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New aspects in the use of canonoid transformations

Mihai Boleanţu and Mircea Crasmareanu

To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. Canonoid transformations with respect to a locally Hamiltonian vector field are studied through the concept of generating function and the Helmholtz theory of the inverse problem. The case of dimension two is connected with the Liouville equation. The use of such transformations for determining first integrals is illustrated with two examples: the Whittaker system (in dimension four) and the damped harmonic oscillator (in the dimension two).

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1. Introduction

The theory of *canonoid transformations* is, by now, a well-known approach in geometrical dynamics. Introduced by Saletan in his famous book [20] as a generalization of classical notion of *canonical transformation*, the concept of canonoid diffeomorphism has its roots in the work of Sophus Lie [16] as it is pointed out by P. Havas in the MR review of [7]: "the most general canonoid transformation for a particular Hamiltonian is given in Lie Theorem III." Important contributions to this theory are given by Cariñena and co-workers [2]-[5], Negri and co-workers [18], [21] as well as in [9] and [15]. A careful analysis of this concept was performed recently in [11] and for Nambu mechanics in [8].

The aim of the present paper is to point out new features of canonoid transformations, for example in order to obtain conservation laws (first integrals) of a given dynamical system. The framework consists in a pair (M, X) with M a smooth manifold of even dimension, $\dim M = 2n$, and $X \in \mathcal{X}(M)$ a vector field on M generating the ODE system:

$$\dot{x}^i = \frac{dx^i}{dt} = X^i \left(x^1, ..., x^{2n}\right) \tag{1.1}$$

where $\left(x^i\right)_{i=\overline{1,2n}}$ are local coordinates on M and X has the local expression

$$X = X^i \frac{\partial}{\partial x^i}.$$

We call X a *locally Hamiltonian vector field* if there exists a symplectic form $\omega \in \Omega^2 (M)$ such that:

$$\mathcal{L}_X \omega = 0, \tag{1.2}$$

in other words, ω is a symplectic structure *associated to* X. Then, $\phi \in Diff(M)$ is called *canonoid* with respect to the pair (X, ω) (conform [1, p. 155]) if the new vector field $Y = \phi_* (X)$ is locally Hamiltonian with the same associated symplectic structure i.e. $\mathcal{L}_Y \omega = 0$. It follows n first integrals $\alpha_0, ..., \alpha_{n-1} \in C^\infty (M)$ for X, or for the system (1), given by [3]:

$$(\phi^* \omega)^{n-k} \wedge \omega^k = \alpha_k \omega^n. \tag{1.3}$$

An important remark here is that α's can be independent or not, trivial or not.

A canonoid transformation may be locally found in the classical way [4] by solving the system of partial differential equations which results from projecting both sides of the equality:

$$\mathcal{L}_X (\theta - \phi^* \theta) = dF \tag{1.4}$$

on the canonical-Darboux base $(dq^a, dp_a)_{a=\overline{1,n}}$; here θ is a local *potential* of ω, i.e. $\omega = d\theta$, and $F \in C^\infty (M)$ is called *the generating function* of the diffeomorphism ϕ.

Let us recall that in [5] a coordinate-free description of canonoid transformations is included, but we prefer here local computations in order to handle concrete examples. More precisely, in the following section we set a pair (M, X) and build, using the Helmholtz method of integrating factor in solving *the inverse problem*, a local symplectic form associated to X. In the next section, using the obtained geometrical framework, we study the existence of a canonoid transformation and corresponding first integrals. In the last section, the theory is applied to a four dimensional differential system, considered by Whittacker, and to a two dimensional system, namely the damped harmonic oscillator.

Another important remark here is that for $n = 1$ the unique (non-null) coefficient of the associated symplectic structure, which appears as integrating factor in the Helmholtz conditions, is solution of the celebrated *Liouville equation* [17], [10]. This equation is a main tool in statistical mechanics where a solution is called *probability density function* [22], while in mathematics is called *last multiplier* [12], [19]. A feature of this equation is that it does not always admits solutions [13].

