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Some sufficient conditions for starlike and convex functions

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Abstract. Using the technique of differential subordination, in particular, a lemma due to Miller and Mocanu [1], we study a certain differential operator to obtain some sufficient conditions for starlike, convex, strongly starlike and strongly convex functions. In particular, we prove that if $f \in \mathcal{A}_n$, $\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^\gamma \neq 0$, $z \in \mathbb{E}$ satisfies

$$\gamma\left(1+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}-\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) \prec \frac{2n(1-\alpha)z}{(1-z)(1+(1-2\alpha)z)},\ 0 \leq \alpha < 1,\ z \in \mathbb{E},$$

then

$$\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^\gamma \prec \frac{1+(1-2\alpha)z}{1-z},\ z \in \mathbb{E},$$

where $\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z),\ 0 \leq \lambda \leq 1$ is univalent and γ is a non-zero complex number.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \left\{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\right\}.$$

The class $\mathcal{A}_n$ of normalized analytic functions is defined as

$$\mathcal{A}_n = \left\{f \in \mathcal{H} : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots\right\}.$$

Let $\phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z); z) \prec h(z),\ \ \phi(p(0), 0; 0) = h(0), \tag{1.1}$$

then p is called a solution of the first order differential subordination (1.1). The univalent function q is called a dominant of the differential subordination (1.1) if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1).

In what follows, only principal values of complex powers are considered.

Irmak and Şan [2] introduced two differential operators $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ defined as under:

$$\mathcal{V}[\gamma, \lambda; f](z) = \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\gamma} \tag{1.2}$$

and

$$\mathcal{W}[\gamma, \lambda; f](z) = \gamma \frac{\mathcal{F}(z)}{\mathcal{F}'(z)} \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)' = \gamma \left(1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) \tag{1.3}$$

where $\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z),\ f \in \mathcal{A}_n,\ 0 \leq \lambda \leq 1$ is univalent and γ is a non-zero complex number and they proved the following results:

Theorem 1.1. *If $f \in \mathcal{A}_n$ satisfies the condition*

$$|\mathcal{W}[\gamma, \lambda; f](z)| < \beta,\ 0 < \beta \leq 1,\ z \in \mathbb{E},$$

then

$$|\arg\{\mathcal{V}[\gamma, \lambda; f](z)\}| < \frac{\pi}{2}\beta,$$

where $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

Theorem 1.2. *If $f \in \mathcal{A}_n$ satisfies*

$$\Re\{\mathcal{W}[\gamma, \lambda; f](z)\} < \frac{nM}{1+M}, M \geq 1,\ z \in \mathbb{E},$$

then

$$|\mathcal{V}[\gamma, \lambda; f](z) - 1| < M,$$

where $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

The main objective of this paper is to determine some sufficient conditions for starlike, convex, strongly starlike and strongly convex functions in terms of the operator defined above in (1.3). We claim that our results improve the above stated results of Irmak and Şan [2] and their consequences.

To prove our main result, we shall use the following lemma of Miller and Mocanu [1, pp. 76].

Lemma 1.3. *Let h be starlike in $\mathbb{E}$, with $h(0) = 0$ and $a \neq 0$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\frac{zp'(z)}{p(z)} \prec h(z),$$

then

$$p(z) \prec q(z) = a \exp\left[\frac{1}{n}\int_0^z \frac{h(t)}{t}\,dt\right],$$

and q is the best (a, n) dominant.

2. Main result and applications

Theorem 2.1. *Let h be starlike in $\mathbb{E}$, with $h(0) = 0$. Let $f \in \mathcal{A}_n$ be such that $\mathcal{V}[\gamma, \lambda; f](z) \neq 0,\ z \in \mathbb{E}$ and satisfy*

$$\mathcal{W}[\gamma, \lambda; f](z) \prec h(z),\ z \in \mathbb{E},$$

then

$$\mathcal{V}[\gamma, \lambda; f](z) \prec q(z) = \exp\left[\frac{1}{n}\int_0^z \frac{h(t)}{t}\, dt\right],$$

and q is the best dominant. Here $0 \leq \lambda \leq 1$, γ is a non-zero complex number and $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

Proof. Write $p(z) = \mathcal{V}[\gamma, \lambda; f](z)$, then a little calculation yields

$$\frac{zp'(z)}{p(z)} = \mathcal{W}[\gamma, \lambda; f](z).$$

Now the proof follows from Lemma 1.3. □

Select $h(z) = \dfrac{2n\beta z}{1 - z^2}$, $0 < \beta \leq 1$, $z \in \mathbb{E}$ in the above theorem, we obtain the following result.

