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Multivariate generalised fractional Pólya type integral inequalities

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Abstract. Here we present a set of multivariate generalised fractional Pólya type integral inequalities on the ball and shell. We treat both the radial and non-radial cases in all possibilities. We give also estimates for the related averages.

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1. Introduction

We mention the following famous Pólya's integral inequality, see [9], [10, p. 62], [11] and [12, p. 83].

Theorem 1.1. *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then the exists at least one point $\xi \in [a,b]$ such that*

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x)\,dx. \tag{1.1}$$

In [13], Feng Qi presents the following very interesting Pólya type integral inequality (1.2), which generalizes (1.1).

Theorem 1.2. *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a) = f(b) = 0$ and $M = \sup_{x\in[a,b]} |f'(x)|$. Then*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4} M, \tag{1.2}$$

where $\frac{(b-a)^2}{4}$ in (1.2) is the best constant.

The above motivate the current paper.

In this article we present multivariate fractional Pólya type integral inequalities in various cases, similar to (1.2).

For the last we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$ ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a, b])$, $[a, b] \subset \mathbb{R}$, $x \in [a, b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{1.3}$$

$a \leq x \leq b$. We define the subspace $C_{a+}^\alpha([a.b])$ of $C^m([a, b])$:

$$C_{a+}^\alpha([a, b]) = \left\{ f \in C^m([a, b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a, b]) \right\}. \tag{1.4}$$

For $f \in C_{a+}^\alpha([a, b])$, we define the left generalized α-fractional derivative of f over $[a, b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)}\right)', \tag{1.5}$$

see [1], p. 24. Canavati first in [5], introduced the above over $[0, 1]$.

Notice that $D_{a+}^\alpha f \in C([a, b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [5] the same over $[0, 1]$ that appeared first.

Theorem 1.3. *Let $f \in C_{a+}^\alpha([a, b])$.*

(i) If $\alpha \geq 1$, then

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \ldots + f^{(m-1)}(a)\frac{(x-a)^{m-1}}{(m-1)!} \tag{1.6}$$

$$+\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \qquad \text{all } x \in [a, b].$$

(ii) If $0 < \alpha < 1$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \qquad \text{all } x \in [a, b]. \tag{1.7}$$

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a, b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{1.8}$$

$x \in [a, b]$, see also [2], [6], [7], [8], [15]. Define the subspace of functions

$$C_{b-}^\alpha([a, b]) := \left\{ f \in C^m([a, b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a, b]) \right\}. \tag{1.9}$$

Define the right generalized α-fractional derivative of f over $[a, b]$ as

$$D_{b-}^\alpha f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)}\right)', \tag{1.10}$$

see [2]. We set $D_{b-}^{0}f = f$. Notice that $D_{b-}^{\alpha}f \in C([a,b])$.

From [2], we need the following right Taylor fractional formula.

Theorem 1.4. *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

(i) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x), \quad \text{all } x \in [a,b]. \tag{1.11}$$

(ii) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(t)\, dt, \quad \text{all } x \in [a,b]. \tag{1.12}$$

We need from [3]

Definition 1.5. *Let $f \in C([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a+}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{1.13}$$

In [4] we proved the following fractional Pólya type integral inequality without any boundary conditions.

Theorem 1.6. *Let $0 < \alpha < 1$, $f \in C([a,b])$. Assume $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ and $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. Set*

$$M_1(f) = \max\left\{ \left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}. \tag{1.14}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq \tag{1.15}$$

$$\frac{\left(\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2}\right)^{\alpha+1} \leq M_1(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\, 2^{\alpha}}. \tag{1.16}$$

Inequalities (1.15), (1.16) are sharp, namely they are attained by

$$f_*(x) = \left\{ \begin{array}{ll} (x-a)^{\alpha}, & x \in \left[a, \frac{a+b}{2}\right] \\ (b-x)^{\alpha}, & x \in \left[\frac{a+b}{2}, b\right] \end{array} \right\}, \quad 0 < \alpha < 1. \tag{1.17}$$

Clearly here non zero constant functions f are excluded.

The last result also motivates this work.

Remark 1.7. (see [4]) When $\alpha \geq 1$, thus $m = [\alpha] \geq 1$, and by assuming that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, we can prove the same statements (1.15), (1.16), (1.17) as in Theorem 1.6. If we set there $\alpha = 1$ we derive exactly Theorem 1.2. So we have generalized Theorem 1.2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.

We present Pólya type integral inequalities on the ball and shell.

2. Main results

We need

Remark 2.1. We define the ball $B(0,R) = \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, and the sphere

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\},$$

where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$. Note that $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ is the Lebesgue measure of the ball.

Following [14, pp. 149-150, exercise 6] and [16, pp. 87-88, Theorem 5.2.2] we can write $F : \overline{B(0,R)} \to \mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr \right) d\omega; \tag{2.1}$$

we use this formula a lot.

Initially the function $f : \overline{B(0,R)} \to \mathbb{R}$ is radial; that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $\forall\, x \in \overline{B(0,R)}$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Here we assume that $g \in C([0,R])$ with $g \in C_{0+}^{\alpha}\left(\left[0, \frac{R}{2}\right]\right)$ and $g \in C_{R-}^{\alpha}\left(\left[\frac{R}{2}, R\right]\right)$, such that $g^{(k)}(0) = g^{(k)}(R) = 0$, $k = 0, 1, ..., m-1$. In case of $0 < \alpha < 1$ then the last boundary conditions are void.

By assumption here and Theorem 1.3 we have

$$g(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} \left(D_{0+}^{\alpha} g\right)(t)\, dt, \tag{2.2}$$

all $s \in \left[0, \frac{R}{2}\right]$,

also it holds, by assumption and Theorem 1.4, that

$$g(s) = \frac{1}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} \left(D_{R-}^{\alpha} g\right)(t)\, dt, \tag{2.3}$$

all $s \in \left[\frac{R}{2}, R\right]$.

