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Semi-infinite optimization problems and their approximations

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Abstract. In this paper, to the semi-infinite optimization problem (P), we attach the approximated semi-infinite optimization problems $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$ and some connections between the optimal solutions of the problems (P), $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$ are given.

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1. Introduction

We consider the optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{such that:} & \\ & x \in X \\ & g_t(x) \leq 0,\ t \in T \\ & h_s(x) = 0,\ s \in S \end{array} \tag{P}$$

where X is a subset of $\mathbb{R}^n$, T and S are nonempty sets, and $f : X \to \mathbb{R}$, $g_t : X \to \mathbb{R}$, $t \in T$ and $h_s : X \to \mathbb{R}$, $s \in S$ are functions.

Let

$$\mathcal{F}(P) := \{x \in X : \ g_t(x) \leq 0, \ (t \in T), \ h_s(x) = 0, \ (s \in S)\},$$

denote the set of all feasible solutions of Problem (P).

Depending on the sets T and S, we can have the following problems: if the sets T and S are finite, then the Problem (P) is a classic optimization problem, otherwise, the Problem (P) is a semi-infinite optimization problem with infinite number of constraints.

The field of semi-infinite programming appeared in 1924, but the name was coined in 1962 by Kortanek, Cooper and Charnes in the papers [3, 4]. Optimization problems in this area are characterized with a finite number of variables and an infinite number of constraints, or an infinite number of variables and a finite number of constraints. This class of optimization problems contains both convex and nonconvex optimization problems.

In recent years, in this domain over 10 books and 1000 articles have been published, treating both theoretical and practical issues, e.g., Hettich and Kortanek in [9].

We can find, in the literature, many semi-infinite optimization models from mechanical stress of materials, robot trajectory planning, economics [13], optimal signal sets, production planning, digital filter design, time minimal heating or cooling of a ball [11], air pollution control, minimal norm problem in the space of polynomial, robust optimization, system and control [8], reverse Chebyshev approximation [10]. The stability analysis in semi-infinite optimization (SIO) became an important issue, e.g., [2, 6, 7]. Authors who have treated (SIO) problem would be: Rückmann and Shapiro [16], Dinh The Luc [14], Polak [15], Still [17], Krabs [12].

Among the assumptions of necessary, respectively sufficient conditions for the solutions of semi-infinite optimization problem, appears the compactness of the sets T and S. The results obtained in this paper do not require that the sets T and S to be compact. The idea is to replace the Problem (P) with another simple problem and to establish the implications between the optimal solutions of the two problems.

Let $\eta : X \times X \to X$ be a function, x^0 be an interior point of X. Assume that the functions $f : X \to \mathbb{R}$, $g_t : X \to \mathbb{R}$, $t \in T$ and $h_s : X \to \mathbb{R}$, $s \in S$ are differentiable at x^0.

In this paper, we propose to attach to Problem (P), the following three approximated problems:

The first problem is:

$$\begin{array}{ll} \min & f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \\ \text{such that:} & \\ & x \in X \\ & g_t(x) \leq 0,\ t \in T \\ & h_s(x) = 0,\ s \in S \end{array} \tag{P$_{1,0}$}$$

called $(1,0)$-η approximated optimization problem.

The second problem is:

$$\begin{array}{ll} \min & f(x) \\ \text{such that:} & \\ & x \in X \\ & g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) \leq 0,\ t \in T \\ & h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) = 0,\ s \in S \end{array} \tag{P$_{0,1}$}$$

called $(0,1)$-η approximated optimization problem.

The third problem is

$$\begin{array}{ll} \min & f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) \\ \text{such that:} & \\ & x \in X \\ & g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x,x^0)\right) \leq 0,\ t \in T \\ & h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x,x^0)\right) = 0,\ s \in S \end{array} \tag{P$_{1,1}$}$$

called $(1,1)$-η approximated optimization problem.

In the case where T and S are finite, the idea of approximating the Problem (P) appeared in several papers, e.g. [1, 5].

After presenting some definitions, in paragraph 3 some connections between the optimal solutions of the four problems: (P), $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$ are given.

