

# An existence result for nonlinear hemivariational-like inequality systems

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**Abstract.** In this paper, we introduce and study a system of nonlinear hemivariational-like inequality system in reflexive Banach spaces. The novelty of our approach is that we prove the existence of at least one solution for our system without imposing any monotonicity assumptions or using nonsmooth critical point theory. The main tool is a fixed theorem due to T.C. Lin [Bull. Aust. Math. Soc. **34** (1986), 107-117.] Two applications are also given: the first one prove the existence of Nash generalized derivative points, while the second one the existence of at least one solution of a Schrödinger-type problem.

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## 1. Introduction

Hemivariational inequalities were introduced by Panagiotopoulos in the early eighties of the 20<sup>th</sup> century (see Panagiotopoulos [20],[21]) as a generalization of variational inequalities. Actually, they are more than simple generalizations, since a hemivariational inequality is not equivalent to a minimum problem. The hemivariational inequality problem simplifies to a variational inequality problem, if we restrict the involved functionals to be convex (see for instance: Fichera [5]; Lions and Stampacchia [11]; Hartman and Stampacchia [6]).

A useful tool to study the hemivariational inequalities is the critical point theory for locally Lipschitz functions, which study was started by K. Chang in his pioneering work [3]. These types of inequalities have created a new field in nonsmooth analysis, because they are based on the concept of the Clarke's generalized gradient of locally Lipschitz functions. Since its appearance, the theory of hemivariational inequalities has given important results both in pure and applied mathematics and it is very useful to understand several problems of mechanics and engineering for non-convex, nonsmooth energy functionals. For a better understanding of the theory of

hemivariational inequalities and other types of inequalities of hemivariational type (variational-hemivariational, quasi-hemivariational, etc.) as well as for various applications to Nonsmooth Mechanics, Economics, Engineering and Geometry, the reader is referred to the monographs of Naniewicz and Panagiotopoulos [17], Motreanu and Panagiotopoulos [15], Motreanu and Rădulescu [16], Kristály, Rădulescu and Varga [9] and their references.

In this paper we introduce a new type of inequality systems, which we call *nonlinear hemivariational-like inequality system*. In the aforementioned works, the solvability of the hemivariational inequality systems was studied via critical point theory for locally Lipschitz functions. The main goal of this paper is to study inequality problems in a general and unified framework (as nonlinear hemivariational-like inequalities can be reduced to variational-like inequalities of standard hemivariational inequalities) and establish an existence result without employing the nonsmooth critical point theory or imposing any monotonicity assumptions.

Concerning the applicability of our abstract result, in the last section of this paper, we present two possible applications: the first is related to the existence of Nash generalized derivative points, while the second deals with a Schrödinger-type problem. The notion of Nash generalized derivative point can be found in the paper of Kristály [8] who studied the existence of such points in the case of Riemannian manifolds. One year later, Repovš and Varga generalized this notion for systems of hemivariational inequalities (see Varga, Repovš [22]). We point out the fact that, in paper [22], the authors established two different proofs for the existence result, in the first proof using Ky Fan's version of Knaster-Kuratowsky-Mazurkiewicz fixed point theorem, while in the second one using Tarafdar's fixed point theorem. Our approach is slightly different, as we use a fixed point theorem due to Lin to prove our main result.

The paper is structured as follows. In Section 2, we recall some definitions and properties of locally Lipschitz functions and generalized gradients. Also we present Lin's fixed point theorem for set-valued mappings which will play an essential role in the proof of the main result. In Section 3, we describe the abstract framework in which we work, we formulate the system of nonlinear hemivariational-like inequalities and prove the main result. The last section is dedicated to some applications as we already mentioned above.

## 2. Preliminaries

Let  $E_1, E_2, \dots, E_n$  be Banach spaces for  $i \in \{1, \dots, n\}$ . Throughout the paper we denote by  $E_i^*$  the topological dual space of the Banach space  $E_i$ , while  $\|\cdot\|_i$  and  $\langle \cdot, \cdot \rangle_i$  denote the norm in  $E_i$  and the duality pairing between  $E_i^*$  and  $E_i$ , respectively for every  $i \in \{1, \dots, n\}$ .

In the following, we recall some basic definitions and properties from the theory developed by Clarke [4].

**Definition 2.1.** Let  $E$  be a Banach space and  $f : E \rightarrow \mathbb{R}$  be a functional.  $f$  is called locally Lipschitz if every point  $v \in E$  possesses a neighborhood  $V$  such that

$$|f(z) - f(w)| \leq K_v \|z - w\|, \quad \forall w, z \in V,$$

for a constant  $K_v > 0$  which depends on  $V$ .