By (2.2) we get

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} \left|\left(D_{0+}^{\alpha} g\right)(t)\right| dt$$

$$\leq \left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]} \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} dt = \frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\Gamma(\alpha+1)} s^{\alpha}, \tag{2.4}$$

for any $s \in \left[0, \frac{R}{2}\right]$.

That is

$$|g(s)| \leq \frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\Gamma(\alpha+1)} s^{\alpha}, \tag{2.5}$$

for any $s \in \left[0, \frac{R}{2}\right]$.

Similarly we obtain

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} \left|\left(D_{R-}^{\alpha} g\right)(t)\right| dt$$

$$\leq \frac{\left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} dt = \frac{\left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha+1)} (R-s)^{\alpha}, \tag{2.6}$$

for any $s \in \left[\frac{R}{2}, R\right]$.

I.e. it holds

$$|g(s)| \leq \frac{\left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha+1)} (R-s)^{\alpha}, \tag{2.7}$$

for any $s \in \left[\frac{R}{2}, R\right]$.

Next we observe that

$$\left|\int_{B(0,R)} f(y)\, dy\right| \leq \int_{B(0,R)} |f(y)|\, dy \overset{(2.1)}{=}$$

$$\int_{S^{N-1}} \left(\int_0^R |g(s)|\, s^{N-1} ds\right) d\omega = \left(\int_0^R |g(s)|\, s^{N-1} ds\right) \int_{S^{N-1}} d\omega =$$

$$\left(\int_0^R |g(s)|\, s^{N-1} ds\right) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} = \tag{2.8}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{\int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^R |g(s)|\, s^{N-1} ds\right\} \overset{\text{(by (2.5) and (2.7))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)} \left\{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]} \int_0^{\frac{R}{2}} s^{\alpha+N-1} ds +\right.$$

$$\left.\left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]} \int_{\frac{R}{2}}^R (R-s)^{\alpha} \left(\left(s-\frac{R}{2}\right)+\frac{R}{2}\right)^{N-1} ds\right\} = \tag{2.9}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)} \left\{\frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)} \left(\frac{R}{2}\right)^{\alpha+N} + \left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]} \cdot\right.$$

$$\left.\left[\sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{R}{2}\right)^k \int_{\frac{R}{2}}^R (R-s)^{(\alpha+1)-1} \left(s-\frac{R}{2}\right)^{N-k-1} ds\right]\right\} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)} \left\{\frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)} \left(\frac{R}{2}\right)^{\alpha+N} +\right. \tag{2.10}$$

$$\left.\left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]} \left[\sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{R}{2}\right)^k \frac{\Gamma(\alpha+1)\Gamma(N-k)}{\Gamma(\alpha+N+1-k)} \left(\frac{R}{2}\right)^{\alpha+N-k}\right]\right\} =$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\,\Gamma(\alpha+1)}+\right.$$

$$\left.\left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]}(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}. \tag{2.11}$$

We have proved that

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \int_{B(0,R)} |f(y)|\,dy \leq$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\,\Gamma(\alpha+1)}+\right. \tag{2.12}$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}.$$

Consider now

$$g_*(s)=\begin{cases} s^{\alpha}, & s\in\left[0,\frac{R}{2}\right], \\ (R-s)^{\alpha}, & s\in\left[\frac{R}{2},R\right],\ \alpha>0. \end{cases} \tag{2.13}$$

We have as in [4] that

$$D_{0+}^{\alpha} s^{\alpha}=\Gamma(\alpha+1),\ \text{ all } s\in\left[0,\frac{R}{2}\right], \tag{2.14}$$

and

$$\left\|D_{0+}^{\alpha} s^{\alpha}\right\|_{\infty,\left[0,\frac{R}{2}\right]}=\Gamma(\alpha+1).$$

Similarly as in [4] we get

$$D_{R-}^{\alpha}(R-s)^{\alpha}=\Gamma(\alpha+1),\ \text{ all } s\in\left[\frac{R}{2},R\right], \tag{2.15}$$

and

$$\left\|D_{R-}^{\alpha}(R-s)^{\alpha}\right\|_{\infty,\left[\frac{R}{2},R\right]}=\Gamma(\alpha+1). \tag{2.16}$$

That is

$$\left\|D_{0+}^{\alpha} g_*\right\|_{\infty,\left[0,\frac{R}{2}\right]}=\left\|D_{R-}^{\alpha} g_*\right\|_{\infty,\left[\frac{R}{2},R\right]}=\Gamma(\alpha+1). \tag{2.17}$$

Consequently we find that

$$\text{R.H.S. (2.12)}=\frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{1}{(\alpha+N)}+\right.$$

$$\left.(N-1)!\Gamma(\alpha+1)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}. \tag{2.18}$$

Let $f_*:\overline{B(0,R)}\to\mathbb{R}$ be radial such that $f_*(x)=g_*(s)$, $s=|x|$, $s\in[0,R]$, $\forall\, x\in\overline{B(0,R)}$.

Then we have

$$\text{L.H.S. } (2.12) = \int_{B(0,R)} f_*(y)\,dy \overset{(2.1)}{=}$$

$$\left(\int_0^R g_*(s)\,s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_0^{\frac{R}{2}} s^{\alpha+N-1}ds + \int_{\frac{R}{2}}^R (R-s)^\alpha s^{N-1}ds\right\} = \tag{2.19}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\left(\frac{R}{2}\right)^{\alpha+N}\frac{1}{(\alpha+N)} + \int_{\frac{R}{2}}^R (R-s)^\alpha\left(\left(s-\frac{R}{2}\right)+\frac{R}{2}\right)^{N-1}ds\right\} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\frac{R^{\alpha+N}}{2^{\alpha+N}(\alpha+N)}+\right.$$

$$\left.\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^k\int_{\frac{R}{2}}^R (R-s)^{(\alpha+1)-1}\left(s-\frac{R}{2}\right)^{N-k-1}ds\right\} = \tag{2.20}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\frac{R^{\alpha+N}}{2^{\alpha+N}(\alpha+N)}+\right.$$

$$\left.\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^k\frac{\Gamma(\alpha+1)\Gamma(N-k)}{\Gamma(\alpha+N+1-k)}\left(\frac{R}{2}\right)^{\alpha+N-k}\right\} =$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\left\{\frac{1}{(\alpha+N)}+\right.$$

$$\left.(N-1)!\Gamma(\alpha+1)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\} \overset{(2.18)}{=} \text{R.H.S. } (2.12), \tag{2.21}$$

proving (2.12) sharp, in fact it is attained.

