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On a generalization of Szász operators by multiple Appell polynomials

Serhan Varma

Abstract. In this paper, we define a form of positive linear operators by means of multiple Appell polynomials. Also, Kantorovich type generalization and simultaneous approximation of these operators are given. Convergence properties of our operators are verified with the help of the universal Korovkin-type property and the order of approximation is calculated by using classical modulus of continuity.

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1. Introduction

Szász [8] introduced the well-known operators in 1950 by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{1.1}$$

where $x \in [0,\infty)$ and $f \in C[0,\infty)$. It may be mentioned that these operators are also the examples of positive approximation processes discovered by Korovkin [6]. Many authors deal with the generalizations of Szász operators.

Jakimovski and Leviatan [5] gave a generalization of Szász operators involving the Appell polynomials. The Appell polynomials $p_k(x)$ have the following generating relation

$$g(u)\, e^{ux} = \sum_{k=0}^{\infty} p_k(x)\, u^k \ , \tag{1.2}$$

where $g(u) = \sum\limits_{k=0}^{\infty} a_k u^k$ is an analytic function in the disc $|u| < R$, $(R > 1)$ and $g(1) \neq 0$. Under the assumptions $p_k(x) \geq 0$ for $x \in [0,\infty)$, they proposed the linear positive operators as

$$P_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)\,f\left(\frac{k}{n}\right)\quad . \tag{1.3}$$

For the special case $g(u)=1$, from the generating functions (1.2) one can easily find $p_k(x)=\frac{x^k}{k!}$ and from this fact (1.3) reduces Szász operators (1.1). The detailed approximation properties of the operators (1.3) were given by Wood ([9],[10]) and Ciupa ([2]-[4]).

In this paper, we construct an operator by the help of multiple Appell polynomials. First of all, we need the following results from the paper of Lee [7].

Definition 1.1. *Multiple polynomial system(multiple PS for short) means a set of polynomials* $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ *with* $\deg(p_{k_1,k_2})=k_1+k_2$ *for* $k_1,k_2\geq 0$.

Definition 1.2. *A multiple PS* $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ *is called multiple Appell if it has a generating function of the form*

$$A(t_1,t_2)\,e^{x(t_1+t_2)}=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}(x)}{k_1!k_2!}t_1^{k_1}t_2^{k_2} \tag{1.4}$$

where A has a series expansion

$$A(t_1,t_2)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{a_{k_1,k_2}}{k_1!k_2!}t_1^{k_1}t_2^{k_2} \tag{1.5}$$

with $A(0,0)=a_{0,0}\neq 0$.

Theorem 1.3. *Let* $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ *be a multiple PS. Then the following statements are all equivalent.*

(a) $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ *is a set of multiple Appell polynomials.*

(b) There exists a sequence $\left\{a_{k_1,k_2}\right\}_{k_1,k_2=0}^{\infty}$ *with* $a_{0,0}\neq 0$ *such that*

$$p_{k_1,k_2}(x)=\sum_{r_1=0}^{k_1}\sum_{r_2=0}^{k_2}\binom{k_1}{r_1}\binom{k_2}{r_2}a_{k_1-r_1,k_2-r_2}x^{r_1+r_2}.$$

(c) For every $k_1+k_2\geq 1$, *we have*

$$p'_{k_1,k_2}(x)=k_1p_{k_1-1,k_2}(x)+k_2p_{k_1,k_2-1}(x).$$

In view of these results, let us restrict the multiple Appell polynomials satisfying

(i) $A(1,1)\neq 0$ and $\frac{a_{k_1,k_2}}{A(1,1)}\geq 0$ for $k_1,k_2\in\mathbb{N}$,

(ii) (1.4) and (1.5) converge for $|t_1|<R_1,\ |t_2|<R_2\ \ (R_1,R_2>1)$.

Under the above restrictions, we introduce the following positive linear operators for $x\in[0,\infty)$

$$K_n(f;x)=\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}f\left(\frac{k_1+k_2}{n}\right) \tag{1.6}$$

whenever the right-hand side of (1.6) exists.

