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The group of isometries of the French rail ways metric

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper we study the isometry groups $\text{Iso}_{d_{F,p}}(\mathbb{R}^n)$, where $d_{F,p}$ is the French rail ways metric given by (2.1). We derive some general properties of $\text{Iso}_{d_{F,p}}(\mathbb{R}^n)$ and we discuss the particular situation when $X = \mathbb{R}^n$ and $d = d_2$ is the Euclidean metric.

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1. Introduction

Let (X, d) be a metric space. The map $f : X \to X$ is called an *isometry* with respect to the metric d (or a *d-isometry*), if it is surjective and it preserves the distances. That is for any points $x, y \in X$ the relation $d(f(x), f(y)) = d(x, y)$ holds. From this relation it follows that f is injective, hence it is bijective. Denote by $\text{Iso}_d(X)$ the set of all isometries of the metric space (X, d). It is clear that $(\text{Iso}_d(X), \circ)$ is a subgroup of $(S(X), \circ)$, where $S(X)$ denotes the group of all bijective transformations $f : X \to X$. We will call $(\text{Iso}_d(X), \circ)$ the *group of isometries* of the metric space (X, d). A general, important and complicated problem is to described the group $(\text{Iso}_d(X), \circ)$. This problem was formulated in the paper [2] for metric spaces with a metric that is not given by a norm defined by an inner product.

Some results in direction to solve this problem for some particular metrics are the following. D.J. Schattschneider [17] found an elementary proof for the property that group $\text{Iso}_{d_1}(\mathbb{R}^2)$ is the semi-direct product of D_4 and $T(2)$, where d_1 is the "taxicab metric" defined by (1.2) (for $p = 1$ and $n = 2$) and D_4 and $T(2)$ are the symmetry group of the square and the translations group of $\mathbb{R}^2$, respectively. A similar result holds for the group $\text{Iso}_{d_1}(\mathbb{R}^3)$, i.e. this group is isomorphic to the semi-direct product of groups D_h and $T(3)$, where D_h is the symmetry group of the Euclidean octahedron and $T(3)$ is the translations group of $\mathbb{R}^3$. This was recently proved by O. Gelisgen,

R. Kaya [7]. In fact the "taxicab metric" generates many interesting non-Euclidean geometric properties (see the book of E.F. Krause [10] or the thesis of G. Chen [4]). Another result concerning the isometry group of the plane $\mathbb{R}^2$ with respect to the "Chinese checker metric" d_C, where

$$d_C(x,y) = \max\{|x^1 - x^2|, |y^1 - y^2|\} + (\sqrt{2}-1)\min\{|x^1 - x^2|, |y^1 - y^2|\}, \qquad (1.1)$$

was recently obtained by R. Kaya, O. Gelisgen, S. Ekmekci, A. Bayar [9].They have showed that this group is isomorphic to the semi-direct product the Dihedral group D_8, the Euclidean symmetry group of the regular octagon and $T(2)$.

In the paper [1] we have considered $X = \mathbb{R}^n$, and for any real number $p \geq 1$ the metric d_p defined by

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x^i - y^i|^p\right)^{1/p}, \qquad (1.2)$$

where $x = (x^1, ..., x^n), y = (y^1, ..., y^n) \in \mathbb{R}^n$. If $p = \infty$, then the metric d_∞ is defined by

$$d_\infty(x,y) = \max\{|x^1 - y^1|, ..., |x^n - y^n|\}. \qquad (1.3)$$

In the case $p = 2$, we get the well-known Euclidean metric on $\mathbb{R}^n$. In this case we have the Ulam's Theorem which states that $\mathrm{Iso}_{d_2}(\mathbb{R}^n)$ is isomorphic to the semi-direct product of orthogonal group $O(n)$ and $T(n)$, where $T(n)$ is the group of translations of $\mathbb{R}^n$ (see for instance the references [5], [16]). The situation $p \neq 2$ is very interesting. The main result of [1] consists in a complete description of the groups $\mathrm{Iso}_{d_p}(\mathbb{R}^n)$ for $p \geq 1, p \neq 2$, and $p = \infty$. We have proved that in case $p \neq 2$ all these groups are isomorphic and consequently, they are not depending on number p. The main ingredient in the proof of the above result is the Mazur-Ulam Theorem about the isometries between normed linear spaces (see the original reference [12], the monograph [6], the references [14], [15] for some extensions, and [18], [19] for new proofs).

In this paper we study some properties of the isometry group of the so called French railroad metric.