We have proved the following main result.

Theorem 2.2. *Let* $f : \overline{B(0,R)} \to \mathbb{R}$ *be radial; that is, there exists a function* g *such that* $f(x) = g(s)$, $s = |x|$, $s \in [0,R]$, $\forall\, x \in \overline{B(0,R)}$, $\alpha > 0$. *Assume that* $g \in C([0,R])$, *with* $g \in C_{0+}^\alpha\left(\left[0,\frac{R}{2}\right]\right)$ *and* $g \in C_{R-}^\alpha\left(\left[\frac{R}{2},R\right]\right)$, *such that* $g^{(k)}(0) = g^{(k)}(R) = 0$, $k = 0,1,...,m-1$, $m = \lceil\alpha\rceil$. *When* $0 < \alpha < 1$ *the last boundary conditions are void. Then*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \le \int_{B(0,R)} |f(y)|\,dy \le$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^\alpha g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right. \tag{2.22}$$

$$\left.(N-1)!\left\|D_{R-}^\alpha g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}.$$

Inequalities (2.22) are sharp, namely they are attained by a radial function f_* such that $f_*(x) = g_*(s)$, all $s \in [0, R]$, where

$$g_*(s) = \begin{cases} s^\alpha, & s \in \left[0, \frac{R}{2}\right], \\ (R-s)^\alpha, & s \in \left[\frac{R}{2}, R\right]. \end{cases} \tag{2.23}$$

We continue with

Remark 2.3. (Continuation of Remark 2.1) Here we assume that $\alpha \geq 1$. By (2.2) we get

$$|g(s)| \leq \frac{s^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{0+}^\alpha g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)}, \tag{2.24}$$

all $s \in \left[0, \frac{R}{2}\right]$.

Also, by (2.3), we obtain

$$|g(s)| \leq \frac{(R-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{R-}^\alpha g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)}, \tag{2.25}$$

all $s \in \left[\frac{R}{2}, R\right]$.

Hence as in (2.8) we get

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left(\int_0^R |g(s)|\, s^{N-1} ds \right) = \tag{2.26}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^R |g(s)|\, s^{N-1} ds \right\} \overset{\text{(by (2.24), (2.25))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)} \left\{ \left(\int_0^{\frac{R}{2}} s^{N+\alpha-2} ds \right) \left\| D_{0+}^\alpha g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)} + \right. \tag{2.27}$$

$$\left. \left(\int_{\frac{R}{2}}^R (R-s)^{\alpha-1} s^{N-1} ds \right) \left\| D_{R-}^\alpha g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)} \right\} =$$

(acting the same as before, see (2.9)-(2.11))

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D_{0+}^\alpha g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D_{R-}^\alpha g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\} \overset{(1.13)}{=} \tag{2.28}$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D^\alpha g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D^\alpha g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\}. \tag{2.29}$$

We have proved

Theorem 2.4. *Here all terms and assumptions as in Theorem* 2.2, *however with* $\alpha \geq 1$. *Then*

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\,\Gamma(\alpha)} + \right.$$

$$\left. (N-1)!\left\|D_{R-}^{\alpha} g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)}\right] \right\}. \tag{2.30}$$

We continue with

Remark 2.5. (Also a continuation of Remark 2.1) Let here $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. By (2.2) we have

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} \left|\left(D_{0+}^{\alpha} g\right)(t)\right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_0^s (s-t)^{p(\alpha-1)}\, dt\right)^{\frac{1}{p}} \left(\int_0^s \left|\left(D_{0+}^{\alpha} g\right)(t)\right|^q dt\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{s^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{0+}^{\alpha} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}, \tag{2.31}$$

all $s \in \left[0, \frac{R}{2}\right]$.

Similarly by (2.3) we obtain

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} \left|\left(D_{R-}^{\alpha} g\right)(t)\right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_s^R (t-s)^{p(\alpha-1)}\, dt\right)^{\frac{1}{p}} \left(\int_s^R \left|\left(D_{R-}^{\alpha} g\right)(t)\right|^q dt\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{(R-s)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{R-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}, \tag{2.32}$$

all $s \in \left[\frac{R}{2}, R\right]$.

Hence it holds

$$\int_{B(0,R)} |f(y)|\, dy \overset{(2.8)}{=}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^R |g(s)|\, s^{N-1} ds \right\} \overset{\text{(by (2.31), (2.32))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\,\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \left(\int_0^{\frac{R}{2}} s^{\alpha+N-2+\frac{1}{p}} ds\right) \left\|D_{0+}^{\alpha} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)} + \right. \tag{2.33}$$

$$\left. \left(\int_{\frac{R}{2}}^R (R-s)^{\alpha-1+\frac{1}{p}} s^{N-1} ds\right) \left\|D_{R-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \right\} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{\left(\frac{R}{2}\right)^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right.$$

$$\left[\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^{k}\left(\int_{\frac{R}{2}}^{R}(R-s)^{\left(\alpha+\frac{1}{p}-1\right)}\left(s-\frac{R}{2}\right)^{N-k-1}ds\right)\right]$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{R^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)2^{\left(\alpha+N-\frac{1}{q}\right)}}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right. \tag{2.34}$$

$$\left[\sum_{k=0}^{N-1}\frac{(N-1)!}{k!(N-k-1)!}\left(\frac{R}{2}\right)^{k}\frac{\Gamma\left(\alpha+\frac{1}{p}\right)\Gamma(N-k)}{\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\left(\frac{R}{2}\right)^{\alpha+\frac{1}{p}+N-k-1}\right]$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{R^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)2^{\left(\alpha+N-\frac{1}{q}\right)}}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right. \tag{2.35}$$

$$(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left(\frac{R^{\alpha+N-\frac{1}{q}}}{2^{\alpha+N-\frac{1}{q}}}\right)\cdot$$

$$\left.\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$\left.(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}. \tag{2.36}$$