2. Definitions and preliminary results

Definition 2.1. *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $f: X \to \mathbb{R}$ be a differentiable function at x^0 and $\eta: X \times X \to X$ be a function. We say that:*
(a) *the function f is invex at x^0 with respect to (w.r.t.) η if*

$$f(x) - f(x^0) \geq [\nabla f(x^0)](\eta(x,x^0)), \text{ for all } x \in X,$$

(b) *the function f is incave at x^0 with respect to (w.r.t.) η if $(-f)$ is invex at x^0 w.r.t. η,*
(c) *the function f is avex at x^0 with respect to (w.r.t.) η if f is both invex and incave at x^0 w.r.t. η,*
(d) *the function f is pseudoinvex at x^0 with respect to (w.r.t.) η if*

$$[\nabla f(x^0)](\eta(x,x^0)) \geq 0 \Rightarrow f(x) - f(x^0) \geq 0, \text{ for all } x \in X,$$

(e) *the function f is quasi-incave at x^0 with respect to (w.r.t.) η if*

$$f(x) - f(x^0) \geq 0 \Rightarrow [\nabla f(x^0)](\eta(x,x^0)) \geq 0, \text{ for all } x \in X.$$

In the following two theorems establish connections between the sets of feasible solutions of the problem (P) and the problems $(P_{0,1})$, $(P_{1,1})$.

Theorem 2.2. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta: X \times X \to X$ and f, g_t, $h_s: X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) *for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,*
(b) *for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η.*
If

$$\begin{aligned} \mathcal{L} := \{x \in X: \ & g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x,x^0)\right) \leq 0,\ t \in T \text{ and} \\ & h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x,x^0)\right) = 0,\ s \in S\}, \end{aligned}$$

then

$$\mathcal{F}(P) \subseteq \mathcal{L}.$$

Proof. Let $x \in \mathcal{F}(P)$. This is equivalent with

$$g_t(x) \leq 0,\ t \in T,$$

and

$$h_s(x) = 0,\ s \in S.$$

From (a) and (b) we have

$$g_t(x) - g_t(x^0) \geq \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right),\ t \in T,$$

and

$$h_s(x) - h_s(x^0) = \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right),\ s \in S.$$

But

$$g_t(x) \leq 0,\ t \in T,$$

and

$$h_s(x) = 0,\ s \in S,$$

so

$$g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) \leq 0,\ t \in T,$$
$$h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) = 0,\ s \in S.$$

Consequently,

$$x \in \mathcal{L}. \qquad \square$$

Theorem 2.3. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,
(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η.

If

$$\mathcal{L} := \{x \in X : \quad g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) \leq 0,\ t \in T \text{ and } h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) = 0,\ s \in S\},$$

then

$$\mathcal{L} \subseteq \mathcal{F}(P).$$

Proof. Let $x \in \mathcal{L}$. This is equivalent with

$$\left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) + g_t(x^0) \leq 0,\ t \in T, \tag{2.1}$$

$$\left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) + h_s(x^0) = 0,\ s \in S. \tag{2.2}$$

From the hypotheses (a) and (b) we have

$$g_t(x) - g_t(x^0) \leq \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right),\ t \in T,$$

and

$$h_s(x) - h_s(x^0) = \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right), s \in S.$$

Now, from (2.1) and (2.2), we obtain

$$g_t(x) \leq 0,\ t \in T,$$

and

$$h_s(x) = 0,\ s \in S.$$

Hence,

$$x \in \mathcal{F}(P). \qquad \square$$

3. Main results

In this paragraph we present some connections between the optimal solutions of semi-infinite optimization problems (P) and $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$.

3.1. Approximate problem $(P_{1,0})$

For $(1,0)$-η approximated type we have the following results:

Theorem 3.1. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,
(b) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,0})$, then x^0 is an optimal solution for Problem (P).