Theorem 2.2. *Let γ be a non-zero complex number and $0 \leq \lambda \leq 1$. Let $f \in \mathcal{A}_n$ be such that $\mathcal{V}[\gamma, \lambda; f](z) \neq 0,\ z \in \mathbb{E}$ and satisfy*

$$\mathcal{W}[\gamma, \lambda; f](z) \prec \frac{2n\beta z}{1 - z^2} = h(z),\ 0 < \beta \leq 1,$$

where $h(\mathbb{E}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq n\beta\}$, then

$$\mathcal{V}[\gamma, \lambda; f](z) \prec \left(\frac{1+z}{1-z}\right)^{\beta},\ z \in \mathbb{E},$$

here $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

Remark 2.3. Note that the differential operator $\mathcal{W}[\gamma, \lambda; f](z)$ takes values in an extended region of the complex plane than the result of Irmak and Şan [2] stated in Theorem 1.1 to get the same conclusion.

Taking $\gamma - 1 = \lambda = 0$ and $\gamma = \lambda = 1$ in Theorem 2.2, we, respectively, obtain the following results.

Corollary 2.4. *(i) If $f \in \mathcal{A}_n$, $\dfrac{zf'(z)}{f(z)} \neq 0$, satisfies the inequality*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec h(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta},\ 0 < \beta \leq 1,\ z \in \mathbb{E},\ i.e.$$

f is strongly starlike of order β where $h(\mathbb{E}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq n\beta\}$.

(ii) If $f \in \mathcal{A}_n,\ 1 + \dfrac{zf''(z)}{f'(z)} \neq 0$, satisfies the inequality

$$1 + z\left(\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right) \prec h(z),$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta},\ 0 < \beta \leq 1,\ z \in \mathbb{E},\ i.e.$$

f is strongly convex of order β where $h(\mathbb{E}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq n\beta\}$.

Remark 2.5. Note that the results in above corollary extend the corresponding results of Irmak and Şan [2] (Corollary 2.1) in the sense that the differential operators

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \text{ and } 1 + z\left(\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right)$$

now take values in largely extended region of the complex plane as compared to their results.

When $h(z)$ in Theorem 2.1 is taken as $h(z) = \dfrac{n\alpha z}{1+\alpha z},\ 0 < \alpha \leq 1,\ z \in \mathbb{E}$, we have the following result.

Theorem 2.6. *For non-zero complex γ and $0 \leq \lambda \leq 1,\ 0 < \alpha \leq 1$, if $f \in \mathcal{A}_n$ with $\mathcal{V}[\gamma, \lambda; f](z) \neq 0,\ z \in \mathbb{E}$, satisfies*

$$\mathcal{W}[\gamma, \lambda; f](z) \prec \frac{n\alpha z}{1+\alpha z},\ z \in \mathbb{E},$$

then

$$|\mathcal{V}[\gamma, \lambda; f](z) - 1| < \alpha,\ z \in \mathbb{E},$$

where $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

Selecting $\gamma - 1 = \lambda = 0$ and $\gamma = \lambda = 1$ in Theorem 2.6, we, respectively, obtain:

Corollary 2.7. *(i) If $f \in \mathcal{A}_n,\ \dfrac{zf'(z)}{f(z)} \neq 0$, satisfies the inequality*

$$\Re\left\{z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right\} < \frac{(n-1)\alpha - 1}{1+\alpha},\ 0 < \alpha \leq 1,\ z \in \mathbb{E},$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \alpha.$$

(ii) If $f \in \mathcal{A}_n,\ 1 + \dfrac{zf''(z)}{f'(z)} \neq 0$, satisfies the inequality

$$\Re\left\{z\left(\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right)\right\} < \frac{(n-1)\alpha - 1}{1+\alpha},\ 0 < \alpha \leq 1,\ z \in \mathbb{E},$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < \alpha.$$

Write $h(z) = \dfrac{2n(1-\alpha)z}{(1-z)(1+(1-2\alpha)z)},\ 0 \le \alpha < 1,\ z \in \mathbb{E}$ in Theorem 2.1 to get the following result.

