := \mathbb{RP}^2 \setminus (D_1 \cup D_1'), \ldots, M_g := \mathbb{RP}^2 \setminus (D_g \cup D_g'),$$

along the circular boundaries

$$\partial D_i' := f_i^{-1}(f_i(q)-\varepsilon) \subset M_i \text{ and } \partial D_{i+1} := f_{i+1}^{-1}(f_{i+1}(p)+\varepsilon) \subset M_{i+1}$$

of

$$D_i' := f_i^{-1}(f_i(q)-\varepsilon, \infty) \text{ and } D_{i+1} := f_{i+1}^{-1}(-\infty, f_{i+1}(p)+\varepsilon),$$

where

$$f_i := f + iL : \mathbb{RP}^2 \longrightarrow \mathbb{R}, \quad (i = 1, \ldots, g-1)$$

and

$$L := \operatorname{length}([f(p)+\varepsilon, f(q)-\varepsilon]) = f(q) - f(p) - 2\varepsilon.$$

The obtained surface is $g\mathbb{RP}^2 \setminus (D_1 \cup D_g')$. Note that f_i is a Morse function with one saddle point which is constant on each of the circular boundaries $\partial D_i = f_i^{-1}(f_i(p)+\varepsilon)$ and $\partial D_i' = f_i^{-1}(f_i(q)-\varepsilon)$ of M_i. Moreover, the equalities $f_i\big|_{\partial D_i'} = f_{i+1}\big|_{\partial D_{i+1}}$ hold for every $i = 1, \ldots, g-1$, which shows that the function

$$F : g\mathbb{RP}^2 \setminus (D_1 \cup D_g') \longrightarrow \mathbb{R},\ F\big|_{M_i} := f_i$$

is well defined. In fact, F is a Morse function with g saddle points which is constant on the circle boundaries

$$\partial D_1 = f_1^{-1}(f_1(p)+\varepsilon) \subset M_1 \text{ and } \partial D_g' = f_g^{-1}(f_g(q)-\varepsilon) \subset M_g.$$

Identifying the circle boundaries ∂D_1 and $\partial D_g'$ of $g\mathbb{RP}^2 \setminus (D_1 \cup D_g')$, via a suitable diffeomorphism $\varphi : \partial D_1 \longrightarrow \partial D_g'$, we get the non-orientable surface $(g+2)\mathbb{RP}^2$. Identifying $\min F$ with $\max F$ in $\operatorname{Im}(F)$ we obtain the circle S^1. Also, the Morse function

$$g\mathbb{RP}^2 \setminus (D_1 \cup D_g') \longrightarrow \operatorname{Im}(F),\ x \mapsto F(x)$$

descends to a circular Morse function

$$f_0 : (g+2)\mathbb{RP}^2 = g\mathbb{RP}^2 \setminus (D_1 \cup D'_g)\Big/\{x = \varphi(x)\} \to S^1 = \mathrm{Im}(F)\Big/\{\min F = \max F\}$$

with g saddle points. This shows that the inequality $\gamma_{S^1}\left((g+2)\mathbb{RP}^2\right) \leq g$ holds for all $g \geq 1$.

Therefore, we provided the second proof of Theorem 2.1 in the non-orientable cases $g\mathbb{RP}^2$ with $g \geq 3$. On the other hand the Klein Bottle $2\mathbb{RP}^2$ is a fibration over S^1 with fiber S^1, which shows that $\gamma_{S^1}\left(2\mathbb{RP}^2\right) = 0 = |\chi(2\mathbb{RP}^2)|$. □

3. On the number of characteristic points

The horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$ is $\mathcal{H} = \mathrm{span}(X, Y) = \{\mathcal{H}_p := \mathrm{span}(X_p, Y_p)\}_{p \in \mathbb{H}^1}$, where $X = \partial_x + 2y_i\partial_t$ and $Y = \partial_y - 2x\partial_t$. Let us consider a surface $S \subseteq \mathbb{R}^3$ which is C^1 smooth. The *characteristic set* [5, 6] of S with respect to $\mathcal{H}$ is defined as

$$C(S, \mathcal{H}) := \{p \in S \,:\, T_pS = \mathcal{H}_p\}.$$

Definition 3.1. *If S is a C^1 smooth surface which can be embedded into $\mathbb{R}^3$, then the* minimum characteristic number *of S relative to $\mathcal{H}$ on $\mathbb{R}^3$ is defined as*

$$mcn(S, \mathcal{H}) := \min\{\mathrm{card}\,(C(f(S), \mathcal{H})) : f \in \mathrm{Embed}(S, \mathbb{R}^3)\},$$

where $\mathrm{Embed}(S, \mathbb{R}^3)$ *stands for the set of all embeddings of S into $\mathbb{R}^3$.*