**Definition 2.2.** Let  $E$  be a Banach space,  $E^*$  be its topological dual space and  $f : E \rightarrow \mathbb{R}$  be a functional. The generalized derivative of a locally Lipschitz functional  $f : E \rightarrow \mathbb{R}$  at the point  $v \in E$  along the direction  $w \in E$  is denoted by  $f^0(v; w)$ , i.e.

$$f^0(v; w) = \limsup_{\substack{z \rightarrow v \\ t \searrow 0}} \frac{f(z + tw) - f(z)}{t}.$$

As follows, we recall here a result for locally Lipschitz functions (for the proof of this result see Clarke [4]).

**Proposition 2.1.** If we consider a function  $f : E \rightarrow \mathbb{R}$  on a Banach space  $E$ , which is locally Lipschitz of rank  $K_w$  near the point  $w \in E$ , then

- (a) the function  $w \rightsquigarrow f^0(v; w)$  is finite, subadditive, positively homogeneous and satisfies

$$|f^0(v; w)| \leq K_v \|w\|;$$

- (b)  $f^0(v; w)$  is upper semicontinuous as a function of  $(v, w)$ .

**Definition 2.3.** The generalized gradient of  $f$  at the point  $v \in E$ , which is a subset of  $E^*$ , is defined by

$$\partial f(v) = \{y^* \in E^* : \langle y^*, w \rangle \leq f^0(v; w), \text{ for each } w \in E\}.$$

**Remark 2.1.** Using the Hahn-Banach theorem (see, for example Brezis [2]), it is easy to see that the set  $\partial f(v)$  is nonempty for every  $v \in E$ .

Similarly, we can define the partial generalized derivative and the partial generalized gradient of a locally Lipschitz functional in the  $i^{th}$  variable.

**Definition 2.4.** Consider the function  $f : E_1 \times \dots \times E_i \times \dots \times E_n \rightarrow \mathbb{R}$  which is locally Lipschitz in the  $i^{th}$  variable. The partial generalized derivative of  $f(v_1, \dots, v_i, \dots, v_n)$  at the point  $v_i \in E_i$  in the direction  $w_i \in E_i$ , denoted by  $f_{,i}^0(v_1, \dots, v_i, \dots, v_n; w_i)$ , is

$$f_{,i}^0(v_1, \dots, v_i, \dots, v_n; w_i) = \limsup_{\substack{z_i \rightarrow v_i \\ t \searrow 0}} \frac{f(v_1, \dots, z_i + tw_i, \dots, v_n) - f(v_1, \dots, z_i, \dots, v_n)}{t},$$

while the partial generalized gradient of the mapping  $v_i \rightsquigarrow f(v_1, \dots, v_i, \dots, v_n)$ , denoted by  $\partial_i f(v_1, \dots, v_i, \dots, v_n)$ , is

$$\partial_i f(v_1, \dots, v_i, \dots, v_n) = \{y_i^* \in E_i^* : f_{,i}^0(v_1, \dots, v_i, \dots, v_n; w_i) \geq \langle y_i^*, w_i \rangle_{E_i}, \forall w_i \in E_i\}.$$

**Definition 2.5.** Let  $E$  be a Banach space and  $f : E \rightarrow \mathbb{R}$  be a locally Lipschitz functional.  $f$  is regular at the point  $u \in E$ , if for each  $w \in E$  there exists the usual one-sided directional derivative  $f'(v; w)$  and it coincides with the generalized directional derivative, i.e.

$$f'(v; w) = f^0(v; w).$$

$f$  is called regular if the above condition holds at every point  $v \in E$ .

We recall the following elementary lemma due to Clarke [4].

**Lemma 2.1.** *Let  $f : Y_1 \times \dots \times Y_n \rightarrow \mathbb{R}$  be a regular, locally Lipschitz functional. Then the following assertions are satisfied:*

- (i)  $\partial f(u_1, \dots, u_i, \dots, u_n) \subseteq \partial_1 f(u_1, \dots, u_i, \dots, u_n) \times \dots \times \partial_n f(u_1, \dots, u_i, \dots, u_n)$ ;
- (ii)  $f^0(u_1, \dots, u_i, \dots, u_n; v_1, \dots, v_i, \dots, v_n) \leq \sum_{i=1}^n f_{,i}^0(u_1, \dots, u_i, \dots, u_n; v_i)$ ;
- (iii)  $f^0(u_1, \dots, u_i, \dots, u_n; 0, \dots, v_i, \dots, 0) \leq f_{,i}^0(u_1, \dots, u_i, \dots, u_n; v_i)$ .

At the end of this section we present Lin's fixed point theorem for set-valued mappings (see Lin [10]), which we shall use later to prove our main result.