We have proved the following

Theorem 2.6. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$. *All other terms and assumptions as in Theorem* 2.2. *Then*

$$\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\Bigg\}. \tag{2.37}$$

Combining Theorems 2.2, 2.4, 2.6 we derive

Theorem 2.7. *Let any* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$ *and* $\alpha\geq 1$. *And let* $f:\overline{B(0,R)}\to\mathbb{R}$ *be radial; that is, there exists a function* g *such that* $f(x)=g(s)$, $s=|x|$, $s\in[0,R]$, $\forall\ x\in\overline{B(0,R)}$. *Assume that* $g\in C([0,R])$, *with* $g\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ *and* $g\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, *such that* $g^{(k)}(0)=g^{(k)}(R)=0$, $k=0,1,...,m-1$, $m=[\alpha]$. *When* $0<\alpha<1$ *the last boundary conditions are void. Then*

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\min\left\{\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right.\right.$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\},$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)}+\right.$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\},$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$\left.\left.(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}\right\}. \tag{2.38}$$

Note 2.8. It holds

$$Vol(B(0,R))=\frac{2\pi^{\frac{N}{2}}R^{N}}{\Gamma\left(\frac{N}{2}\right)N}. \tag{2.39}$$

The corresponding estimate on the average follows

Corollary 2.9. *Let all terms and assumptions as in Theorem 2.7. Then*

$$\left|\frac{1}{Vol(B(0,R))}\int_{B(0,R)}f(y)\,dy\right|\leq\frac{1}{Vol(B(0,R))}\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\min\left\{\frac{NR^{\alpha}}{2^{\alpha+N}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right.\right.$$

$$(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\Bigg\},$$

$$\frac{NR^{\alpha-1}}{2^{\alpha+N-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)}+\right.$$

$$(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\Bigg\},$$

$$\frac{NR^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\Bigg\}\Bigg\}. \quad (2.40)$$

We continue with Pólya type inequalities on the ball for non-radial functions.

Theorem 2.10. *Let $f\in C\left(\overline{B(0,R)}\right)$ that is not necessarily radial, $0<\alpha<2$. Assume for any $\omega\in S^{N-1}$, that $f(\cdot\omega)\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $f(\cdot\omega)\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $f(0)=f(R\omega)=0$. When $0<\alpha<1$ the last boundary conditions are void. We further assume that*

$$\left\|\frac{\partial_{0+}^{\alpha}f(r\omega)}{\partial r^{\alpha}}\right\|_{\infty,\left(r\in\left[0,\frac{R}{2}\right]\right)},\ \left\|\frac{\partial_{R-}^{\alpha}f(r\omega)}{\partial r^{\alpha}}\right\|_{\infty,\left(r\in\left[\frac{R}{2},R\right]\right)}\leq K, \quad (2.41)$$

for every $\omega\in S^{N-1}$, where $K>0$.

Then

(i)

$$\int_{B(0,R)}|f(y)|\,dy\leq\frac{K\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\cdot \quad (2.42)$$

$$\left\{\frac{1}{(\alpha+N)\Gamma(\alpha+1)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\},$$

and

(ii)

$$\left|\frac{1}{Vol(B(0,R))}\int_{B(0,R)}f(y)\,dy\right|\leq\frac{1}{Vol(B(0,R))}\int_{B(0,R)}|f(y)|\,dy\leq \quad (2.43)$$

$$\frac{KNR^{\alpha}}{2^{\alpha+N}}\left\{\frac{1}{(\alpha+N)\Gamma(\alpha+1)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}.$$

Proof. In Remark 2.1, see (2.8), (2.9), (2.10), (2.11), we proved that

$$\int_0^R |g(s)| s^{N-1} ds \leq \left(\frac{R}{2}\right)^{\alpha+N} \cdot$$

$$\left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)} + \left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]} (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)}\right] \right\}. \tag{2.44}$$

In the above (2.44) we plug in $g(\cdot) = f(\cdot\omega)$, for $\omega \in S^{N-1}$ fixed, and we get

$$\int_0^R |f(s\omega)| s^{N-1} ds \overset{(2.41)}{\leq} K \left(\frac{R}{2}\right)^{\alpha+N} \cdot$$

$$\left\{ \frac{1}{(\alpha+N)\Gamma(\alpha+1)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)}\right] \right\} =: \lambda_1. \tag{2.45}$$

Consequently we obtain

$$\int_{B(0,R)} |f(y)| dy = \int_{S^{N-1}} \left(\int_0^R |f(s\omega)| s^{N-1} ds \right) d\omega \leq$$

$$\lambda_1 \int_{S^{N-1}} d\omega = \lambda_1 \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{2.46}$$

proving the claims. □

We continue with

Theorem 2.11. *Let $f \in C\left(\overline{B(0,R)}\right)$ that is not necessarily radial, $1 \leq \alpha < 2$. Assume for any $\omega \in S^{N-1}$, that $f(\cdot\omega) \in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $f(\cdot\omega) \in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $f(0) = f(R\omega) = 0$. We further assume*

$$\left\| \frac{\partial_{0+}^{\alpha} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}, \left\| \frac{\partial_{R-}^{\alpha} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \leq M, \tag{2.47}$$

for every $\omega \in S^{N-1}$, where $M > 0$.