Theorem 3.2. *If $g \geq 2$, then $2g - 2 \leq mcn(\Sigma_g, \mathcal{H}) \leq 4g - 4$.*

For the lower bound $2g - 2$ of $mcn(\Sigma_g, \mathcal{H})$ we refer the reader to [4] and for the upper bound $4g - 4$ we need to construct an embedding of Σ_g in $\mathbb{R}^3$ with $4g - 4$ characteristic points with respect to the horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$. In this respect we shall use the possibility to embed Σ_1 in $\mathbb{R}^3$ as a revolution surface and construct a suitable embedding of Σ_g out of Σ_1 by performing some surgery on Σ_1. The handles we plan to glue are surfaces of revolution as well. In fact, we shall use the embedding of Σ_g described in the previous section. Therefore we need to investigate the size of the characteristic sets of revolution surfaces $S \subset \mathbb{R}^3$ with respect to the horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$.

3.1. Revolution surfaces in $\mathbb{H}^1$ with low number of horizontal points

Every revolution surface S obtained by rotating a plane curve $x = \alpha(v), z = v$ $(\alpha > 0)$ around the vertical line $x = x_0, y = y_0$ admits a local parametrization of type

$$\begin{array}{l} x = x_0 + \alpha(v)\cos u \\ y = y_0 + \alpha(v)\sin u \\ z = v \end{array} \;,\; u \in I, v \in J,$$

where I is an open interval of length 2π and J will be symmetric with respect to the origin, i.e. $J = (-a, a)$. The function f is subject to the following requirements:

$$\alpha \text{ is bounded },\; \alpha'' > 0 \text{ and } \lim_{v \to \pm a} \alpha'(v) = \pm\infty. \tag{3.1}$$

The vector equation of our revolution surface is

$$\vec{r} = (x_0 + \alpha(v)\cos u)\partial_x + (x_0 + \alpha(v)\sin u)\partial_y + v\partial_t$$

and

$$\begin{array}{rcl} \vec{r}_u & = & -(\alpha(v)\sin u)\partial_x + (\alpha(v)\cos u)\partial_y \\ \vec{r}_v & = & (\alpha'(v)\cos u)\partial_x + (\alpha'(v)\sin u)\partial_y + \partial_t \\ \vec{r}_u \wedge \vec{r}_v & = & (\alpha(v)\cos u)\partial_x + (\alpha(v)\sin u)\partial_y - \alpha(v)\alpha'(v)\partial_t. \end{array}$$

On the other hand the horizontal vector fields of the distribution $\mathcal{H}$ are $X = \partial_x + 2y\partial_t$, $Y = \partial_y - 2x\partial_t$ and their vector product is

$$X \wedge Y = -2y\partial_x + 2x\partial_y + \partial_t.$$

Thus, the point $r(u,v) := (x(u,v), y(u,v), z(u,v)) \in S$ is a horizontal point if and only if the vectors $\vec{r}_u \wedge \vec{r}_v, X \wedge Y$ are linearly dependent at $r(u,v)$, i.e.

$$\begin{array}{rcrcl} \sin u & + & 2\alpha(v)\alpha'(v)\cos u & = & -2x_0\alpha'(v) \\ 2\alpha(v)\alpha'(v)\sin u & - & \cos u & = & -2y_0\alpha'(v). \end{array}$$

Thus

$$\begin{array}{rcl} \sin u & = & -2\alpha'(v)\dfrac{x_0 + 2y_0\alpha(v)\alpha'(v)}{1 + 4\alpha^2(v)(\alpha'(v))^2} \\ \cos u & = & -2\alpha'(v)\dfrac{2x_0\alpha(v)\alpha'(v) - y_0}{1 + 4\alpha^2(v)(\alpha'(v))^2}. \end{array} \tag{3.2}$$

Remark 3.3. *No revolution surface around the z-axis has $\mathcal{H}$-tangency points, as the equations (3.2) have no solutions at all for $x_0 = y_0 = 0$.*