2. The main results

Let (X, d) be a metric space and let $f : X \to X$ be a map. The minimal displacement of f, denoted by $\lambda(f)$, is the greatest lower bound of the displacement function of f, that is

$$\lambda(f) = \inf_{x \in X} d(x, f(x)).$$

The minimal set of f, denoted by $\mathrm{Min}(f)$, is the subset of X defined as

$$\mathrm{Min}(f) = \{x \in X : d(x, f(x)) = \lambda(f)\}.$$

According to the monograph of A. Papadopoulos [13], we have the following general classification of the isometries of a metric space X in terms of the invariants $\lambda(f)$ and $\mathrm{Min}(f)$. Consider $f : X \to X$ to be an isometry. Then f is said to be

1. *Parabolic* if $\mathrm{Min}(f) = \Phi$.
2. *Elliptic* if $\mathrm{Min}(f) \neq \Phi$ and $\lambda(f) = 0$. Thus, f is elliptic if and only if $\mathrm{Fix}(f) \neq \Phi$.

3. *Hyperbolic* if $\mathrm{Min}(f) \neq \Phi$ and $\lambda(f) > 0$.

France is a centralized country: every train that goes from one French city to another has to pass through Paris. This is slightly exaggerated, but not too much. This motivates the name French railroad metric for the following construction. Let (X, d) be a metric space, and fix $p \in X$. Define a new metric $d_{F,p}$ on X by letting

$$d_{F,p}(x,y) = \begin{cases} 0, & \text{if and only if } x = y \\ d(x,p) + d(p,y), & \text{if } x \neq y \end{cases} \tag{2.1}$$

The French railroad metric generates a very poor geometry. In fact, the geometric properties are concentrated at the point p. For instance, consider X to be the plane $\mathbb{R}^2$, $p = O$, the origin, and d is the standard Euclidean metric d_2. Then for two distinct points A and B the perpendicular bisector of the segment $[AB]$ exists if and only if $AO = BO$. Similarly, the Menger segment $[AB]$ (see the monograph of W. Benz [3, p.43]) has a midpoint, if and only if $AO = BO$. In the case $AO \neq BO$, the segment $[AB]$ consists only in the points A and B.

The main purpose of this paper is to derive some general properties of the group $\mathrm{Iso}_{d_{F,p}}(X)$. We will give an equivalent property for $\mathrm{Iso}_{d_{F,0}}(\mathbb{R}^n)$, the isometry group of the space $\mathbb{R}^n$ with the fixed point the origin $p = 0$, and the Euclidean metric d_2, and we will discuss the complexity of an effective description in the cases $n = 1$ and $n = 2$.

Let $\mathrm{Iso}_d^{(p)}(X)$ be the subgroup of $\mathrm{Iso}_d(X)$ consisting in all isometries fixing the point p, i.e.

$$\mathrm{Iso}_d^{(p)}(X) = \{h \in \mathrm{Iso}_d(X) : h(p) = p\}.$$

Theorem 2.1. *The following relation* $\mathrm{Iso}_d^{(p)}(X) \subseteq \mathrm{Iso}_{d_{F,p}}(X)$ *holds, that is* $\mathrm{Iso}_d^{(p)}(X)$ *is a subgroup of* $\mathrm{Iso}_{d_{F,p}}(X)$.

Proof. Let $f \in \mathrm{Iso}_d^{(p)}(X)$. For every $x, y \in X, x \neq y$, we have

$$d_{F,p}(f(x), f(y)) = d(f(x), p) + d(p, f(y)) = d(f(x), f(p)) + d(f(p), f(y))$$
$$= d(x,p) + d(p,y) = d_{F,p}(x,y),$$

hence, f belongs to $\mathrm{Iso}_{d_{F,p}}(X)$. □

For $x \neq y$, the relation $d_{F,p}(f(x), f(y)) = d_{F,p}(x, y)$ is equivalent to

$$d(f(x), p) + d(p, f(y)) = d(x,p) + d(p,y). \tag{2.2}$$

Theorem 2.2. *For every isometry* $f \in \mathrm{Iso}_{F,p}(X)$*, the point* p *is fixed, that is* $f(p) = p$.

Proof. Let start to investigate the topology generated by metric $d_{F,p}$. Consider the ball of radius r and centered in the point C, that is the set

$$B_{F,p}(C; r) = \{x \in X : d_{F,p}(C, x) \leq r\}.$$

The inequality $d_{F,p}(C, x) \leq r$ is equivalent to $d(C,p) + d(p,x) \leq r$. If $C \neq p$, then we obtain $d(p,x) \leq r - d(C,p)$. This means that $B_{F,p}(C; r) = B_d(p; r - d(C,p)$ if $d(C,p) < r$ and $B_{F,p}(C; r) = \{C\}$ if $d(C,p) \geq r$. Therefore, the neighborhoods basis at the point p generated by $d_{F,p}$ coincides with the the neighborhoods basis at the point p generated by the metric d.

