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Certain subclasses of analytic univalent functions generated by harmonic univalent functions

Santosh Joshi and Girish Shelake

Abstract. In this paper we define and investigate subclasses of analytic univalent functions generated by harmonic univalent and sense-preserving mappings.We obtain some inclusion theorems and convolution characterizations for above subclasses of univalent functions.

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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we write

$$f = h + \bar{g} \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D, see [6].

Every harmonic function $f = h + \bar{g}$ is uniquely determined by the coefficients of power series expansions in the unit disk $U = \{z : |z| < 1\}$ given by

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad z \in U, |B_1| < 1, \tag{1.2}$$

where $A_n \in \mathbb{C}$ for $n = 2, 3, 4, \ldots$ and $B_n \in \mathbb{C}$ for $n = 1, 2, 3, \ldots$. For further information about these mappings, one may refer to [4, 6, 7].

In 1984, Clunie and Sheil-Small [6] studied the family S_H of all univalent sense-preserving harmonic functions f of the form (1.1) in U, such that h and g are represented by (1.2). Note that S_H reduces to the well-known family S, the class of all normalized analytic univalent functions h given in (1.2), whenever the co-analytic part g of f is zero. Let K and K_H be the subclasses of S and S_H respectively such that images of $f(U)$ are convex.

In last two decades, several researchers have defined various subclasses of S using subordination. For the functions h and F analytic in U, we say h is subordinate to F $(h \prec F)$, if there exists an analytic function w in the unit disk U, with $w(0) = 0$ and $|w(z)| < 1$ such that $h(z) = F(w(z))$ for all z in U. Using subordination, we define two subclasses of S as follows:

$$S^*[A, B, \alpha, \gamma] = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \frac{1 + \gamma[B + (A - B)(1 - \alpha)]z}{1 + \gamma Bz}, z \in U \right\},$$

$$K[A, B, \alpha, \gamma] = \left\{ f \in S : \frac{(zf'(z))'}{f'(z)} \prec \frac{1 + \gamma[B + (A - B)(1 - \alpha)]z}{1 + \gamma Bz}, z \in U \right\},$$

where $0 \leq \alpha < 1, 0 < \gamma \leq 1, -1 \leq B < \gamma(B + (A - B)(1 - \alpha)) < A \leq 1$. Note that the condition $|B| \leq 1$ implies that the function $[1 + \gamma(B + (A - B)(1 - \alpha))z][1 + \gamma Bz]^{-1}$ is convex and univalent in U. For different values of parameters A, B, α and γ one can obtain several subclasses of S. For $\gamma = 1$ we get the subclasses defined by S. Joshi et.al[8].

Note that the convex domains are those domains that are convex in every direction. The following lemma will motivate us to construct certain analytic univalent function associated with $f \in S_H$.

Lemma 1.1 ([5, 6])**.** *A harmonic function $f = h + \bar{g}$ locally univalent in U is a univalent mapping of U and $f \in K_H$ if and only if $h - g$ is an analytic univalent mapping of U onto a domain convex in the direction of the real axis.*

For $f = h + \bar{g}$ in S_H, where h and g are given by (1.2), Lemma 1.1 led us to construct the function t with suitable normalization, given by

$$t(z) = \frac{h(z) - g(z)}{1 - B_1} = z + \sum_{n=2}^{\infty} \frac{A_n - B_n}{1 - B_1} z^n, \quad z \in U. \tag{1.3}$$

Since $f \in S_H$ is sense-preserving, it follows that $|B_1| < 1$. Hence the function t belongs to S. This observation has prompted us to define the following classes:

$$S_H[A, B, \alpha, \gamma] := \{f = h + \bar{g} \in S_H : t \in S^*[A, B, \alpha, \gamma]\},$$

$$K_H[A, B, \alpha, \gamma] := \{f = h + \bar{g} \in S_H : t \in K[A, B, \alpha, \gamma]\}.$$

In [2], Ahuja O.P connected hypergeometric functions with harmonic mappings $f = h + \bar{g}$ by defining the convolution operator Ω by

$$\Omega(f) := f \tilde{*} (\phi_1 + \bar{\phi}_2) = h * \phi_1 + \overline{g * \phi_2},$$

where $*$ denotes the convolution product of two power series and ϕ_1, ϕ_2 are defined by

$$\phi_1(z) = zF(a_1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^n,$$

$$\phi_2(z) = zF(a_2, b_2; c_2; z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} z^n.$$