Remark 1.4. We have to note the following special cases.

Case 1. For $t_2 = 0$, the generating functions given by (1.4) reduce to the generating functions for the Appell polynomials given by (1.2). By the help of Appell polynomials, Jakimovski and Leviatan constructed the operators (1.3) and gave the approximation properties in [5].

Case 2. For $t_2 = 0$ and $A(t_1, 0) = 1$, from the generating functions given by (1.4) we easily find $p_k(x) = x^k$. From this fact, one can obtain Szász operators (1.1).

The outline of the paper is as follows. In the following section, we obtain the uniform convergence of the operators (1.6) by using the universal Korovkin-type property and the order of approximation by virtue of classical modulus of continuity. Also, simultaneous approximation of the operators (1.6) is derived. In section 3, Kantorovich type generalization of our operators and the approximation properties are given.

2. Approximation properties of K_n operators

In this section, we state our main theorem with the help of the universal Korovkin-type property and calculate the order of approximation by modulus of continuity. We note that throughout the paper we use the following abbreviations for the partial derivatives

$$\frac{\partial A}{\partial t_i} = A_{t_i} \quad \text{and} \quad \frac{\partial^2 A}{\partial t_i \partial t_j} = A_{t_i t_j} \quad i,j = 1,2 \ .$$

Lemma 2.1. *The operators given by* (1.6) *satisfy the following equalities*

$$K_n(1;x) = 1 \tag{2.1}$$

$$K_n(s;x) = x + \frac{A_{t_1}(1,1) + A_{t_2}(1,1)}{nA(1,1)} \tag{2.2}$$

$$\begin{aligned} K_n\left(s^2;x\right) &= x^2 + \frac{x}{n}\left(1 + \frac{2\left(A_{t_1}(1,1) + A_{t_2}(1,1)\right)}{A(1,1)}\right) \\ &\quad + \frac{1}{n^2 A(1,1)}\left\{A_{t_1}(1,1) + A_{t_2}(1,1)\right. \\ &\quad \left. + A_{t_1 t_1}(1,1) + 2A_{t_1 t_2}(1,1) + A_{t_2 t_2}(1,1)\right\} \end{aligned} \tag{2.3}$$

where $x \geq 0$.

Proof. For $f(s) = 1$, we get from (1.6)

$$K_n(f;x) = \frac{e^{-nx}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!}.$$

By taking $t_1 = 1$, $t_2 = 1$ and replacing x by $\frac{nx}{2}$ in the generating function (1.4), one can easily get (2.1).

For $f(s)=s$, we obtain from (1.6)

$$K_n(s;x)=\frac{e^{-nx}}{nA(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(k_1+k_2)\,p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}.$$

Taking the partial derivatives of the generating function (1.4) with respect to t_1 and t_2 and later by taking $t_1=1$, $t_2=1$ and replacing x by $\frac{nx}{2}$, one can get (2.2).

Taking into account the second order partial derivatives of the generating function (1.4) with respect to t_1 and t_2, one can get (2.3) by using similar technique. □

Let us define the class of E as follows

$$E:=\left\{f:x\in[0,\infty)\,,\ \frac{f(x)}{1+x^2}\text{ is convergent as }x\to\infty\right\}.$$

Theorem 2.2. *Let $f\in C[0,\infty)\cap E$. Then*

$$\lim_{n\to\infty}K_n(f;x)=f(x)\,,$$

the convergence being uniform in each compact subset of $[0,\infty)$.

Proof. According to (2.1)-(2.3), we have

$$\lim_{n\to\infty}K_n\left(s^i;x\right)=x^i\ ,\quad i=0,1,2\ .$$

Since the above convergences are verified uniformly in each compact subset of $[0,\infty)$, we obtain the desired result by applying the universal Korovkin-type property (vi) of Theorem 4.1.4 in [1]. □

Let us recall the following definition.