**Theorem 2.1.** *Let  $E$  be a Hausdorff topological vector space and  $D \subset E$  be a nonempty convex set. Let  $\mathcal{U} \subseteq D \times D$  be a subset such that*

- (a)  $(u, u) \in \mathcal{U}$  for each  $u \in D$ ;
- (b) for each  $v \in D$  the set  $\mathcal{V}(v) = \{w \in D : (v, w) \in \mathcal{U}\}$  is closed in  $D$ ;
- (c) for each  $w \in D$  the set  $\mathcal{W}(w) = \{v \in D : (v, w) \notin \mathcal{U}\}$  is either convex or empty;
- (d)  $D$  has a nonempty compact convex subset  $D_0$  such that the set  $M$  is compact, where  $M$  is defined as follows:

$$M = \{w \in D : (v, w) \in \mathcal{U} \text{ for each } v \in D_0\}.$$

Then there exists a point  $w_0 \in M$  such that  $D \times \{w_0\}$  is a subset of  $\mathcal{U}$ .

### 3. Formulation of the problem

Let us consider the reflexive Banach spaces  $X_1, \dots, X_n$ , the Banach spaces  $Y_1, \dots, Y_n$  and the bounded, closed, convex sets  $D_i \subseteq X_i$  for  $i \in \{1, \dots, n\}$ , where  $n \in \mathbb{Z}_+$ . We suppose that for  $i \in \{1, \dots, n\}$  there exist linear compact operators  $A_i : X_i \rightarrow Y_i$ , the single-valued functions  $\eta_i : X_i \times X_i \rightarrow X_i$  and the nonlinear functionals  $\phi_i : X_1 \times \dots \times X_i \times \dots \times X_n \times X_i \rightarrow \mathbb{R}$ . We also assume that  $J : Y_1 \times \dots \times Y_n \rightarrow \mathbb{R}$  is a regular locally Lipschitz functional. We introduce the following notations:

- $X = X_1 \times \dots \times X_n$ ,  $Y = Y_1 \times \dots \times Y_n$  and  $D = D_1 \times \dots \times D_n$ ;
- $\bar{u}_i = A_i(u_i)$ ,  $\bar{\eta}_i(u_i, v_i) = A_i(\eta_i(u_i, v_i))$ , for each  $i = \overline{1, n}$ ;
- $u = (u_1, \dots, u_n)$  and  $\bar{u} = (\bar{u}_1, \dots, \bar{u}_n)$ ;
- $\eta(u, v) = (\eta_1(u_1, v_1), \dots, \eta_n(u_n, v_n))$  and  $\bar{\eta}(u, v) = (\bar{\eta}_1(u_1, v_1), \dots, \bar{\eta}_n(u_n, v_n))$ ;
- $\Phi : X \times X \rightarrow \mathbb{R}$ ,  $\Phi(u, v) = \sum_{i=1}^n \phi_i(u_1, \dots, u_i, \dots, u_n, \eta_i(u_i, v_i))$ .

We are now in the position to formulate our problem. As we mentioned in the Introduction we will call this *nonlinear hemivariational-like inequality system*:

**(NHLIS)** Find  $(u_1, \dots, u_n) \in D_1 \times \dots \times D_n$  such that

$$\begin{cases} \phi_1(u_1, \dots, u_n, \eta_1(u_1, v_1)) + J_{,1}^0(\bar{u}_1, \dots, \bar{u}_n; \bar{\eta}_1(u_1, v_1)) \geq 0 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \phi_n(u_1, \dots, u_n, \eta_n(u_n, v_n)) + J_{,n}^0(\bar{u}_1, \dots, \bar{u}_n; \bar{\eta}_n(u_n, v_n)) \geq 0 \end{cases}$$

for all  $(v_1, \dots, v_n) \in D_1 \times \dots \times D_n$ .

In order to obtain our result, we need the following assertions:

- $(\mathcal{H})$  For each  $i \in \{1, \dots, n\}$  the mapping  $\eta_i(\cdot, \cdot) : X_i \times X_i \rightarrow X_i$  satisfies the following conditions:
  - (i)  $\eta_i(u_i, u_i) = 0$ , for all  $u_i \in X_i$ ;
  - (ii)  $\eta_i(u_i, \cdot)$  is linear operator, for each  $u_i \in X_i$ ;
  - (iii) for each  $v_i \in X_i, \eta_i(u_i^m, v_i) \rightharpoonup \eta(u_i, v_i)$ , whenever  $u_i^m \rightharpoonup u_i$ .
- $(\Phi)$  For every  $i \in \{1, \dots, n\}$ , the functional  $\phi_i : X_1 \times \dots \times X_i \times \dots \times X_n \times X_i \rightarrow \mathbb{R}$  satisfies
  - (i)  $\phi_i(u_1, \dots, u_i, \dots, u_n, 0) = 0$ , for all  $u_i \in X_i$ ;
  - (ii) for all  $v_i \in X_i$ , the mapping  $(u_1, \dots, u_n) \rightsquigarrow \phi_i(u_1, \dots, u_n; \eta_i(u_i, v_i))$  is weakly upper semicontinuous;
  - (iii) the mapping  $v_i \rightsquigarrow \sum_{i=1}^n \phi_i(u_1, \dots, u_n; \eta_i(u_i, v_i))$  is convex, for each  $(u_1, \dots, u_n) \in X_1 \times \dots \times X_n$ .