Here $F(a, b; c; z)$ is a well-known hypergeometric function and a's, b's, c's are complex parameters with $c \neq 0, -1, -2, \ldots$. Corresponding to any function $f = h + \bar{g}$ given by (1.2), we have $\Omega(f) = H + \bar{G}$, where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n \quad \text{and}$$
$$G(z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n z^n, |B_1| < 1. \tag{1.4}$$

We will frequently use the Gauss summation formula

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \mathrm{Re}(c-a-b) > 0.$$

In the present paper, we study certain connections of the mappings $f = h + \bar{g}$ in S_H with the corresponding analytic functions in the classes $S^*[A, B, \alpha, \gamma]$ and $K[A, B, \alpha, \gamma]$. More precisely, we obtain some inclusion theorems and convolution characterization theorems for the classes $S_H[A, B, \alpha, \gamma]$ and $K_H[A, B, \alpha, \gamma]$.

2. Lemmas

Lemma 2.1. *A function h defined by the first equation in (1.2) is in $S^*[A, B, \alpha, \gamma]$ if*

$$\sum_{n=2}^{\infty} \{(n-1)(1+\gamma|B|) + \gamma(A-B)(1-\alpha)\}|A_n| \leq (A-B)(1-\alpha)\gamma.$$

Proof. In view of definition of $S^*[A, B, \alpha, \gamma]$, it follows that $h \in S^*[A, B, \alpha, \gamma]$ if and only if there exists an analytic function w such that

$$\frac{zh'(z)}{h(z)} = \frac{1 + \gamma[B + (A-B)(1-\alpha)]w(z)}{1 + \gamma B w(z)},$$

with $w(0) = 0$ and $|w(z)| < |z|$. Since $|w(z)| < 1$, the above equation is equivalent to

$$\left| \frac{\frac{zh'(z)}{h(z)} - 1}{\gamma[B + (A-B)(1-\alpha)] - \gamma B \frac{zh'(z)}{h(z)}} \right| < 1, \ z \in U.$$

On the other hand, on $|z| = 1$ we have

$$
\begin{aligned}
&|zh'(z) - h(z)| - |[B + (A-B)(1-\alpha)]h(z) - Bzh'(z)|\gamma \\
&= \left|\sum_{n=2}^{\infty}(n-1)A_n z^n\right| \\
&\quad - \gamma\left|(A-B)(1-\alpha)z - \sum_{n=2}^{\infty}[(n-1)B - (A-B)(1-\alpha)]A_n z^n\right| \\
&\le \sum_{n=2}^{\infty}[(n-1)(1+\gamma|B|) + \gamma(A-B)(1-\alpha)]|A_n| - (A-B)(1-\alpha)\gamma \\
&\le 0,
\end{aligned}
$$

provided the given condition holds. Hence from the maximum modulus Theorem it follows that $h \in S^*[A, B, \alpha, \gamma]$. □

Lemma 2.2. *A function h defined by the first equation in (1.2) is in $K[A, B, \alpha, \gamma]$ if*

$$\sum_{n=2}^{\infty} n\{(n-1)(1+\gamma|B|) + \gamma(A-B)(1-\alpha)\}|A_n| \le (A-B)(1-\alpha)\gamma.$$

Proof. From the definition of $K[A, B, \alpha, \gamma]$, it follows that $h \in K[A, B, \alpha, \gamma]$ if and only if there exists an analytic function w such that

$$\frac{(zh'(z))'}{h'(z)} = \frac{1 + \gamma[B + (A-B)(1-\alpha)]w(z)}{1 + \gamma B w(z)},$$

with $w(0) = 0$ and $|w(z)| < |z| < 1$. This equality is equivalent to

$$\left|\frac{\frac{(zh'(z))'}{h'(z)} - 1}{\gamma[B + (A-B)(1-\alpha)] - \gamma B\frac{(zh'(z))'}{h'(z)}}\right| < 1, \ z \in U$$

□

The remaining steps of the proof are similar to the proof of Lemma 2.1.

Lemma 2.3 ([2]). *Let $f = h + \bar{g}$ where h and g are analytic functions of the form (1.2). If $a_j, b_j \in \mathbb{C}\backslash\{0\}, c_j \in \mathbb{R}$ are such that $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$ and the following inequalities*

(i) $\displaystyle\sum_{n=2}^{\infty}|A_n| + \sum_{n=1}^{\infty}|B_n| \le 1, |B_1| < 1,$

(ii) $\displaystyle\sum_{j=1}^{2}\left(\frac{|a_j b_j|}{c_j - |a_j| - |b_j| - 1} + 1\right)F(|a_j|, |b_j|; c_j; 1) \le 2$

are satisfied, then $\Omega(f)$ is sense-preserving harmonic and univalent in U; and so $\Omega(f) \in S_H$.