Theorem 2.8. *Let $f \in \mathcal{A}_n$ be such that $\mathcal{V}[\gamma,\lambda;f](z) \neq 0,\ z \in \mathbb{E}$ and satisfy*

$$\mathcal{W}[\gamma,\lambda;f](z) \prec \frac{2n(1-\alpha)z}{(1-z)(1+(1-2\alpha)z)},\ 0 \le \alpha < 1,\ z \in \mathbb{E},$$

then

$$\mathcal{V}[\gamma,\lambda;f](z) \prec \frac{1+(1-2\alpha)z}{1-z},\ z \in \mathbb{E},$$

for $0 \le \lambda \le 1$, γ a non-zero complex number and $\mathcal{V}[\gamma,\lambda;f](z)$ and $\mathcal{W}[\gamma,\lambda;f](z)$ are given by (1.2) and (1.3) respectively.

Note that above theorem, in turn, gives the following result of Irmak and Şan [2].

Corollary 2.9. *Let $0 \le \alpha < 1,\ 0 \le \lambda \le 1$ and γ be a non-zero complex number. Let $\mathcal{V}[\gamma,\lambda;f](z)$ and $\mathcal{W}[\gamma,\lambda;f](z)$ be given by (1.2) and (1.3) respectively. If $f \in \mathcal{A}_n$ with $\mathcal{V}[\gamma,\lambda;f](z) \neq 0,\ z \in \mathbb{E}$, satisfies*

$$\Re\{\mathcal{W}[\gamma,\lambda;f](z)\} > \begin{cases} \frac{n\alpha}{2(\alpha-1)}, & 0 \le \alpha \le 1/2, \\ \frac{n(\alpha-1)}{2\alpha}, & 1/2 \le \alpha < 1 \end{cases},$$

then

$$\Re\{\mathcal{V}[\gamma,\lambda;f](z)\} > \alpha,\ z \in \mathbb{E}.$$

Setting $\gamma - 1 = \lambda = 0$ and $\gamma = \lambda = 1$ in above corollary, we obtain, respectively, the following results which offer a correct version of Corollary 2.3 of Irmak and Şan [2].

Corollary 2.10. *(i) Let $0 \le \alpha < 1$ and let $f \in \mathcal{A}_n$, $\dfrac{zf'(z)}{f(z)} \neq 0$, satisfy the inequality*

$$\Re\left\{z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right\} > \begin{cases} \frac{(n-2)\alpha+2}{2(\alpha-1)}, & 0 \le \alpha \le 1/2, \\ \frac{(n-2)\alpha-n}{2\alpha}, & 1/2 \le \alpha < 1 \end{cases},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ \textit{i.e. } f \textit{ is starlike of order } \alpha.$$

(ii) Let $0 \le \alpha < 1$ and let $f \in \mathcal{A}_n$, $1 + \dfrac{zf''(z)}{f'(z)} \neq 0$, satisfy the inequality

$$\Re\left\{z\left(\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right)\right\} > \begin{cases} \frac{(n-2)\alpha+2}{2(\alpha-1)}, & 0 \le \alpha \le 1/2, \\ \frac{(n-2)\alpha-n}{2\alpha}, & 1/2 \le \alpha < 1 \end{cases},$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ \textit{i.e. } f \textit{ is convex of order } \alpha.$$

Setting $\gamma-1=\lambda=0,\ \gamma=\lambda=1$ and $\alpha=0$ in the Theorem 2.8, we, respectively, obtain the following results.

Corollary 2.11. *(i) Suppose* $f\in\mathcal{A}_n,\ \dfrac{zf'(z)}{f(z)}\neq 0$, *satisfies the condition*

$$z\left(\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right)\prec G(z),\ \textit{where } G(\mathbb{E})=\mathbb{C}\setminus\{w\in\mathbb{C}:\Re(w)=-1,|\Im(w)|\geq n\},$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{1+z}{1-z},\ z\in\mathbb{E},\ \textit{i.e. } f \textit{ is starlike in } \mathbb{E}.$$

(ii) Suppose $f\in\mathcal{A}_n,\ 1+\dfrac{zf''(z)}{f'(z)}\neq 0$, *satisfies the condition*

$$z\left(\frac{(zf'(z))''}{(zf'(z))'}-\frac{(zf'(z))'}{zf'(z)}\right)\prec G(z),$$

then

$$1+\frac{zf''(z)}{f'(z)}\prec\frac{1+z}{1-z},\ z\in\mathbb{E},\ \textit{i.e. } f \textit{ is convex in } \mathbb{E},$$

where $G(\mathbb{E})=\mathbb{C}\setminus\{w\in\mathbb{C}:\Re(w)=-1,|\Im(w)|\geq n\}$.

References

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