Then

(i)

$$\int_{B(0,R)} |f(y)| dy \leq \frac{M\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \cdot \tag{2.48}$$

$$\left\{ \frac{1}{(\alpha+N-1)\Gamma(\alpha)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)}\right] \right\},$$

and

(ii)

$$\frac{1}{Vol(B(0,R))} \int_{B(0,R)} |f(y)| dy \leq \tag{2.49}$$

$$\frac{MNR^{\alpha-1}}{2^{\alpha+N-1}}\left\{\frac{1}{(\alpha+N-1)\,\Gamma(\alpha)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}.$$

Proof. In Remark 2.3, see (2.26), (2.27), (2.28), we proved that

$$\int_0^R |g(s)|\, s^{N-1}ds \leq \left(\frac{R}{2}\right)^{\alpha+N-1}\cdot$$

$$\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\,\Gamma(\alpha)}+\left\|D_{R-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}. \quad (2.50)$$

In the above (2.50) we plug in $g(\cdot)=f(\cdot\omega)$, for $\omega\in S^{N-1}$ fixed, and we derive

$$\int_0^R |f(s\omega)|\, s^{N-1}ds \overset{(2.47)}{\leq} M\left(\frac{R}{2}\right)^{\alpha+N-1}\cdot$$

$$\left\{\frac{1}{(\alpha+N-1)\,\Gamma(\alpha)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}=:\lambda_2. \quad (2.51)$$

Hence

$$\int_{B(0,R)}|f(y)|\,dy=\int_{S^{N-1}}\left(\int_0^R |f(s\omega)|\, s^{N-1}ds\right)d\omega\leq$$

$$\lambda_2\int_{S^{N-1}}d\omega=\lambda_2\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \quad (2.52)$$

proving the claims. □

We further have

Theorem 2.12. *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\frac{1}{q}<\alpha<2$. Let $f\in C\left(\overline{B(0,R)}\right)$ that is not necessarily radial. Assume for any $\omega\in S^{N-1}$, that $f(\cdot\omega)\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $f(\cdot\omega)\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $f(0)=f(R\omega)=0$. When $\frac{1}{q}<\alpha<1$ the last boundary condition is void. We further assume*

$$\left\|\frac{\partial_{0+}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)},\ \left\|\frac{\partial_{R-}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\leq\Phi, \quad (2.53)$$

for every $\omega\in S^{N-1}$, where $\Phi>0$.

Then

(i)

$$\int_{B(0,R)}|f(y)|\,dy\leq\frac{\Phi\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\,\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,2^{\alpha+N-\frac{1}{q}-1}}\cdot \quad (2.54)$$

$$\left\{\frac{1}{\left(\alpha+N-\frac{1}{q}\right)}+(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\right\},$$

and

(ii)

$$\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)}\left|f\left(y\right)\right|dy\leq\frac{\Phi NR^{\alpha-\frac{1}{q}}}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}}}\cdot \tag{2.55}$$

$$\left\{\frac{1}{\left(\alpha+N-\frac{1}{q}\right)}+\left(N-1\right)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\right\}.$$

Proof. In Remark 2.5, see (2.33), (2.34), (2.35), (2.36), we proved that

$$\int_0^R\left|g\left(s\right)\right|s^{N-1}ds\leq$$

$$\left(\frac{R}{2}\right)^{\alpha+N-\frac{1}{q}}\frac{1}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\cdot\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right. \tag{2.56}$$

$$\left.\left(N-1\right)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}.$$

In the above (2.56) we plug in $g\left(\cdot\right)=f\left(\cdot\omega\right)$, for $\omega\in S^{N-1}$ fixed, and we find

$$\int_0^R\left|f\left(s\omega\right)\right|s^{N-1}ds\overset{(2.53)}{\leq}\Phi\left(\frac{R}{2}\right)^{\alpha+N-\frac{1}{q}}\frac{1}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\cdot$$

$$\left\{\frac{1}{\left(\alpha+N-\frac{1}{q}\right)}+\left(N-1\right)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\right\}=:\lambda_3. \tag{2.57}$$

Thus

$$\int_{B(0,R)}\left|f\left(y\right)\right|dy=\int_{S^{N-1}}\left(\int_0^R\left|f\left(s\omega\right)\right|s^{N-1}ds\right)d\omega\leq$$

$$\lambda_3\int_{S^{N-1}}d\omega=\lambda_3\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{2.58}$$

proving the claims. $\square$

We make

Remark 2.13. Let the spherical shell $A:=B\left(0,R_2\right)-\overline{B\left(0,R_1\right)}$, $0<R_1<R_2$, $A\subseteq\mathbb{R}^N$, $N\geq2$, $x\in\overline{A}$. Consider first that $f:\overline{A}\to\mathbb{R}$ is radial; that is, there exists g such that $f\left(x\right)=g\left(r\right)$, $r=|x|$, $r\in\left[R_1,R_2\right]$, $\forall\ x\in\overline{A}$. Here x can be written uniquely as $x=r\omega$, where $r=|x|>0$ and $\omega=\frac{x}{r}\in S^{N-1}$, $|\omega|=1$, see ([14], p. 149-150 and [1], p. 421), furthermore for general $F:\overline{A}\to\mathbb{R}$ Lebesgue integrable function we have that

$$\int_A F\left(x\right)dx=\int_{S^{N-1}}\left(\int_{R_1}^{R_2}F\left(r\omega\right)r^{N-1}dr\right)d\omega. \tag{2.59}$$

Let $d\omega$ be the element of surface measure on S^{N-1}, then

$$\omega_N := \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{2.60}$$

Here

$$Vol\left(A\right) = \frac{\omega_N\left(R_2^N - R_1^N\right)}{N} = \frac{2\pi^{\frac{N}{2}}\left(R_2^N - R_1^N\right)}{N\Gamma\left(\frac{N}{2}\right)}. \tag{2.61}$$

We assume that $g \in C\left([R_1, R_2]\right)$, and $\alpha > 0$, $m = [\alpha]$, such that $g \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $g \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, with $g^{(k)}\left(R_1\right) = g^{(k)}\left(R_2\right) = 0$, $k = 0, 1, ..., m-1$. When $0 < \alpha < 1$ the last boundary conditions are void.