Lemma 2.4 ([2]). *If $a, b, c > 0$, then*

(i) $\displaystyle\sum_{n=1}^{\infty}(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{ab}{c-a-b-1}F(a,b;c;1)$ *if* $c > a+b+1$,

(ii) $\displaystyle\sum_{n=2}^{\infty}(n-1)^2\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \left(\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right)F(a,b;c;1)$
if $c > a+b+2$.

3. Main results

Theorem 3.1. *Let $f = h + \bar{g}$ be of the form (1.2), and for $j = 1, 2$, suppose $a_j, b_j \in \mathbb{C}\backslash\{0\}, c_j \in \mathbb{R}$ are such that $c_j > |a_j| + |b_j| + 1$ and $\Omega(f) \in S_H$. If the coefficient conditions*

(i) $\displaystyle\sum_{n=2}^{\infty}|A_n| + \sum_{n=1}^{\infty}|B_n| \leq 1$

(ii) $\displaystyle\sum_{j=1}^{2}\left(\frac{(1+\gamma|B|)}{\gamma(A-B)(1-\alpha)}\frac{|a_j b_j|}{c_j-|a_j|-|b_j|-1} + 1\right)F(|a_j|,|b_j|;c_j;1)$
$\leq (2 + |1-B_1|) < 4$

are satisfied, then $\Omega(f) \in S_H[A, B, \alpha, \gamma]$.

Proof. In order to prove that $\Omega(f) \in S_H[A, B, \alpha, \gamma]$, it suffices to prove that the function

$$\begin{aligned} T(z) &:= \frac{H(z)-G(z)}{1-B_1} \\ &= z + \sum_{n=2}^{\infty}\left[\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_n - \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}B_n\right]\frac{1}{1-B_1}z^n \end{aligned} \tag{3.1}$$

is in $S^*[A, B, \alpha, \gamma]$. Note that $|A_n| \leq 1$ and $|B_n| \leq 1$, by the condition (i). As an application of the Lemma 2.1, the function $T \in S^*[A, B, \alpha, \gamma]$ provided that $Q_1 \leq 1$, where

$$\begin{aligned} Q_1 &:= \sum_{n=2}^{\infty}\left[\frac{(n-1)(1+\gamma|B|)+\gamma(A-B)(1-\alpha)}{(A-B)(1-\alpha)\gamma}\right] \\ &\quad\cdot\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\frac{A_n}{1-B_1} - \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\frac{B_n}{1-B_1}\right| \\ &\leq \sum_{n=2}^{\infty}\left[\frac{(n-1)(1+\gamma|B|)+\gamma(A-B)(1-\alpha)}{\gamma(A-B)(1-\alpha)|1-B_1|}\right] \\ &\quad\cdot\left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right) \\ &= \frac{(1+\gamma|B|)}{|1-B_1|(A-B)(1-\alpha)\gamma} \end{aligned}$$

$$\cdot \sum_{n=2}^{\infty}(n-1)\left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right)$$
$$+\frac{1}{|1-B_1|}\sum_{n=2}^{\infty}\left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right)$$
$$=\frac{(1+\gamma|B|)}{|1-B_1|(A-B)(1-\alpha)\gamma}\left(\frac{|a_1b_1|}{c_1-|a_1|-|b_1|-1}F(|a_1|,|b_1|;c_1;1)\right.$$
$$\left.+\frac{|a_2b_2|}{c_2-|a_2|-|b_2|-1}F(|a_2|,|b_2|;c_2;1)\right)$$
$$+\frac{1}{|1-B_1|}(F(|a_1|,|b_1|;c_1;1)+F(|a_2|,|b_2|;c_2;1)-2)$$

by Lemma 2.3. Therefore, it follows that $T \in S^*[A,B,\alpha,\gamma]$ if the inequality

$$\frac{1}{|1-B_1|}\sum_{j=1}^{2}\left(\frac{(1+\gamma|B|)}{(A-B)(1-\alpha)\gamma}\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right)$$
$$\cdot F(|a_j|,|b_j|;c_j;1)-\frac{2}{|1-B_1|}\leq 1$$

holds. But this inequality is true because of given condition (ii). □

Theorem 3.2. *Let $f = h + \bar{g}$ given by (1.2) be in S_H. If the inequality*

$$\sum_{n=2}^{\infty}\{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma\}|A_n|$$
$$+\sum_{n=1}^{\infty}\{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma\}|B_n|\leq (A-B)(1-\alpha)\gamma|1-B_1|$$

is satisfied, then $f \in S_H[A,B,\alpha,\gamma]$.