Definition 2.3. *Let $f\in\tilde{C}[0,\infty)$ and $\delta>0$. The modulus of continuity $\omega(f;\delta)$ of the function f is defined by*

$$\omega(f;\delta):=\sup_{\substack{x,y\in[0,\infty)\\|x-y|\le\delta}}|f(x)-f(y)|$$

where $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$.

Next, for the order of approximation we express the following.

Theorem 2.4. *Let $f\in\tilde{C}[0,\infty)\cap E$. We have the following inequality for the operators* (1.6)

$$\begin{aligned}&|K_n(f;x)-f(x)|\\&\quad\le\left\{1+\sqrt{\begin{array}{c}x+\frac{1}{nA(1,1)}\left\{A_{t_1}(1,1)+A_{t_2}(1,1)+A_{t_1t_1}(1,1)\right.\\\left.+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\right\}\end{array}}\right\}\\&\quad\times\omega\left(f;\frac{1}{\sqrt{n}}\right).\end{aligned}\tag{2.4}$$

Proof. According to (2.1) and the property of modulus of continuity, the left-hand side of (2.4) leads to

$$\begin{aligned}
&\left|K_n(f;x)-f(x)\right| \\
&\leq \frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\left|f\left(\frac{k_1+k_2}{n}\right)-f(x)\right| \\
&\leq \left\{1+\frac{1}{\delta}\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\left|\frac{k_1+k_2}{n}-x\right|\right\}\omega(f;\delta)\ .
\end{aligned} \tag{2.5}$$

By applying the Cauchy-Schwarz inequality for each sums, (2.5) becomes

$$\begin{aligned}
&\left|K_n(f;x)-f(x)\right| \\
&\leq \left\{1+\frac{1}{\delta}\left(\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\right)^{1/2}\right. \\
&\quad\left.\times\left(\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\left(\frac{k_1+k_2}{n}-x\right)^2\right)^{1/2}\right\}\omega(f;\delta) \\
&= \left\{1+\frac{1}{\delta}\left(K_n(1;x)\right)^{1/2}\left(K_n\left((s-x)^2;x\right)\right)^{1/2}\right\}\omega(f;\delta)
\end{aligned} \tag{2.6}$$

In view of Lemma 2.1, (2.6) gives (2.4) by taking $\delta=\delta_n=\frac{1}{\sqrt{n}}$. □

In the end of this section, we will give the following theorem for the simultaneous approximation. Let $\tilde{C}^r[0,\infty)$ be the space of r-times differentiable function such that $f^{(r)}$ is uniformly continuous on $[0,\infty)$.

Theorem 2.5. *Let $f\in\tilde{C}^r[0,\infty)\cap E$. We have the following inequality for the derivatives of the operators* (1.6)

$$\begin{aligned}
\left|K_n^{(r)}(f;x)-f^{(r)}(x)\right| \leq&\ \left\{1+\sqrt{\begin{array}{c}x+\frac{1}{nA(1,1)}\{A_{t_1}(1,1)+A_{t_2}(1,1)\\+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\}\end{array}}\right\} \\
&\times\omega\left(f^{(r)};\frac{1}{\sqrt{n}}+\frac{r}{n}\right)+\omega\left(f^{(r)};\frac{r}{n}\right).
\end{aligned} \tag{2.7}$$

Proof. By virtue of the result (c) from Theorem 1.3, we deduce that

$$K_n^{(r)}(f;x)=n^r\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\Delta_{1/n}^r f\left(\frac{k_1+k_2}{n}\right) \tag{2.8}$$

where $\Delta_{1/n}^{r} f\left(\frac{k_1+k_2}{n}\right)$ is the difference of order r of f with the step $\frac{1}{n}$. Taking into account the relation between finite difference and divided difference, (2.8) leads to