By assumption here and Theorem 1.3 we have

$$g\left(s\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_{R_1}^{s} \left(s-t\right)^{\alpha-1} \left(D^{\alpha}_{R_1+}g\right)\left(t\right) dt, \tag{2.62}$$

all $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$,

also it holds, by assumption and Theorem 1.4, that

$$g\left(s\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_{s}^{R_2} \left(t-s\right)^{\alpha-1} \left(D^{\alpha}_{R_2-}g\right)\left(t\right) dt, \tag{2.63}$$

all $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

By (2.62) we get

$$\left|g\left(s\right)\right| \leq \frac{1}{\Gamma\left(\alpha\right)} \int_{R_1}^{s} \left(s-t\right)^{\alpha-1} \left|\left(D^{\alpha}_{R_1+}g\right)\left(t\right)\right| dt \tag{2.64}$$

$$\leq \left\|D^{\alpha}_{R_1+}g\right\|_{\infty,\left[R_1, \frac{R_1+R_2}{2}\right]} \frac{\left(s-R_1\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}, \tag{2.65}$$

for any $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$.

Similarly we obtain by (2.63) that

$$\left|g\left(s\right)\right| \leq \frac{1}{\Gamma\left(\alpha\right)} \int_{s}^{R_2} \left(t-s\right)^{\alpha-1} \left|\left(D^{\alpha}_{R_2-}g\right)\left(t\right)\right| dt \tag{2.66}$$

$$\leq \left\|D^{\alpha}_{R_2-}g\right\|_{\infty,\left[\frac{R_1+R_2}{2}, R_2\right]} \frac{\left(R_2-s\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}, \tag{2.67}$$

for any $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

Next we observe that

$$\left|\int_A f\left(y\right) dy\right| \leq \int_A \left|f\left(y\right)\right| dy \overset{(2.59)}{=} \tag{2.68}$$

$$\int_{S^{N-1}} \left(\int_{R_1}^{R_2} \left|g\left(s\right)\right| s^{N-1} ds\right) d\omega = \left(\int_{R_1}^{R_2} \left|g\left(s\right)\right| s^{N-1} ds\right) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} = \tag{2.69}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{\int_{R_1}^{\frac{R_1+R_2}{2}} \left|g\left(s\right)\right| s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} \left|g\left(s\right)\right| s^{N-1} ds\right\} \overset{\text{(by (2.65) and (2.67))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\alpha+1\right)}\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\int_{R_1}^{\frac{R_1+R_2}{2}}\left(s-R_1\right)^{\alpha}s^{N-1}ds\right.$$

$$\left.+\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\int_{\frac{R_1+R_2}{2}}^{R_2}\left(R_2-s\right)^{\alpha}s^{N-1}ds\right\}= \tag{2.70}$$

$$\frac{\pi^{\frac{N}{2}}\left(N-1\right)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\right.$$

$$\left(\sum_{k=0}^{N-1}\left(-1\right)^{N+k-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N-k+\alpha}}{k!\Gamma\left(N-k+\alpha+1\right)}\right)+$$

$$\left.\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{\alpha+N-k}}{k!\Gamma\left(\alpha+1+N-k\right)}\right]\right\}. \tag{2.71}$$

We have proved that

$$\left|\int_A f\left(y\right)dy\right|\leq\int_A\left|f\left(y\right)\right|dy\leq\frac{\pi^{\frac{N}{2}}\left(N-1\right)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{\left(-1\right)^{N+k-1}\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N-k+\alpha}}{k!\Gamma\left(N-k+\alpha+1\right)}\right)+\right.$$

$$\left.\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{\alpha+N-k}}{k!\Gamma\left(\alpha+1+N-k\right)}\right]\right\}. \tag{2.72}$$

Consider now $f_*:\overline{A}\to\mathbb{R}$ be radial such that $f_*\left(x\right)=g_*\left(s\right)$, $s=\left|x\right|$, $s\in\left[R_1,R_2\right]$, $\forall$ $x\in\overline{A}$, where

$$g_*\left(s\right)=\begin{cases}\left(s-R_1\right)^{\alpha}, & s\in\left[R_1,\frac{R_1+R_2}{2}\right],\\ \left(R_2-s\right)^{\alpha}, & s\in\left[\frac{R_1+R_2}{2},R_2\right],\ \alpha>0.\end{cases} \tag{2.73}$$

We have, as in [4], that

$$\left\|D_{R_1+}^{\alpha}g_*\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}=\Gamma\left(\alpha+1\right),\quad\text{and}\quad\left\|D_{R_2-}^{\alpha}g_*\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}=\Gamma\left(\alpha+1\right). \tag{2.74}$$

Hence

$$\text{R.H.S. (2.72) (applied on } g_*)=\frac{\Gamma\left(\alpha+1\right)\pi^{\frac{N}{2}}\left(N-1\right)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\sum_{k=0}^{N-1}\left(1+\left(-1\right)^{N+k-1}\right)\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{\alpha+N-k}}{k!\Gamma\left(\alpha+1+N-k\right)}\right\}. \tag{2.75}$$

Furthermore we find

$$\text{L.H.S. (2.72) (applied on } f_*)=\int_A f_*\left(y\right)dy=$$

$$\left(\int_{R_1}^{R_2}g_*\left(s\right)s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_{R_1}^{\frac{R_1+R_2}{2}}(s-R_1)^{\alpha}s^{N-1}ds+\int_{\frac{R_1+R_2}{2}}^{R_2}(R_2-s)^{\alpha}s^{N-1}ds\right\}= \tag{2.76}$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma(\alpha+1)}{\Gamma\left(\frac{N}{2}\right)2^{N+\alpha-1}}\left\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right)+\right. \tag{2.77}$$

$$\left.\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\}=$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma(\alpha+1)}{\Gamma\left(\frac{N}{2}\right)2^{N+\alpha-1}}\left\{\sum_{k=0}^{N-1}\left((-1)^{N+k-1}+1\right)\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right\}. \tag{2.78}$$

So that we find

$$\text{R.H.S. (2.72) (applied on } g_*) = \text{L.H.S. (2.72) (applied on } f_*), \tag{2.79}$$

proving sharpness of (2.72).