Proof. From the definition of $S_H[A,B,\alpha,\gamma]$, it suffices to prove that the function t given by (1.3) is in the class $S^*[A,B,\alpha,\gamma]$. As an application of Lemma 2.1, we only need to show that $Q_2 \leq 1$, where

$$Q_2 := \sum_{n=2}^{\infty}\frac{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma}\left|\frac{A_n-B_n}{1-B_1}\right|.$$

But

$$Q_2 \leq \sum_{n=2}^{\infty}\frac{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma}\left[\frac{|A_n|+|B_n|}{|1-B_1|}\right]$$

and thus $Q_2 \leq 1$ holds because of the given condition. □

Theorem 3.3. *Let $f = h + \bar{g}$ be of the form (1.2) and for $j = 1, 2$, suppose $a_j, b_j \in \mathbb{C}\backslash\{0\}, c_j \in \mathbb{R}$ such that $c_j > |a_j|+|b_j|+2$ and $\Omega(f) \in S_H$. If the coefficient conditions*

(i) $\sum_{n=2}^{\infty}|A_n| + \sum_{n=1}^{\infty}|B_n| \leq 1,$

(ii) $\sum_{j=1}^{2}\left\{\frac{(1+\gamma B)}{(A-B)(1-\alpha)\gamma}\frac{(|a_j|)_2(|b_j|)_2}{(c_j-|a_j|-|b_j|-2)_2}+\left(\frac{2(1+\gamma|B|)}{(A-B)(1-\alpha)\gamma}+1\right)\right.$
$\left.\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right\}F(|a_j|,|b_j|;c_j;1)\leq 2+|1-B_1|<4$
are satisfied, then $\Omega(f)\in K_H[A,B,\alpha,\gamma]$.

Proof. In view of the definition of $K_H[A,B,\alpha,\gamma]$ and the fact that $\Omega(f)\in S_H$, it suffices to prove that the function T given by (3.1) is in $K[A,B,\alpha,\gamma]$. Note that by the condition (i) we have $|A_n|\leq 1$ and $|B_n|\leq 1$. In the view of Lemma (2.2), the function $T\in K[A,B,\alpha,\gamma]$ provided that $Q_3\leq 1$, where

$$Q_3:=\sum_{n=2}^{\infty}n\left[\frac{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma}\right]$$
$$\cdot\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\frac{A_n}{1-B_1}-\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\frac{B_n}{1-B_1}\right|$$
$$\leq\sum_{n=2}^{\infty}n\left[\frac{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma|1-B_1|}\right]$$
$$\cdot\left[\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right]$$
$$=\frac{1+\gamma|B|}{|1-B_1|(A-B)(1-\alpha)\gamma}\sum_{n=2}^{\infty}[(n-1)^2+(n-1)](D_1+D_2)$$
$$+\frac{1}{|1-B_1|}\sum_{n=2}^{\infty}(n-1)(D_1+D_2)+\frac{1}{|1-B_1|}\sum_{n=2}^{\infty}(D_1+D_2)$$
$$=\frac{1}{|1-B_1|}\left[\begin{array}{l}\frac{1+\gamma|B|}{(A-B)(1-\alpha)\gamma}\sum_{n=2}^{\infty}(n-1)^2(D_1+D_2)\\+\left(\frac{1+\gamma|B|}{(A-B)(1-\alpha)\gamma}+1\right)\sum_{n=2}^{\infty}(n-1)(D_1+D_2)\\+\sum_{n=2}^{\infty}(D_1+D_2)\end{array}\right],$$

where $D_j=\frac{(|a_j|)_{n-1}(|b_j|)_{n-1}}{(c_j)_{n-1}(1)_{n-1}}$ for $j=1,2$.

Using Lemma 2.4, we find that

$$Q_3\leq\frac{1}{|1-B_1|}\sum_{j=1}^{2}\left\{\frac{(1+\gamma|B|)}{(A-B)(1-\alpha)\gamma}\frac{(|a_j|)_2(|b_j|)_2}{(c_j-|a_j|-|b_j|-2)_2}+\left(\frac{2(1+\gamma|B|)}{(A-B)(1-\alpha)\gamma}+1\right)\right.$$
$$\left.\cdot\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right\}F(|a_j|,|b_j|;c_j;1)-\frac{2}{|1-B_1|},$$

and this proves that $Q_3 \leq 1$, if the condition (ii) holds. □

The proof of the next theorem is similar to the proof of Theorem 3.2 and hence it is omitted.