**Remark 3.1.** Because  $J_{,i}^0(u_1, \dots, u_n; v_i)$  is convex and  $\eta_i(u_i, \cdot)$  is linear for each  $i \in \{1, \dots, n\}$  and for each  $(u_1, \dots, u_n) \in X_1 \times \dots \times X_n$ , it follows that the mapping  $v_i \rightsquigarrow J_{,i}^0(u_1, \dots, u_n; \eta_i(u_i, v_i))$  is convex.

**Example 3.1. 1.** For  $i \in \{1, \dots, n\}$  let us choose the function  $\eta_i : X_i \times X_i \rightarrow X_i$  as follows:

$$\eta_i(u_i, v_i) = v_i - u_i, \text{ for all } u_i, v_i \in X_i.$$

In this case,  $\eta_i(u_i, v_i)$  satisfies the assumptions (i), (ii) from  $(\mathcal{H})$ . Notice that, with this choice we get back the problem formulated in Varga and Repovš's paper [22].

2. Let  $B_i : X_i \rightarrow X_i$  be a linear compact operator,  $\alpha_i > 0, \beta_i \in X_i, i \in \{1, \dots, n\}$  and define a function  $f : X_i \rightarrow X_i$  by  $f_i(x) = \alpha_i B_i(x) + \beta_i$ . If we take the function  $\eta_i : X_i \times X_i \rightarrow X_i$  as follows:

$$\eta_i(u_i, v_i) = f_i(v_i) - f_i(u_i), \text{ for all } u_i, v_i \in X_i, i \in \{1, \dots, n\},$$

then it is clear that the conditions  $(\mathcal{H})$  (i)-(iii) hold for  $\eta_i(u_i, v_i)$ .

Now we present the main result of this paper. We deal with the case when the sets  $D_i$  are nonempty, bounded, closed and convex.

**Theorem 3.1.** Let us consider the nonempty, bounded, closed and convex sets  $D_i \subset X_i$  for each  $i \in \{1, \dots, n\}$ . If the conditions  $(\mathcal{H})$  and  $(\Phi)$  are fulfilled, then the system of nonlinear hemivariational-like inequalities (NHLIS) admits at least one solution.

Next let us formulate the following hemivariational inequality:

**(VHI)** Find  $u \in D$  such that for all  $v \in D$  we have

$$\Phi(u, v) + J^0(\bar{u}; \bar{\eta}(u, v)) \geq 0.$$

**Proposition 3.1.** If  $u^0 = (u_1^0, \dots, u_n^0) \in D_1 \times \dots \times D_n$  is a solution of the inequality (VHI) and the assumptions  $(\mathcal{H})$ -(i) and  $(\Phi)$ -(i) hold, then  $u^0$  is also a solution of the system (NHLIS).

*Proof.* First of all, we fix a point  $v_i \in D_i$ , for each  $i \in \{1, \dots, n\}$  and we assume that  $v_j = u_j^0, j \neq i$ . Let us suppose that  $u^0$  solves **(VHI)**. Then, from Lemma 2.1 (ii), **(H)**-(i) and **(Φ)**-(i) we obtain

$$\begin{aligned} 0 &\leq \Phi(u^0, v) + J^0(\bar{u}^0; \bar{\eta}(u^0, v)) \\ &\leq \sum_{j=1}^n \phi_j(u_1^0, \dots, u_j^0, \dots, u_n^0, \eta_j(u_j^0, v_j)) + \sum_{j=1}^n J_{,j}^0(\bar{u}_1^0, \dots, \bar{u}_n^0; \bar{\eta}_j(u_j^0, v_j)) \\ &= \phi_i(u_1^0, \dots, u_i^0, \dots, u_n^0, \eta_i(u_i^0, v_i)) + J_{,i}^0(\bar{u}_1^0, \dots, \bar{u}_n^0, \bar{\eta}_i(u_i^0, v_i)), \forall i = \overline{1, n}. \end{aligned}$$

Hence, it follows that  $(u_1^0, \dots, u_n^0) \in D_1 \times \dots \times D_n$  is a solution of our system **(NHLIS)** too. □

*Proof of Theorem 3.1.* If we take into consideration the Proposition 3.1, it is enough to prove that problem **(VHI)** has at least one solution. Toward this, let us introduce the set

$$\mathcal{U} = \{(v, u) \in D \times D : \Phi(u, v) + J^0(\bar{u}; \bar{\eta}(u, v)) \geq 0\} \subset D \times D.$$

We will show that  $\mathcal{U}$  satisfies all the conditions of Lin’s theorem for the weak topology of the space  $X$ .