We have proved the following

Theorem 2.14. *Let $f:\overline{A}\to\mathbb{R}$ be radial; that is, there exists a function g such that $f(x)=g(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, $\alpha>0$, $m=[\alpha]$. We assume that $g\in C([R_1,R_2])$, such that $g\in C^{\alpha}_{R_1+}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ and $g\in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$,with $g^{(k)}(R_1)=g^{(k)}(R_2)=0$, $k=0,1,...,m-1$. When $0<\alpha<1$ the last boundary conditions are void. Then*

$$\left|\int_A f(y)\,dy\right|\le\int_A|f(y)|\,dy\le\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)+\right. \tag{2.80}$$

$$\left.\left\|D^{\alpha}_{R_2-}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\}.$$

Inequalities (2.80) are sharp, namely they are attained by the radial function $f_*:\overline{A}\to\mathbb{R}$ such that $f_*(x)=g_*(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, where

$$g_*(s)=\begin{cases}(s-R_1)^{\alpha}, & s\in\left[R_1,\frac{R_1+R_2}{2}\right],\\ (R_2-s)^{\alpha}, & s\in\left[\frac{R_1+R_2}{2},R_2\right].\end{cases} \tag{2.81}$$

We continue with

Remark 2.15. Here $\alpha\ge1$. By (2.64) we get

$$|g(s)|\le\frac{(s-R_1)^{\alpha-1}}{\Gamma(\alpha)}\left\|D^{\alpha}_{R_1+}g\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}, \tag{2.82}$$

for any $s\in\left[R_1,\frac{R_1+R_2}{2}\right]$.

And by (2.66) we derive

$$|g(s)| \leq \frac{(R_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{R_2-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)}, \tag{2.83}$$

for any $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

Hence

$$\int_A |f(y)|\, dy \overset{(2.69)}{=}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)|\, s^{N-1} ds \right\} \overset{\text{(by (2.82) and (2.83))}}{\leq} \tag{2.84}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)} \left\{ \left\| D_{R_1+}^{\alpha} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\int_{R_1}^{\frac{R_1+R_2}{2}} (s - R_1)^{\alpha-1} s^{N-1} ds \right) + \right. \tag{2.85}$$

$$\left. \left\| D_{R_2-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\int_{\frac{R_1+R_2}{2}}^{R_2} (R_2 - s)^{\alpha-1} s^{N-1} ds \right) \right\} =$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \left\{ \left\| D_{R_1+}^{\alpha} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \right.$$

$$\left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1 + R_2)^k (R_2 - R_1)^{N+\alpha-k-1}}{k!\Gamma(N + \alpha - k)} \right) + \tag{2.86}$$

$$\left. \left\| D_{R_2-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1 + R_2)^k (R_2 - R_1)^{N+\alpha-k-1}}{k!\Gamma(N + \alpha - k)} \right) \right\}.$$

We have proved that

Theorem 2.16. *All terms and assumptions here as in Theorem* 2.14, *but with* $\alpha \geq 1$. *Then*

$$\int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \cdot$$

$$\left\{ \left\| D_{R_1+}^{\alpha} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1 + R_2)^k (R_2 - R_1)^{N+\alpha-k-1}}{k!\Gamma(N + \alpha - k)} \right) + \right.$$

$$\left. \left\| D_{R_2-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1 + R_2)^k (R_2 - R_1)^{N+\alpha-k-1}}{k!\Gamma(N + \alpha - k)} \right) \right\}. \tag{2.87}$$

We continue with

Remark 2.17. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha > \frac{1}{q}$. By (2.64) we get

$$|g(s)| \leq \frac{(s - R_1)^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{R_1+}^{\alpha} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)}, \tag{2.88}$$

for any $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$.

Similarly by (2.66) we derive

$$|g(s)| \leq \frac{(R_2-s)^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{R_2-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)}, \tag{2.89}$$

for any $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

Hence

$$\int_A |f(y)|\,dy =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)|\, s^{N-1} ds \right\} \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left\{ \left\|D_{R_1+}^{\alpha} g\right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\int_{R_1}^{\frac{R_1+R_2}{2}} (s-R_1)^{\alpha-1+\frac{1}{p}} s^{N-1} ds \right) + \right.$$

$$\left. \left\|D_{R_2-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\int_{\frac{R_1+R_2}{2}}^{R_2} (R_2-s)^{\alpha-1+\frac{1}{p}} s^{N-1} ds \right) \right\} = \tag{2.90}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \left\|D_{R_1+}^{\alpha} g\right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\frac{(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{2^{\alpha+N-\frac{1}{q}}} \right) \cdot \right.$$

$$\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N-k+\alpha-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \tag{2.91}$$

$$\left\|D_{R_2-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\frac{(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{2^{\alpha+N-\frac{1}{q}}} \right) \cdot$$

$$\left. \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)} \right) \right\} =$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\|D_{R_1+}^{\alpha} g\right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left. \left\|D_{R_2-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\}. \tag{2.92}$$

We have proved

Theorem 2.18. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$. *All terms and assumptions as in Theorem* 2.14. *Then*

$$\int_A |f(y)| \, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)! \Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right) \Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\}. \tag{2.93}$$

Combining Theorems 2.14, 2.16, 2.18 we derive

Theorem 2.19. *Let any* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *And let* $f : \overline{A} \to \mathbb{R}$ *be radial; that is, there exists a function* g *such that* $f(x) = g(s)$, $s = |x|$, $s \in [R_1, R_2]$, $\forall\, x \in \overline{A}$; $\alpha \geq 1$, $m = [\alpha]$. *We assume that* $g \in C([R_1, R_2])$, *such that* $g \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ *and* $g \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, *with* $g^{(k)}(R_1) = g^{(k)}(R_2) = 0$, $k = 0, 1, ..., m-1$. *Then*

$$\left| \int_A f(y) \, dy \right| \leq \int_A |f(y)| \, dy \leq \min \left\{ \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \cdot \right.$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{\infty, \left[R_1, \frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N-k+\alpha}}{k! \Gamma(N-k+\alpha+1)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{\infty, \left[\frac{R_1+R_2}{2}, R_2\right]} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k}}{k! \Gamma(\alpha+1+N-k)} \right) \right\},$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \cdot \right.$$

$$\left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right) +$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right) \right\},$$

$$\frac{\pi^{\frac{N}{2}} (N-1)! \Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right) \Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left\|D_{R_2-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)}\right)\Bigg\}\Bigg\}. \quad (2.94)$$