2. The inverse problem

Let M be a real, smooth and orientable, $2n$-dimensional manifold, $C^\infty (M)$ the real algebra of smooth real functions on M, $\mathcal{X} (M)$ the Lie algebra of vector fields and $\Omega^k(M)$ the $C^\infty(M)$-module of k-differential forms, $0 \le k \le n$. Fix $X \in \mathcal{X}(M)$ which we suppose that it is not locally Hamiltonian with respect to the 2-form

$$\omega_0 = \sum_{a=1}^{n} dx^a \wedge dx^{n+a}.$$

In order to build a symplectic structure associated to X we follows the approach of Helmholtz based on the notion of *integrating factor* namely a set $c_{ij} = c_{ij}(t,x) \in C^\infty(\mathbb{R} \times M)$ such that the equivalent to (1.1) system $c_{ij}\left(\dot{x}^j - X^j\right) = 0$ is *self-adjoint.* The self-adjoint conditions are given by:

$$\left\{\begin{array}{c} c_{ij} + c_{ji} = 0 \\ \frac{\partial c_{ij}}{\partial x^h} + \frac{\partial c_{jh}}{\partial x^i} + \frac{\partial c_{hi}}{\partial x^j} = 0 \\ \frac{\partial c_{ij}}{\partial t} = \frac{\partial D_i}{\partial x^j} - \frac{\partial D_j}{\partial x^i}, \quad D_i = -c_{ij}X^j. \end{array}\right. \tag{2.1}$$

A global formulation of the Helmholtz conditions can be derived in terms of differential forms; namely if we consider the time-dependent 2-forms:

$$\left\{\begin{array}{c} \omega = \frac{1}{2}c_{ij}dx^i \wedge dx^j \\ \Omega = \omega + i_X\omega \wedge dt = \frac{1}{2}c_{ij}dx^i \wedge dx^j + D_i dx^i \wedge dt \end{array}\right. \tag{2.2}$$

then the Helmholtz conditions reduce to the closedeness of Ω: $d\Omega = 0$.

The following result is straightforward: *The vector field X admits a locally Hamiltonian description if and only if the system* (2.1) *admits an autonomous, i.e. time-independent, and non-degenerate, i.e.* $\det(c_{ij}) \neq 0$, *solution. In this case, the associated symplectic form is ω given by* (2.2).

Actually, the determination of the form ω comes from the integration of the system formed by the first two equations of (2.1) and by the equation:

$$X^h \frac{\partial c_{ij}}{\partial x^h} + c_{ih}\frac{\partial X^h}{\partial x^j} + c_{hj}\frac{\partial X^h}{\partial x^i} = 0 \tag{2.3}$$

which may be obtained from self-adjointness conditions (2.1).

3. Canonoid transformations and associated first integrals

Suppose that we found a symplectic 2-form ω such that X is locally Hamiltonian with respect to ω. If $\theta = A_j dx^j \in \Omega^1(M)$ is a local potential of ω, i.e. $d\theta = \omega$, and $F \in C^\infty(M)$ is a given function then the canonoid transformation ϕ having F as generating function is determined by the relation:

$$\mathcal{L}_X \phi^* \theta = dF. \tag{3.1}$$

Searching ϕ with local expression $\overline{x}^i = \varphi^i\left(x^1, ..., x^{2n}\right)$ the previous equation becomes:

$$\overline{c}_{ij}\frac{\partial \varphi^i}{\partial x^h}\frac{\partial \varphi^j}{\partial x^r}X^h + \frac{\partial}{\partial x^r}\left(\overline{A}_j X^s \frac{\partial \varphi^j}{\partial x^s}\right) = \frac{\partial F}{\partial x^r}, \tag{3.2}$$

with $\overline{c}_{ij}(x) = c_{ij}(\phi(x))$ and $\overline{A}_j(x) = A_j(\phi(x))$. Considering the vector field $Y = \phi_*(X)$, the equation (3.2) may be rewritten as:

$$\overline{c}_{ij}\frac{\partial\varphi^j}{\partial x^r}\overline{Y}^r + \frac{\partial}{\partial x^r}\left(\overline{A}_j\overline{X}^j\right) = \frac{\partial F}{\partial x^r} \tag{3.3}$$

if Y has the expression $Y = Y^i\frac{\partial}{\partial x^i}$ and $\overline{Y}^i(x) = Y^i(\phi(x))$.