$$
\begin{aligned}
K_n^{(r)}(f;x) &= r!\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\frac{\Delta_{1/n}^{r} f\left(\frac{k_1+k_2}{n}\right)}{r!\left(\frac{1}{n}\right)^r} \\
&= r!\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!} \\
&\quad \times\left[\frac{k_1+k_2}{n},\frac{k_1+k_2+1}{n},...,\frac{k_1+k_2+r}{n};f\right] \\
&= r!\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}h\left(\frac{k_1+k_2}{n}\right)=r!K_n(h;x)
\end{aligned}
$$

where $h(t)=\left[t,t+\frac{1}{n},...,t+\frac{r}{n};f\right]$. By using the inequality (2.4), we have

$$
\begin{aligned}
&\left|K_n^{(r)}(f;x)-f^{(r)}(x)\right| \\
&\leq r!\left|K_n(h;x)-h(x)\right|+\left|r!h(x)-f^{(r)}(x)\right| \\
&\leq r!\left\{1+\sqrt{\begin{array}{c} x+\frac{1}{nA(1,1)}\left\{A_{t_1}(1,1)+A_{t_2}(1,1)\right. \\ \left.+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\right\}\end{array}}\right\} \\
&\quad \times\omega\left(h;\frac{1}{\sqrt{n}}\right)+\left|r!h(x)-f^{(r)}(x)\right| \qquad (2.9)
\end{aligned}
$$

From the property of the modulus of continuity, we get

$$
\begin{aligned}
&\left|h(t+\delta)-h(t)\right| \\
&= \left|\left[t+\delta,t+\delta+\frac{1}{n},...,t+\delta+\frac{r}{n};f\right]-\left[t,t+\frac{1}{n},...,t+\frac{r}{n};f\right]\right| \\
&= \frac{1}{r!}\left|f^{(r)}\left(t+\delta+\frac{r}{n}\theta_1\right)-f^{(r)}\left(t+\frac{r}{n}\theta_2\right)\right| \\
&\leq \frac{1}{r!}\omega\left(f^{(r)};\delta+\frac{r}{n}\left|\theta_1-\theta_2\right|\right) \\
&\leq \frac{1}{r!}\omega\left(f^{(r)};\delta+\frac{r}{n}\right)
\end{aligned}
$$

where θ_1, $\theta_2\in(0,1)$. For $\delta=\frac{1}{\sqrt{n}}$, we obtain

$$
\omega\left(h;\frac{1}{\sqrt{n}}\right)\leq\frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{n}}+\frac{r}{n}\right) \quad . \qquad (2.10)
$$

Otherwise,

$$\begin{aligned}\left|r!h(x)-f^{(r)}(x)\right| &= \left|r!\left[x,x+\frac{1}{n},...,x+\frac{r}{n};f\right]-f^{(r)}(x)\right| \\ &= \left|f^{(r)}\left(x+\frac{r}{n}\theta_3\right)-f^{(r)}(x)\right| \\ &\leq \omega\left(f^{(r)};\frac{r}{n}\theta_3\right)\leq\omega\left(f^{(r)};\frac{r}{n}\right) \qquad (2.11)\end{aligned}$$

where $\theta_3\in(0,1)$. Combining (2.10) and (2.11) with (2.9), we reach the desired result. □

3. Kantorovich type generalization of K_n operators

In this section, we give a Kantorovich type generalization of the operators K_n (1.6) with same restrictions

$$K_n^*(f;x)=\frac{ne^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}f(s)\,ds\ . \qquad (3.1)$$

Now, let us give the following lemma which is used in the sequel.