The identity $\sin^2 u + \cos^2 u = 1$ leads us to the equation

$$(\alpha'(v))^2 = \frac{1}{4\left(||(x_0, y_0)||^2 - \alpha^2(v)\right)}, \tag{3.3}$$

which has at least two solutions on the interval $J = (-a, a)$, as the right hand side of (3.3) is bounded and $(\alpha')^2$ covers the positive real half line $[0, \infty)$ twice, once on the interval $(-a, o]$ and once on the interval $[0, a)$. For suitable choices of the function α, the equation (3.3) has precisely two solutions. Such a choice is

$$\alpha(v) = 2 - \sqrt{\frac{2 - v^2}{2}} \tag{3.4}$$

for $a = \sqrt{2}$ and $||(x_0, y_0)|| = 3$. Indeed, for the choice (3.4) of the function α the equation (3.3) becomes:

$$4v^2\sqrt{2(2 - v^2)} = -v^4 - 9v^2 + 2. \tag{3.5}$$

Note that the equation (3.5) has precisely two solutions, as can be easily checked.

Proof of Theorem 3.2. The closed convex curve in the plane xOz described at the beginning of the section (2.1.1) is supposed to have its unique center at the point $(3, 0, 0)$. The coordinates of the points p_i and q_i have the forms (x_i, y_i, z_i) and $(x_i, y_i, -z_i)$ respectively, for $i = 1, \ldots, g-1$. Moreover $||(x_i, y_i)||^2 := x_i^2 + y_i^2 = 3$

for all $i = 1, \ldots, g-1$. The handles we use within our surgery process are revolution surfaces around the vertical lines $x = x_i, y = y_i$ of parametrized equations

$$\begin{array}{l} x = x_i + \alpha(v)\cos u \\ y = y_i + \alpha(v)\sin u \\ z = v \end{array}, \ u \in I, v \in J,$$

We denote by v_i and v_i' the roots of the equations

$$(\alpha'(v))^2 = \frac{1}{4(||(x_i, y_i)||^2 - \alpha^2(v))}, \tag{3.6}$$

with the choice (3.4) for the function f. The equations which corresponds to (3.2)

$$\begin{cases} \sin u &= -2\alpha'(v_i)\dfrac{x_i + 2y_i\alpha(v_i)\alpha'(v_i)}{1 + 4\alpha^2(v_i)(\alpha'(v_i))^2} \\ \\ \cos u &= -2\alpha'(v_i)\dfrac{2x_i\alpha(v_i)\alpha'(v_i) - y_i}{1 + 4\alpha^2(v_i)(\alpha'(v_i))^2}, \end{cases} \tag{3.7}$$

$$\begin{cases} \sin u &= -2\alpha'(v_i')\dfrac{x_i + 2y_i\alpha(v_i')\alpha'(v_i')}{1 + 4\alpha^2(v_i')(\alpha'(v_i'))^2} \\ \\ \cos u &= -2\alpha'(v_i')\dfrac{2x_i\alpha(v_i')\alpha'(v_i') - y_i}{1 + 4\alpha^2(v_i')(\alpha'(v_i'))^2}. \end{cases} \tag{3.8}$$

Since the graphs of the sine and cosine functions on each interval of length 2π are intersected at most twice by any straight line parallel to the u-axis, it follows that the equations (3.7) as well as (3.8) have at most two roots for each $i = 1, \ldots, g-1$. On the other hand the surface Σ_g embedded in $\mathbb{H}^1$ the way described right after Theorem 3.2 has no other $\mathcal{H}$-characteristic points. Indeed, on the two annuli $\mathcal{A}$ and $\mathcal{A}'$ the tangent planes to Σ_g are parallel to the xOy plane, a parallelism relation which happens for the planes of the distribution $\mathcal{H}$ just along the z-axis. This shows that Σ_g, embedded in $\mathbb{R}^3 \setminus Oz$ as described before, has no extra characteristic points as two annuli have no common points with the z-axis. The remaining part of our embedded Σ_g is completely contained in Σ_1 which is, in its turn, a revolution surface around the z-axis and has no $\mathcal{H}$-tangency points, as we saw in Remark 3.3. Thus, our embedded surface Σ_g has at most $4(g-1)$ $\mathcal{H}$-tangency points. □

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