Now, let us consider the Hamiltonian function H [6] defined by $i_X\omega = dH$ and the Poisson structure [6] defined by the symplectic form ω. The associated Poisson bracket is expressed as $\{f,g\} = c^{jk}\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^k}$ and the local components of the vector field $X = X_H$ are $X^k = c^{kj}\frac{\partial H}{\partial x^j}$ where $\left(c^{jk}\right)$ is the inverse matrix of the matrix (c_{jk}).

Taking into account that $\overline{Y}^i = \{\varphi^i, H\}$ the last equation reads:

$$\overline{c}_{ij}\frac{\partial\varphi^j}{\partial x^r}\{\varphi^i, H\} + \frac{\partial}{\partial x^r}\left(\overline{A}_j\left\{\varphi^j, H\right\}\right) = \frac{\partial F}{\partial x^r}. \tag{3.4}$$

A straightforward computation give us the first integral:

$$\alpha_{n-1} = \overline{c}_{ij}\left\{\varphi^i, \varphi^j\right\}. \tag{3.5}$$

There is also another way to find a canonoid transformation. If X is locally Hamiltonian with respect to ω, a vector field $Y = Y^i\frac{\partial}{\partial x^i}$ is locally Hamiltonian with respect to the same 2-form ω if and only if $\left(Y^1, ..., Y^{2n}\right)$ is a solution of the system (2.3). If we find such a solution we have to look for a transformation ψ such that $\psi_*Y = X$. This equality becomes:

$$\frac{\partial\psi^i}{\partial x^j}Y^j = X^i(\psi(x)) \tag{3.6}$$

where $\overline{x}^i = \psi^i\left(x^1, ..., x^{2n}\right)$ is the local expression of the diffeomorphism ψ. Obviously, if ψ is canonoid with respect to Y then $\phi = \psi^{-1}$ is canonoid with respect to X.

In the particular case $n = 2$ let α_0, α_1 be determined from the pair (X, ϕ) and β_0, β_1 similarly found and related to the pair (Y, ψ). A direct computation shows that:

$$\alpha_0 = \frac{1}{\phi^*\beta_0}, \alpha_1 = \frac{\phi^*\beta_1}{\phi^*\beta_0}. \tag{3.7}$$

Another particular case is $n = 1$. The equation (2.3) is reduced to:

$$X(\xi) + \xi divX = 0 \tag{3.8}$$

where ξ is the integrating factor and $divX$ is the divergence of the vector field X which is locally Hamiltonian with respect to $\omega = \xi dx^1 \wedge dx^2$. But (3.8) is exactly the Liouville equation discussed at the end of Introduction. The unique first integral associated to the pair (X, ϕ) in this case is $\alpha_0 = \overline{\xi}\left\{\varphi^1, \varphi^2\right\}$ with $\overline{\xi}(x) = \xi(\phi(x))$.

4. Examples

Let us consider the system of second-order ODE of Whittaker, [23]:

$$\begin{cases} \ddot{q}^1 - q^1 = 0 \\ \ddot{q}^2 - \dot{q}^1 = 0 \end{cases} \tag{4.1}$$

which does not admit a classical Lagrangian formulation [14]. If we use the notation $q^1 = x^1, q^2 = x^2, \dot{q}^1 = x^3, \dot{q}^2 = x^4$, we get the first-order equivalent system:

$$\begin{cases} \dot{x}^1 = x^3 \\ \dot{x}^2 = x^4 \\ \dot{x}^3 = x^1 \\ \dot{x}^4 = x^3 \end{cases} \tag{4.2}$$

An admissible symplectic structure for the vector field

$$X = x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}$$

is:

$$\omega = dx^1 \wedge dx^2 + dx^1 \wedge dx^3 + dx^2 \wedge dx^4 - dx^3 \wedge dx^4. \tag{4.3}$$

The Hamiltonian function H and the potential θ are respectively given by:

$$\begin{cases} H = \left(x^3\right)^2 + \frac{1}{2}\left[\left(x^4\right)^2 - \left(x^1\right)^2\right] - x^1 x^4 \\ \theta = -\left(x^2 + x^3\right) dx^1 + \left(x^2 - x^3\right) dx^4. \end{cases} \tag{4.4}$$