Theorem 3.4. *Let $f = h + \bar{g}$ given by (1.2) be in S_H. If the inequality*

$$\sum_{n=2}^{\infty} n\{(n-1)((1+\gamma|B|) + (A-B)(1-\alpha)\gamma\}|A_n|$$

$$+\sum_{n=1}^{\infty} n\{(n-1)(1+\gamma|B|) + (A-B)(1-\alpha)\gamma\}|B_n| \leq (A-B)(1-\alpha)\gamma$$

is satisfied, then $f \in K_H[A, B, \alpha, \gamma]$.

The next two Theorems give characterizations of functions in $S_H[A, B, \alpha, \gamma]$ and $K_H[A, B, \alpha, \gamma]$.

Theorem 3.5. *If $f(z) = h(z) + \overline{g(z)} \in S_H$ then $f \in S_H[A, B, \alpha, \gamma]$ if and only if*

$$\frac{1}{z}[(h(z) - g(z)) * F_1(z)] \neq 0$$

for all z in U and all ξ, such that $|\xi| = 1$, where

$$F_1(z) := \frac{z + \left(\frac{\xi - (B + (A-B)(1-\alpha))\gamma}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^2}.$$

Proof. By definition of $S_H[A, B, \alpha, \gamma]$, it is obvious that $f \in S_H[A, B, \alpha, \gamma]$ if and only if $t(z)$ given by (1.3) belongs to $S^*[A, B, \alpha, \gamma]$. But, $t \in S^*[A, B, \alpha, \gamma]$ if and only if

$$\frac{zt'(z)}{t(z)} \prec \frac{1 + (B + (A-B)(1-\alpha))\gamma z}{1 + \gamma B z},$$

that is

$$\frac{zt'(z)}{t(z)} \neq \frac{1 + (B + (A-B)(1-\alpha))\gamma\varsigma}{1 + \gamma B\varsigma}$$

for $z \in U$ and $|\varsigma| = 1$, which is equivalent to

$$\frac{1}{z}[(1 + \gamma B\varsigma)zt' - (1 + (B + (A-B)(1-\alpha))\gamma\varsigma)t] \neq 0.$$

Since

$$zt' = t * \frac{z}{(1-z)^2}, \quad t = t * \frac{z}{1-z},$$

the above inequality is equivalent to

$$\frac{1}{z}\left[t(z) * \left[\frac{-\gamma(A-B)(1-\alpha)\varsigma z + [1 + (B + (A-B)(1-\alpha))\gamma\varsigma]z^2}{(1-z)^2}\right]\right]$$

$$= \frac{-\gamma(A-B)(1-\alpha)\varsigma}{(1-B_1)z}\left[(h(z) - g(z)) * \left(\frac{z + \left(\frac{\xi - \gamma(B + (A-B)(1-\alpha))}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^2}\right)\right] \neq 0,$$

where $|-1/\varsigma| = |\xi| = 1$, and the result follows. □

Corollary 3.6. *If $f(z) = h(z) + \overline{g(z)} \in S_H$, then $f \in K_H[A, B, \alpha, \gamma]$ if and only if*

$$\frac{1}{z}[(h(z) - g(z)) * F_2(z)] \neq 0,$$

for all z in U and all ξ, such that $|\xi| = 1$, where

$$F_2(z) := \frac{z + \left(\frac{2\xi-(2B+(A-B)(1-\alpha))\gamma}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^3}.$$

Proof. Note that $t \in K[A, B, \alpha, \gamma]$ if and only $zt'(z) \in S_H[A, B, \alpha, \gamma]$. If we let

$$p(z) = \frac{z + \left(\frac{\xi-(B+(A-B)(1-\alpha))\gamma}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^2},$$

we note that

$$zp'(z) = \frac{z + \left(\frac{2\xi-2B\gamma-(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^3}.$$

Using the identity $zt' * p = t * zp'$, the result follows from Theorem 3.5. □

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Santosh Joshi
Department of Mathematics
Walchand College, Sangli-416415
Maharashtra, India
e-mail: joshisb@hotmail.com

Girish Shelake
Department of Mathematics
Willingdon College, Sangli-416415
Maharashtra, India
e-mail: shelakegd@gmail.com