For every $x, y \in X, x \neq y$, we have $d_{F,p}(x, y) = d(x, p) + d(p, y) \geq d(x, y)$, hence

$$d_{F,p}(x, y) \geq d(x, y). \tag{2.3}$$

Because f is a $d_{F,p}$-isometry, from (2.3) it follows

$$d(f(x), f(y)) \leq d_{F,p}(f(x), f(y)) = d_{F,p}(x, y).$$

Take $y = p$ in relation (2.2) and obtain

$$d(f(x), p) + d(p, f(p)) = d(x, p). \tag{2.4}$$

Clearly, f is continuous in the topology generated by the metric $d_{F,p}$. According to the result from the beginning of the proof, we have $x \to p$ in the topology of $d_{F,p}$ if and only if $x \to p$ in the topology of d. That is $d_{F,p}(x, p) \to 0$ if and only if $d(x, p) \to 0$, hence the relation (2.4) implies $d(f(p), p) = 0$, hence $f(p) = p$. □

From the result in Theorem 2.2, it follows :

Corollary 2.3. *For every metric space (X, d) and for every point $p \in X$, the metric space $(X, d_{F,p})$ is of elliptic type, that is every its isometry is elliptic.*

3. Comments on the case $X = \mathbb{R}^n$ and $d = d_2$

Let consider $X = \mathbb{R}^n$ with the Euclidean metric $d = d_2$, and the point p to be the origin of $\mathbb{R}^n$. If f is an isometry with respect to the induced French railroad metric, then f satisfies relation (2.4) hence, for every $x, y \in \mathbb{R}^n$, we have

$$||f(x)|| + ||f(y)|| = ||x|| + ||y||. \tag{3.1}$$

According to Theorem 2.2, the origin of $\mathbb{R}^n$ is a fixed point of f, i.e. we have $f(0) = 0$. Therefore, the relation (3.1) is equivalent to

$$||f(x)|| = ||x||, \quad x \in \mathbb{R}^n. \tag{3.2}$$

Therefore, $f \in \mathrm{Iso}_{d_{F,0}}(\mathbb{R}^n)$ if and only if it is bijective and it satisfies the functional equation (3.2). The equation (3.2) ensures the continuity of f only at the point 0. For the sake of simplicity, let us denote by G_n the isometry group $\mathrm{Iso}_{d_{F,0}}(\mathbb{R}^n)$. If f is linear, then the equation (3.2) means that it is an orthogonal transformation of $\mathbb{R}^n$. It follows that the real orthogonal group $O(n, \mathbb{R})$ is a subgroup of G_n. The group $\{-1_{\mathbb{R}^n}, 1_{\mathbb{R}^n}\}$ is a normal subgroup of G_n.

Also, if we impose some smoothness conditions we obtain interesting situations. Denote by $G_n^k, k = 0, 1, \cdots, \infty$, the subgroup of G_n consisting in all $d_{F,0}$-isometries of class C^k. Clearly, we get

$$G_n^\infty \subset \cdots \subset G_n^k \subset \ldots \subset G_n^1 \subset G_n^0 \subset G_n.$$

On the other hand, if $f \in G_n^k$, then the restriction $f\,|_{S^{n-1}}$ is an element of the group $\mathrm{Diff}^k(S^{n-1})$ of the C^k- diffeomorphisms of the unity $(n-1)$-dimensional sphere of the space $\mathbb{R}^n$. As we expect, a bijective extension of a C^k- diffeomorphism $h : S^{n-1} \to S^{n-1}$ to the space $\mathbb{R}^n$, preserving the property to satisfy (3.2), is not unique. This means that it is very possible that the group G_n^k is larger than $\mathrm{Diff}^k(S^{n-1})$.

In the case $n = 1$, the group G_1 is defined by all bijective maps $f : \mathbb{R} \to \mathbb{R}$, continuous at 0, and satisfying the functional equation $|f(x)| = |x|, x \in \mathbb{R}$. The relation $|f(x)| = |x|$ is equivalent to $f^2(x) - x^2 = 0$, i.e. $(f(x) - x)(f(x) + x) = 0$. Therefore, the general possible form of the maps in G_1 is

$$f_A(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{R} \setminus A \\ -x & \text{if} \quad x \in A \end{cases}$$

where A is an arbitrary subset of $\mathbb{R}$. For all real number $a \neq 0$, the map

$$f_a(x) = \begin{cases} x & \text{if} \quad x \neq -a, a \\ -a & \text{if} \quad x = a \\ a & \text{if} \quad x = -a \end{cases}$$

belongs to G_1. Moreover, we have $f_a \circ f_a = 1_{\mathbb{R}}$ hence $\{1_{\mathbb{R}}, f_a\}$ is a subgroup of G_1. This means that the group G_1 has infinitely many subgroups isomorphic to $\mathbb{Z}_2$. Another subgroup isomorphic to $\mathbb{Z}_2$ is $G_1^{\infty} = \{-1_{\mathbb{R}}, 1_{\mathbb{R}}\}$, and it contains all smooth diffeomorphisms of the real line satisfying the functional equation $|f(x)| = |x|$, $x \in \mathbb{R}$. It is a normal subgroup of G_1. Clearly, $1_{\mathbb{R}}$ preserves the orientation of the 0-dimensional sphere $S^0 = \{-1, 1\}$, and $-1_{\mathbb{R}}$ reverses the orientation of $S^0 = \{-1, 1\}$.