Proof. Obviously $\mathcal{F}(P) = \mathcal{F}(P_{1,0})$. On the other hand, the point x^0 is an optimal solution for $(P_{1,0})$, then $x^0 \in \mathcal{F}(P_{1,0})$ and

$$\begin{aligned} f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right) \leq \\ \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right), \text{ for all } x \in \mathcal{F}(P_{1,0}). \end{aligned} \tag{3.1}$$

From (b) and (3.1) we obtain:

$$\left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,0}) = \mathcal{F}(P). \tag{3.2}$$

Now from (a) and (3.2) it follows:

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P).$$

Hence x^0 is an optimal solution for Problem (P). $\square$

Theorem 3.2. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and quasi-incave at x^0 w.r.t. η,
(b) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{1,0})$.

Proof. Obviously $\mathcal{F}(P) = \mathcal{F}(P_{1,0})$. On the other hand, the point x^0 is an optimal solution for (P), then $x^0 \in \mathcal{F}(P)$ and

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P). \tag{3.3}$$

Suppose that x^0 is not the optimal solution for Problem $(P_{1,0})$, which implies that there exists $x^1 \in \mathcal{F}(P_{1,0})$ such that

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^1, x^0)\right) < f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right). \tag{3.4}$$

From (3.4) and (b) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x^1, x^0)\right) < 0.$$

From (a) we obtain

$$f(x^1) < f(x^0),$$

which contradicts the optimality of x^0 for Problem (P).
Hence x^0 is an optimal solution for Problem $(P_{1,0})$. □

3.2. Approximate problem $(P_{0,1})$

For $(0,1)$-η approximated type we have the following results:

Theorem 3.3. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,
(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(c) $x^1 \in \mathcal{F}(P)$.

If x^1 is an optimal solution for Problem $(P_{0,1})$, then x^1 is an optimal solution for Problem (P).

Proof. The point x^1 is an optimal solution for $(P_{0,1})$, we have

$$f(x^1) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}). \tag{3.5}$$

By Theorem 2.2, we have

$$\mathcal{F}(P) \subseteq \mathcal{F}(P_{0,1}). \tag{3.6}$$

From (c), (3.5) and (3.6) we obtain

$$f(x^1) \leq f(x), \text{for all } x \in \mathcal{F}(P).$$

Hence, x^1 is an optimal solution for Problem (P). □

Theorem 3.4. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,
(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(c) $x^1 \in \mathcal{F}(P_{0,1})$.

If x^1 is an optimal solution for Problem (P), then x^1 is an optimal solution for Problem $(P_{0,1})$.

Proof. The point x^1 is an optimal solution for (P), we have

$$f(x^1) \leq f(x), \text{ for all } x \in \mathcal{F}(P). \tag{3.7}$$

By Theorem 2.3, we have

$$\mathcal{F}(P_{0,1}) \subseteq \mathcal{F}(P). \tag{3.8}$$

From (c), (3.7) and (3.8) we obtain

$$f(x^1) \leq f(x), \text{for all } x \in \mathcal{F}(P_{0,1}).$$

Hence, x^1 is an optimal solution for Problem $(P_{0,1})$. □

3.3. Approximate problem $(P_{1,1})$

For $(1,1)$-η approximated type we have the following results:

Theorem 3.5. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(P)$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,1})$, then x^0 is an optimal solution for Problem (P).

Proof. The point x^0 is an optimal solution for $(P_{1,1})$, we have

$$\begin{aligned} f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right) \leq \\ \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right), \text{ for all } x \in \mathcal{F}(P_{1,1}). \end{aligned} \tag{3.9}$$

By Theorem 2.2, we have

$$\mathcal{F}(P) \subseteq \mathcal{F}(P_{1,1}).$$

From (e) and (3.9) we obtain:

$$\left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,1}). \tag{3.10}$$

Now from (a) and (3.10) it follows:

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{1,1}),$$

then, from (d),

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P).$$

Hence x^0 is an optimal solution for Problem (P). □

Theorem 3.6. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and quasi-incave at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(P_{1,1})$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{1,1})$.

Proof. The point x^0 is an optimal solution for (P), we have

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P). \tag{3.11}$$

By Theorem 2.3, we have

$$\mathcal{F}(P_{1,1}) \subseteq \mathcal{F}(P).$$

From (3.11) and (a) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P),$$

hence

$$\left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,1}).$$

Consequently,

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0,x^0)\right) \leq$$
$$\leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right), \text{for all } x \in \mathcal{F}(P_{1,1}).$$

Hence x^0 is an optimal solution for Problem $(P_{1,1})$. □

Theorem 3.7. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and quasi-incave at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0,
(c) for each $s \in S$, the function h_s is differentiable at x^0,
(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{1,1})$.