At first, we can observe that  $(u, u) \in \mathcal{U}, \forall u \in D$ . Indeed, if we fix a point  $u \in D$ , then from **(H)**-(i) and **(Φ)**-(i) we deduce that

$$\begin{aligned} \Phi(u, u) + J^0(\bar{u}; \bar{\eta}(u, u)) &= \sum_{i=1}^n \phi_i(u_1, \dots, u_i, \dots, u_n, \eta_i(u_i, u_i)) + J^0(\bar{u}; 0) \\ &= \sum_{i=1}^n \phi_i(u_1, \dots, u_i, \dots, u_n, 0) = 0, \end{aligned}$$

that is,  $(u, u) \in \mathcal{U}$ .

Let us consider now the set  $\mathcal{V}(v) = \{u \in D : (v, u) \in \mathcal{U}\}$ . The next step is to justify that this set is weakly closed in  $D$ .

It suffices to prove that for a fixed point  $v \in D$  the function

$$F : D \rightarrow \mathbb{R}, F(u) = \Phi(u, v) + J^0(\bar{u}; \bar{\eta}(u, v))$$

is weakly upper semicontinuous, which is equivalent with the fact that the mappings  $\Phi(u, v)$  and  $J^0(\bar{u}; \bar{\eta}(u, v))$  are both weakly upper semicontinuous for the fixed  $v \in D$ . Let  $\{u^m\} \subset D$  be a sequence which converges weakly to some  $u \in D$ . Then, for every  $i \in \{1, \dots, n\}$ ,  $u_i^m$  converges weakly to  $u_i$ , when  $m \rightarrow +\infty$ . From **(H)**-(iii), it follows that for all  $i \in \{1, \dots, n\}$ ,  $\eta_i(u_i^m, v_i)$  converges weakly to  $\eta_i(u_i, v_i)$ , when  $m \rightarrow +\infty$ .

Taking this fact into account and  $(\Phi)$ -**(ii)**, we observe:

$$\begin{aligned} \limsup_{m \rightarrow \infty} \Phi(u^m, v) &= \limsup_{m \rightarrow \infty} \sum_{i=1}^n \phi_i(u_1^m, \dots, u_n^m; \eta_i(u_i^m, v_i)) \\ &\leq \sum_{i=1}^n \limsup_{m \rightarrow \infty} \phi_i(u_1^m, \dots, u_n^m; \eta_i(u_i^m, v_i)) \\ &\leq \sum_{i=1}^n \phi_i(u_1, \dots, u_n; \eta_i(u_i, v_i)) \\ &= \Phi(u, v), \end{aligned}$$

which means that the mapping  $u \mapsto \Phi(u, v)$  is weakly upper semicontinuous. On the other hand, we assumed that  $A_i$  is a compact operator, for each  $i \in \{1, \dots, n\}$ . Hence, we obtain that  $\bar{u}^m$  converges strongly to some  $\bar{u} \in D$  and for each  $i \in \{1, \dots, n\}$  and  $v_i \in D_i$ ,  $\bar{\eta}_i(u_i^m, v_i)$  converges strongly to  $\bar{\eta}_i(u_i, v_i)$ , so  $\bar{\eta}(u^m, v)$  converges strongly to  $\bar{\eta}(u, v)$ , for each  $v \in D$ . Applying this fact, together with Proposition 2.1 (b), we derive that

$$\limsup_{m \rightarrow \infty} J^0(\bar{u}^m; \bar{\eta}(u^m, v)) \leq J^0(\bar{u}; \bar{\eta}(u, v)).$$

Therefore, the function  $F$  is weakly upper semicontinuous. Hence, the set  $\{u \in D : F(u) \geq \alpha\}$  is weakly closed, for every  $\alpha \in \mathbb{R}$ . If we choose  $\alpha = 0$ , then we obtain that the set  $\mathcal{V}(v)$  is weakly closed, which is exactly what we wanted to prove.