Lemma 3.1. *The operators given by* (3.1) *satisfy the following equalities*

$$\begin{aligned}K_n^*(1;x) &= 1 \qquad (3.2)\\ K_n^*(s;x) &= x+\frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{nA(1,1)}+\frac{1}{2n} \qquad (3.3)\\ K_n^*\left(s^2;x\right) &= x^2+\frac{2x}{n}\left(1+\frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{A(1,1)}\right)+\frac{1}{n^2A(1,1)}\\ &\quad\times\left\{\begin{array}{c}2\left(A_{t_1}(1,1)+A_{t_2}(1,1)+A_{t_1t_2}(1,1)\right)\\ +A_{t_1t_1}(1,1)+A_{t_2t_2}(1,1)\end{array}\right\}+\frac{1}{3n^2} \qquad (3.4)\end{aligned}$$

Proof. Above equalities (3.2)-(3.4) follow from Lemma 2.1 immediately. □

Next, we derive the following two theorems for the uniform convergence and the order of approximation.

Theorem 3.2. *Let $f\in C[0,\infty)\cap E$. Then*

$$\lim_{n\to\infty}K_n^*(f;x)=f(x),$$

the convergence being uniform in each compact subset of $[0,\infty)$.

Proof. From (3.2)-(3.4), we get

$$\lim_{n\to\infty}K_n^*\left(s^i;x\right)=x^i\ ,\quad i=0,1,2\ .$$

The proof is completed by virtue of the above uniform convergences in each compact subset of $[0,\infty)$ and the universal Korovkin-type property (vi) of Theorem 4.1.4 in [1]. $\square$

Theorem 3.3. *Let $f \in \tilde{C}[0,\infty) \cap E$. We have the following inequality for the operators* (3.1)

$$\begin{aligned} &\left| K_n^* (f;x) - f(x) \right| \\ &\le \left\{ 1 + \sqrt{x + \frac{1}{nA(1,1)} \left\{ \begin{array}{c} 2\left(A_{t_1}(1,1) + A_{t_2}(1,1) + A_{t_1 t_2}(1,1) \right) \\ + A_{t_1 t_1}(1,1) + A_{t_2 t_2}(1,1) \end{array} \right\} + \frac{1}{3n}} \right\} \\ &\times \omega\left(f; \frac{1}{\sqrt{n}} \right). \end{aligned} \tag{3.5}$$

Proof. From (3.2) and the property of modulus of continuity, the left-hand side of (3.5) becomes

$$\begin{aligned} &\left| K_n^* (f;x) - f(x) \right| \\ &\le \frac{ne^{-nx}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!} \int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}} \left| f(s) - f(x) \right| ds \\ &\le \left\{ 1 + \frac{1}{\delta} \frac{ne^{-nx}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!} \int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}} |s-x| \, ds \right\} \omega(f;\delta) . \end{aligned}$$

By using the Cauchy-Schwarz inequality for the integral, we have

$$\begin{aligned} &\left| K_n^* (f;x) - f(x) \right| \\ &\le \left\{ 1 + \frac{1}{\delta} \frac{e^{-nx}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!} \left(n \int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}} (s-x)^2 \, ds \right)^{1/2} \right\} \omega(f;\delta) . \end{aligned} \tag{3.6}$$

By applying the Cauchy-Schwarz inequality for each sums, (3.6) leads to

$$\begin{aligned} &\left| K_n^* (f;x) - f(x) \right| \\ &\le \left\{ 1 + \frac{1}{\delta} \left(\frac{e^{-nx}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!} \right)^{1/2} \right. \\ &\quad \left. \times \left(\frac{ne^{-nx}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!} \int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}} (s-x)^2 \, ds \right)^{1/2} \right\} \omega(f;\delta) \\ &= \left\{ 1 + \frac{1}{\delta} \left(K_n(1;x) \right)^{1/2} \left(K_n^* \left((s-x)^2 ; x \right) \right)^{1/2} \right\} \omega(f;\delta) \end{aligned} \tag{3.7}$$

Taking into account Lemma 2.1 and Lemma 3.1 in (3.7), we get the inequality (3.5) for $\delta = \delta_n = \frac{1}{\sqrt{n}}$. □

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Serhan Varma
e-mail: `svarma@science.ankara.edu.tr`
`smartsero@hotmail.com`
Ankara University Faculty of Science
Department of Mathematics
Tandoğan TR-06100
Ankara, Turkey