Proof. Obviously $\mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1})$. On the other hand, the point x^0 is an optimal solution for $(P_{0,1})$, then

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}). \tag{3.12}$$

From (3.12) and (a) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{0,1}). \tag{3.13}$$

Now from (d) and (3.13) it follows

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0,x^0)\right) \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right),$$

for all $x \in \mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1})$.
Hence x^0 is an optimal solution for Problem $(P_{1,1})$. □

Theorem 3.8. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0,
(c) for each $s \in S$, the function h_s is differentiable at x^0,
(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,1})$, then x^0 is an optimal solution for Problem $(P_{0,1})$.

Proof. Obviously $\mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1})$. On the other hand, the point x^0 is an optimal solution for $(P_{1,1})$, then

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0,x^0)\right) \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right), \quad \text{for all } x \in \mathcal{F}(P_{1,1}). \tag{3.14}$$

From (3.14) and (d) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,1}).$$

Now from (a) and (3.14) it follows

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1}).$$

Hence x^0 is an optimal solution for Problem $(P_{0,1})$. □

Theorem 3.9. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0,
(b) for each $t \in T$, the function g_t is differentiable at x^0, and invex at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0, and avex at x^0 w.r.t. η,
(d) $x^1 \in \mathcal{F}(P_{1,0})$.

If x^1 is an optimal solution for Problem $(P_{1,1})$, then x^1 is an optimal solution for Problem $(P_{1,0})$.

Proof. By Theorem 2.2, we have $\mathcal{F}(P_{1,0}) \subseteq \mathcal{F}(P_{1,1})$. Now (d) implies that x^1 is an optimal solution for $(P_{1,0})$. □

Theorem 3.10. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0,
(b) for each $t \in T$, the function g_t is differentiable at x^0, and incave at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0, and avex at x^0 w.r.t. η.
(d) $x^1 \in \mathcal{F}(P_{1,1})$.

If x^1 is an optimal solution for Problem $(P_{1,0})$, then x^1 is an optimal solution for Problem $(P_{1,1})$

Proof. By Theorem 2.3, we have $\mathcal{F}(P_{1,1}) \subseteq \mathcal{F}(P_{1,0})$. Now (d) implies that x^1 is an optimal solution for $(P_{1,1})$. □

Theorem 3.11. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0, and incave at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0, and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(P_{0,1})$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,0})$, then x^0 is an optimal solution for Problem $(P_{0,1})$.

Proof. By Theorem 2.3, we have $\mathcal{F}(P_{0,1}) \subseteq \mathcal{F}(P_{1,0})$. The point x^0 is an optimal solution for $(P_{1,0})$, then

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right) \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right), \text{ for all } x \in \mathcal{F}(P_{1,0}). \tag{3.15}$$

From (3.15) and (e) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,0}). \tag{3.16}$$

Now from (a) and (3.16) it follows

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}).$$

Hence x^0 is an optimal solution for Problem $(P_{0,1})$. □

Theorem 3.12. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and quasi-incave at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(P_{1,0})$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{1,0})$

Proof. By Theorem 2.2, we have $\mathcal{F}(P_{1,0}) \subseteq \mathcal{F}(P_{0,1})$. The point x^0 is an optimal solution for $(P_{0,1})$, then

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}). \tag{3.17}$$

Assume that $x^0 \in \mathcal{F}(P_{1,0})$ is not the optimal solution for $(P_{1,0})$, then there exists $x^1 \in \mathcal{F}(P_{1,0})$ such that

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right) > f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^1, x^0)\right) \tag{3.18}$$

From (3.18) and (e) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x^1, x^0)\right) < 0. \tag{3.19}$$

Now from (a) it follows

$$f(x^1) < f(x^0).$$

which contradicts the optimality of x^0 for Problem $(P_{0,1})$. □

4. Conclusions

In this paper, three problems $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$ are presented, whose solutions give us information about the solutions of semi-infinite optimization problem (P).

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