In the following, we consider the set  $\mathcal{W}(u) = \{v \in D : (v, u) \notin \mathcal{U}\}$ , for all  $u \in D$ , and we will establish that this set is either convex or empty.

We suppose that  $\mathcal{W}(u) \neq \emptyset$  for a fixed  $u \in D$  and we shall demonstrate that this set is convex. In order to obtain this fact, let us choose  $v', v'' \in \mathcal{W}(u)$ ,  $\lambda \in ]0, 1[$  and let  $v^\lambda$  be their convex combination, i.e.  $v^\lambda = (1 - \lambda)v' + \lambda v''$ . Applying  $(\Phi)$ -**(iii)**, we get

$$\begin{aligned} \Phi(u, v^\lambda) &= \sum_{i=1}^n \phi_i(u_1, \dots, u_n, \eta_i(u_i, v_i^\lambda)) \\ &= \sum_{i=1}^n \phi_i(u_1, \dots, u_n, \eta_i(u_i, (1 - \lambda)v'_i + \lambda v''_i)) \\ &\leq (1 - \lambda) \sum_{i=1}^n \phi_i(u_1, \dots, u_n, \eta_i(u_i, v'_i)) + \lambda \sum_{i=1}^n \phi_i(u_1, \dots, u_n, \eta_i(u_i, v''_i)) \\ &= (1 - \lambda)\Phi(u, v') + \lambda\Phi(u, v''), \quad \forall \lambda \in ]0, 1[. \end{aligned}$$

Hence,  $v \rightsquigarrow \Phi(u, v)$  is convex. On the other hand, from Remark 3.1 we deduce that the mapping  $v \rightsquigarrow J^0(\bar{u}; \eta_i(\bar{u}, \bar{v}))$  is convex. Therefore, the set  $\mathcal{W}(u)$  is convex for the fixed  $u \in D$ .

It remains to justify that the set  $M = \{u \in D : (v, u) \in \mathcal{U}, \forall v \in D\}$  is weakly compact. Obviously, we can give the set  $M$  in the following form:

$$M = \bigcap_{v \in D} \mathcal{V}(v).$$

Since  $D$  is bounded, closed and convex subset of the reflexive space  $X$ , it follows that  $D$  also is a weakly compact subset of  $X$ . Taking into consideration that the set  $\mathcal{V}$  is weakly closed in  $D$ , it follows that the set  $M$  is an intersection of weakly closed subsets of  $D$ . Therefore, the set  $M$  is weakly compact.

We proved that the set  $\mathcal{U}$  satisfies the assumptions (1)-(4) of Lin's theorem. Using this theorem, we conclude that there exists  $u^0 \in M \subseteq D$  such that  $K \times \{u^0\} \subset \mathcal{U}$ , which means that

$$\Phi(u^0, v) + J^0(\bar{u}^0; \bar{\eta}(u^0, v)) \geq 0, \forall v \in D.$$

Therefore  $u^0$  is a solution of **(VHI)**, which – applying Proposition 3.1 – is also a solution of **(NHLIS)**. This completes the proof.  $\square$

**Remark 3.2.** *It is known that the solutions of hemivariational inequality systems on unbounded domains exist if we extend the assumptions for the bounded domains with a coercivity condition. Taking this into account, if we impose some coercivity conditions, it will ensure that Theorem 3.1 will also hold when the sets  $D_i$  are unbounded (for details, see Repovš, Varga [22], Remark 3.3).*

### 4. Applications

In this section we give some concrete applications of our main result. First, we give a generalization of Theorem 4.1 from Varga and Repovš's paper [22], then in particular case, we present an existence result for a Schrödinger-type problem.

#### 4.1. Nash generalized derivative points

Let us consider the Banach spaces  $E_1, \dots, E_n$  for each  $i \in \{1, \dots, n\}$  and the nonempty set  $D_i \subset E_i$ . Let  $g_i : D_1 \times \dots \times D_i \times \dots \times D_n \rightarrow \mathbb{R}$  be given functionals for  $i \in \{1, \dots, n\}$ . In the following we define some useful notions referred to Nash-type equilibrium problems which will be used throughout this subsection. Firstly, we recall the well-known notion introduced by Nash (see e.g. Nash [18, 19]):

**Definition 4.1.** *We say that an element  $(u_1, \dots, u_i, \dots, u_n) \in D_1 \times \dots \times D_i \times \dots \times D_n$  is a Nash equilibrium point for the functionals  $g_i, i \in \{1, \dots, n\}$ , if*

$$g_i(u_1, \dots, u_i, \dots, u_n) \leq g_i(u_1, \dots, v_i, \dots, u_n),$$

for all  $i \in \{1, \dots, n\}$  and for all  $(v_1, \dots, v_i, \dots, v_n) \in D_1 \times \dots \times D_i \times \dots \times D_n$ .