In the case $n = 2$, we can identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and then the group G_2 is defined by all bijective maps $f : \mathbb{C} \to \mathbb{C}$, continuous at 0, and satisfying the functional equation $|f(z)| = |z|, z \in \mathbb{C}$. The orthogonal group $O(2, \mathbb{R})$ is the symmetry group of the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. It is isomorphic (as a real Lie group) to the circle group $(S^1, \cdot)$, also known as $U(1)$. In this case, the circle group $(S^1, \cdot)$ is a subgroup of G_2. Consequently, all subgroups of $(S^1, \cdot)$ are subgroups of G_2.

References

[1] Andrica, D., Bulgărean, V., *Some Remarks on the Group of Isometries of a Metric Space*, in "Nonlinear Analysis"(Stability, Approximation, and Inequalities), In honor of Th.M.Rassias on the occassion of his 60th birthday, P.M. Pardalos, P.G. Georgiev, H.M. Srivasava, Eds., Springer, 2012, 57-64.

[2] Andrica, D., Wiesler, H., *On the isometry groups of a metric space* (Romanian), Seminar "Didactica Matematicii", **5**(1989), 1-4.

[3] Benz, W., *Classical Geometries in Modern Contexts.Geometry of Real Inner Product Spaces*, Second Edition, Birkhäuser, Basel-Boston-Berlin, 2007.

[4] Chen, G., *Lines and Circles in Taxicab Geometry*, Master Thesis, Department of Mathematics and Computer Science, Central Missouri State University, 1992.

[5] Clayton, W.D., *Euclidean Geometry and Transformations*, Addison-Wesley, Reading, MA, 1972.

[6] Fleming, R.J., Jamison, J.E., *Isometries on Banach Spaces: Function Spaces*, Chapman Hall / CRC , Monographs and Surveys in Pure and Applied Mathematics No.129, Boca Raton, London, New York, Washington D.C., 2003.

[7] Gelisgen, O., Kaya, R., *The taxicab space group*, Acta Mathematica Hungarica, **122**(2009), No. 1-2, 187-200.

[8] Kaya, R., *Area Formula For Taxicab Triangles*, Pi Mu Epsilon, **12**(2006), 213-220.

[9] Kaya, R., Gelisgen, O., Ekmekci, S., Bayar, A., *On the group of the isometries of the plane with generalized absolute metric*, Rocky Mountain Journal of Mathematics, **39**(2009), no. 2.

[10] Krause, E.F., *Taxicab Geometry*, Addison-Wesley Publishing Company, Menlo Park, CA, 1975.

[11] Ozcan, M., Kaya, R., *Area of a Triangle in Terms of the Taxicab Distance*, Missouri J. of Math. Sci., **15**(2003), 178-185.

[12] Mazur, S., Ulam, S., *Sur les transformationes isometriques d'espaces vectoriels normes*, C. R. Acad. Sci. Paris, **194**(1932), 946-948.

[13] Papadopoulos, A., *Metric Spaces, Convexity and Nonpositive Curvature*, European Mathematical Society, 2005.

[14] Park, C.-G., Rassias, Th.M., *Isometries on linear n-normed spaces*, Journal of Inequalities in Pure and Applied Mathematics, **7**(5)(2006), Article 168, 7 pp.

[15] Park, C.-G., Rassias, Th.M., *Additive isometries on Banach spaces*, Nonlinear Functional Analysis and Applications, **11**(5)(2006),793-803.

[16] Richard, S.M., George, D.P., *Geometry. A Metric Approach with Models*, Springer-Verlag, New York, 1981.

[17] Schattschneider, D.J., *The Taxicab Group*, Amer. Math. Monthly, **91**(1984), 423-428.

[18] Väisälä, J., *A proof of the Mazur-Ulam theorem*, Amer. Math. Monthly, bf 110(7)(2003), 633-635.

[19] Vogt, A., *Maps which preserve equality of distance*, Studia Math., **45**(1973), 4348.

[20] Willard Jr., M., *Symmetry Groups and Their Applications*, Academic Press, New York, 1972.

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