Another vector field which is locally Hamiltonian with respect to the same symplectic form is, for example, $Y = \frac{\partial}{\partial x^1}$. The canonoid transformation ϕ with $Y = \phi_*(X)$ and its generating function F are given respectively by:

$$\begin{cases} \varphi^1 = \ln\left(x^1 + x^3\right) \\ \varphi^2 = \sin\left[\left(x^3\right)^2 - \left(x^1\right)^2\right] \\ \varphi^3 = x^4 - x^1 \\ \varphi^4 = \cos\left[x^2 - x^3 + \left(x^1 - x^4\right)\ln\left(x^1 + x^3\right)\right] \end{cases} \tag{4.5}$$

and:

$$F = \sin\left[\left(x^3\right)^2 - \left(x^1\right)^2\right] - x^1 - x^2 - x^3 + x^4. \tag{4.6}$$

The first integral (3.5) is:

$$\alpha_1 = \cos\left[\left(x^3\right)^2 - \left(x^1\right)^2\right] + \sin\left[x^2 - x^3 + \left(x^1 - x^4\right)\ln\left(x^1 + x^3\right)\right] \times \\ \times \left[\left(x^1 - x^4\right)\cos\left(\left(x^3\right)^2 - \left(x^1\right)^2\right) - 1\right]. \tag{4.7}$$

Now, let us consider the equation of the damped harmonic oscillator:

$$\ddot{x} + c\dot{x} + kx = 0 \tag{4.8}$$

for which we assume that $c > 0, k > 0$ and $c^2 - 4k > 0$. The vector field

$$X = x^2 \frac{\partial}{\partial x^1} - \left(kx^1 + cx^2\right)\frac{\partial}{\partial x^2}$$

where we have used the notations $x = x^1$ and $\dot{x} = x^2$ has the integrating factor:

$$\xi = \left(a_1 x^1 + x^2\right)^{-\frac{c}{a_2}} \tag{4.9}$$

where:

$$a_1 = \frac{c + \sqrt{c^2 - 4k}}{2}, \quad a_2 = \frac{c - \sqrt{c^2 - 4k}}{2}. \tag{4.10}$$

The Hamiltonian function has the form:

$$H = \frac{\left(a_1 x^1 + x^2\right)^{a_1 \sigma}}{\sigma\left(a_1 \sigma + 1\right)} \left(\sigma x^2 - x^1\right), \sigma = -\frac{1}{a_2}. \tag{4.11}$$

A canonoid transformation for the vector field X is :

$$\begin{cases} \varphi^1 = \frac{\sigma}{a_1} \ln\left(a_1 x^1 + x^2\right) \\ \varphi^2 = -\sigma \ln\left(a_1 x^1 + x^2\right) + \left(a_1 x^1 + x^2\right)^A \left(a_2 x^1 + x^2\right)^B \end{cases} \tag{4.12}$$

with:

$$A = \frac{2a_1}{\sqrt{c^2 - 4k}}, B = \frac{-2a_2}{\sqrt{c^2 - 4k}}. \tag{4.13}$$

The vector fields X and $Y = \phi_* X = \frac{1}{a_1} \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}$ are locally Hamiltonian with respect to the same symplectic form $\omega = \xi dx^1 \wedge dx^2$. Finally, the first integral is $\alpha_0 = H^\beta$ with $\beta = \frac{c}{\sqrt{c^2 - 4k}}$.

5. Conclusions

0) The canonoid transformations provides useful information about the geometrical (symplectic structures and therefore volume forms) and dynamical (first integrals and bi-Hamiltonian description) objects which can be naturally associated to a given dynamical system.
1) The theory of these transformations has deep connections with other fundamental theoretical and applied constructions namely the theory of inverse problem and the Liouville equation.
2) The important structures generated by a canonoid transformation can be essential steps toward two remarkable approaches: the complete integrability of Liouville-Arnold type and the numerical integrators.
3) From the previous remarks it seems that this type of transformations to be more adapted than the canonical maps to some "in present" complicated or strange dynamical systems.
4) Due to the connection with the Liouville equation we can call the Helmholtz conditions of self-adjointness as *generalized Liouville equations* or *Liouville equations of higher even dimension* and maybe this fact opens a new way to connect the classical (Newtonian) mechanics to statistical physics.

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Mihai Boleanţu
West University of Timişoara
Faculty of Economics
Timişoara, 300115, Romania
e-mail: mihai.boleantu@feaa.uvt.ro

Mircea Crasmareanu
Al. I. Cuza University, Faculty of Mathematics
Iaşi, 700506, Romania
e-mail: mcrasm@uaic.ro