Let  $D'_i \subset E_i$  be an open set such that for each  $i \in \{1, \dots, n\}$  we have that  $D_i \subset D'_i$ . We impose a condition on the functional  $g_i : D_1 \times \dots \times D'_i \times \dots \times D_n \rightarrow \mathbb{R}$  for all  $i \in \{1, \dots, n\}$ : the mapping  $u_i \rightsquigarrow g_i(u_1, \dots, u_i, \dots, u_n)$  must be continuous and locally Lipschitz. Under these conditions we can give the definition of the Nash generalized derivative point. This notion was introduced by Kristály in [8] for functions defined in Riemannian manifolds. We give a little bit different form of this definition:

**Definition 4.2.** *A point  $(u_1, \dots, u_i, \dots, u_n) \in D_1 \times \dots \times D_i \times \dots \times D_n$  is said to be a Nash generalized derivative point for the functionals  $g_1, \dots, g_i, \dots, g_n$  if for all*



$i \in \{1, \dots, n\}$  and all  $(v_1, \dots, v_i, \dots, v_n) \in D_1 \times \dots \times D_i \times \dots \times D_n$  the following inequality holds:

$$g_{i,i}^0(u_1, \dots, u_i, \dots, u_n; v_i - u_i) \geq 0.$$

**Remark 4.1.** 1. We notice, that every Nash equilibrium point also is a Nash generalized derivative point.

2. In the paper [7], Kassay, Kolumbán and Páles introduced the notion of Nash stationary point. We can show easily that if for every  $i \in \{1, \dots, n\}$  the functionals  $g_i$  are differentiable with respect to the  $i^{\text{th}}$  variable, the notions of Nash generalized derivative point and Nash stationary point coincide.

An application of the main theorem yields the existence result for a type of Nash generalized derivative points. Thus, if we take in Theorem 3.1 the following choices

$$\phi_i(u_1, \dots, u_i, \dots, u_n, \eta_i(u_i, v_i)) = g_{i,i}^0(u_1, \dots, u_i, \dots, u_n; \eta_i(u_i, v_i))$$

for  $i \in \{1, \dots, n\}$  and set  $J = 0$ , and we suppose that the function

$$(u_1, \dots, u_i, \dots, u_n; v_i) \rightsquigarrow g_{i,i}(u_1, \dots, u_i, \dots, u_n; \eta_i(u_i, v_i))$$

is weakly upper semicontinuous for each  $v_i \in D_i, i = \overline{1, n}$ , then we obtain the theorem below:

**Theorem 4.1.** For each  $i \in \{1, \dots, n\}$  let  $D_i \subset X_i$  be a nonempty, bounded, closed and convex set and let us assume that conditions  $(\mathcal{H})$  and  $(\Phi)$  hold true. In these conditions, there exists a point  $(u_1^0, \dots, u_i^0, \dots, u_n^0) \in D_1 \times \dots \times D_n$  such that for all  $(u_1, \dots, u_n) \in D_1 \times \dots \times D_n$  and  $i \in \{1, \dots, n\}$  we have

$$g_{i,i}^0(u_1^0, \dots, u_i^0, \dots, u_n^0; \eta_i(u_i^0, v_i)) \geq 0.$$

In order to highlight the importance of the previous result, we make the following remark:

**Remark 4.2.** If we choose  $\eta(u_i^0, v_i) = v_i - u_i^0$  for  $i \in \{1, \dots, n\}$ , then we give back the existence result for Nash generalized derivative points from Repovš and Varga’s paper [22].

As a second application, we state an existence result for a general hemivariational inequalities system. In order to establish this theorem, in Theorem 2.1 we need to require that the functionals  $\phi_i : Y_1 \times \dots \times Y_i \times \dots \times Y_n \times Y_i \rightarrow \mathbb{R}$  be differentiable in the  $i^{\text{th}}$  variable for  $i \in \{1, \dots, n\}$  and their derivatives  $\phi'_i : Y_1 \times \dots \times Y_i \times \dots \times Y_n \times Y_i \rightarrow \mathbb{R}$  be continuous for  $i \in \{1, \dots, n\}$ .

**Corollary 4.1.** Let us consider the regular, locally Lipschitz function  $J : Y_1 \times Y_2 \times \dots \times Y_i \times \dots \times Y_n \rightarrow \mathbb{R}$  and the nonlinear functionals  $\phi_i : Y_1 \times \dots \times Y_i \times \dots \times Y_n \times Y_i \rightarrow \mathbb{R}$  with their continuous derivatives  $\phi'_i : Y_1 \times \dots \times Y_i \times \dots \times Y_n \times Y_i \rightarrow \mathbb{R}$  for  $i \in \{1, \dots, n\}$ . Let also  $D_i \subset X_i$  be bounded, closed and convex sets for  $i \in \{1, \dots, n\}$  and let us assume that conditions  $(\mathcal{H})$  and  $(\Phi)$  hold true. Then, there exists a point  $u^0 = (u_1^0, \dots, u_n^0) \in D_1 \times \dots \times D_n$  such that

$$\Phi'_i(\bar{u}^0, \bar{\eta}_i(u_i^0, u_i)) + J_i^0(\bar{u}^0; \bar{\eta}_i(u_i^0, u_i)) \geq 0,$$

for each  $u = (u_1, \dots, u_n) \in D_1 \times \dots \times D_n$  and  $i \in \{1, \dots, n\}$ .

**4.2. Schrödinger-type systems**

Let  $a_1, a_2 : \mathbb{R}^n \rightarrow \mathbb{R}$  ( $n > 2$ ) be two continuous functions such that the following assumptions hold:

- $\inf_{x \in \mathbb{R}^n} a_i(x) > 0, i = 1, 2;$
- $\text{meas}(\{x \in \mathbb{R}^n : a_i(x) \leq M_i\}) < \infty,$  for every  $M_i > 0, i = 1, 2.$

For  $i = 1, 2,$  let us consider the Hilbert-spaces

$$X_i := \left\{ u \in W^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla u(x)|^2 + a_i(x)u^2(x)dx < \infty \right\},$$

which are equipped with the inner products

$$\langle u, v \rangle_{X_i} = \int_{\mathbb{R}^n} [\nabla u(x)\nabla v(x) + a_i(x)u(x)v(x)] dx.$$

It is known that for  $q \in [2, 2^*],$  the space  $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$  is continuous, therefore  $X_1 \times X_2 \hookrightarrow L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$  is continuous for  $s, r \in [2, 2^*].$  If  $s, r \in [2, 2^*],$  then  $X_1 \times X_2 \hookrightarrow L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$  is compact (for the proof of the latter result see Bartsch, Wang [1]).

Let  $G : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a regular, locally Lipschitz-function, for which we assume the following:

**(G1)** There exist  $c > 0$  and  $r \in (2, 2^*), s \in (2, 2^*)$  such that

$$\begin{aligned} |w_u| &\leq c(|u| + |v| + |u|^{r-1}), \\ |w_v| &\leq c(|v| + |u| + |v|^{s-1}), \end{aligned}$$

for all  $(u, v) \in \mathbb{R}^2, w_u \in \partial_1 G(u, v), w_v \in \partial_2 G(u, v), i = \overline{1, 2},$  where  $\partial_1 G(u, v)$  and  $\partial_2 G(u, v)$  denote the (partial) generalized gradient of  $G(\cdot, v)$  at the point  $u$  and  $v,$  respectively, while  $2^* = \frac{2N}{N-2}, N > 2$  is the Sobolev critical exponent.

Let  $K_1 \subset X_1, K_2 \subset X_2$  be two nonempty, convex, closed and bounded subsets. In the following let us denote by  $G_1^0(u(x), v(x); w_1(x))$  and  $G_2^0(u(x), v(x); w_2(x))$  the directional derivatives of  $G$  in the first and the second variable along the direction  $w_1$  and, respectively  $w_2.$  Now, we are in the position to formulate the following Schrödinger-type problem:

**(Sch-S)** Find  $(u_1, u_2) \in K_1 \times K_2$  such that for every  $(v_1, v_2) \in K_1 \times K_2$

$$\begin{aligned} \langle \overline{u_1}, \overline{\eta_1}(u_1, v_1) \rangle + \int_{\mathbb{R}^n} G_1^0(\overline{u_1}(x), \overline{v_1}(x), \overline{\eta_1}(u_1(x), v_1(x)))dx &\geq 0, \forall v_1 \in K_1; \\ \langle \overline{u_2}, \overline{\eta_2}(u_2, v_2) \rangle + \int_{\mathbb{R}^n} G_2^0(u_2(x), v_2(x), \overline{\eta_2}(u_2(x), v_2(x)))dx &\geq 0, \forall v_2 \in K_2. \end{aligned}$$

At the end of this section, as an application of the main result we show the existence of at least one solution of the previous problem.

**Corollary 4.2.** *If  $K_1 \subset X_1$  and  $K_2 \subset X_2$  are two nonempty, convex, closed and bounded subsets and  $\eta_1, \eta_2$  satisfy the conditions  $(\mathcal{H})$  and  $G$  satisfies **(G1)**, then the problem **(Sch-S)** has at least one solution.*

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