The corresponding estimate on the average follows

Corollary 2.20. *Let all terms and assumptions as in Theorem* 2.19. *Then*

$$\left|\frac{1}{Vol(A)}\int_A f(y)\,dy\right| \leq \frac{1}{Vol(A)}\int_A |f(y)|\,dy \leq \left(\frac{N!}{2^{\alpha+N}\left(R_2^N-R_1^N\right)}\right)\cdot$$

$$\min\Bigg\{\Bigg\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)$$

$$+\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\Bigg\},$$

$$2\Bigg\{\left\|D_{R_1+}^{\alpha}g\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}\left(\sum_{k=0}^{N-1}(-1)^{N+k-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)$$

$$+\left\|D_{R_2-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)\Bigg\},$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)2^{\frac{1}{q}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\cdot$$

$$\Bigg\{\left\|D_{R_1+}^{\alpha}g\right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+$$

$$\left\|D_{R_2-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)}\right)\Bigg\}\Bigg\}. \quad (2.95)$$

We need

Definition 2.21. *(see* [1], *p. 287) Let* $\alpha > 0$, $m = [\alpha]$, $\beta := \alpha - m$, $f \in C^m\left(\overline{A}\right)$, *and* A *is a spherical shell. Assume that there exists* $\frac{\partial_{R_1+}^{\alpha}f(x)}{\partial r^{\alpha}} \in C\left(\overline{A}\right)$, *given by*

$$\frac{\partial_{R_1+}^{\alpha}f(x)}{\partial r^{\alpha}} := \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial r}\left(\int_{R_1}^{r}(r-t)^{-\beta}\frac{\partial^m f(t\omega)}{\partial r^m}dt\right), \quad (2.96)$$

where $x \in \overline{A}$; *that is,* $x = r\omega$, $r \in [R_1, R_2]$, *and* $\omega \in S^{N-1}$.

We call $\frac{\partial_{R_1+}^{\alpha}f}{\partial r^{\alpha}}$ *the left radial generalised fractional derivative of* f *of order* α.

We also need to introduce

Definition 2.22. *Let $\alpha > 0$, $m = [\alpha]$, $\beta := \alpha - m$, $f \in C^m(\overline{A})$, and A is a spherical shell. Assume that there exists $\frac{\partial^\alpha_{R_2-} f(x)}{\partial r^\alpha} \in C(\overline{A})$, given by*

$$\frac{\partial^\alpha_{R_2-} f(x)}{\partial r^\alpha} := (-1)^{m-1} \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial r} \left(\int_r^{R_2} (t-r)^{-\beta} \frac{\partial^m f(t\omega)}{\partial r^m} dt \right), \tag{2.97}$$

where $x \in \overline{A}$; that is, $x = r\omega$, $r \in [R_1, R_2]$, and $\omega \in S^{N-1}$.

We call $\frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha}$ the right radial generalised fractional derivative of f of order α.

We present

Theorem 2.23. *Let the sperical shells $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$; $A_1 := B\left(0, \frac{R_1+R_2}{2}\right) - \overline{B(0, R_1)}$, $A_2 := B(0, R_2) - \overline{B\left(0, \frac{R_1+R_2}{2}\right)}$. Let $f \in C(\overline{A})$, not necessarily radial, $\alpha > 0$, $m = [\alpha]$. Assume that $\frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \in C(\overline{A_1})$, $\frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \in C(\overline{A_2})$. For each $\omega \in S^{N-1}$, we assume further that $f(\cdot\omega) \in C^\alpha_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega) \in C^\alpha_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, with $\frac{\partial^k f(R_1\omega)}{\partial r^k} = \frac{\partial^k f(R_2\omega)}{\partial r^k} = 0$, $k = 0, 1, ..., m-1$. When $0 < \alpha < 1$ the last boundary conditions are void. Then*

(i)

$$\left| \int_A f(y)\, dy \right| \leq \int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \cdot$$

$$\left\{ \left\| \frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\}, \tag{2.98}$$

and

(ii)

$$\left| \frac{1}{Vol(A)} \int_A f(y)\, dy \right| \leq \frac{1}{Vol(A)} \int_A |f(y)|\, dy \leq \left(\frac{N!}{2^{\alpha+N}\left(R_2^N - R_1^N\right)} \right) \cdot \tag{2.99}$$

$$\left\{ \left\| \frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\}.$$

Proof. By (2.69)-(2.71) we get

$$\int_{R_1}^{R_2} |g(s)|\, s^{N-1} ds \leq \left(\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \right) \left(\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \right) \cdot \tag{2.100}$$

$$\left\{ \left\| D^\alpha_{R_1+} g \right\|_{\infty, \left[R_1, \frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right]\Bigg\}.$$

For fixed $\omega\in S^{N-1}$, $f(\cdot\omega)$ sets like a radial function on $\overline{A}$. Thus plugging $f(\cdot\omega)$ into (2.100), we get

$$\int_{R_1}^{R_2}|f(s\omega)|\,s^{N-1}ds\le\left(\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\right)\left(\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\right)\cdot\tag{2.101}$$

$$\left\{\left\|\frac{\partial_{R_1+}^{\alpha}f}{\partial r^{\alpha}}\right\|_{\infty,\overline{A_1}}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right)+\right.$$

$$\left.\left\|\frac{\partial_{R_2-}^{\alpha}f}{\partial r^{\alpha}}\right\|_{\infty,\overline{A_2}}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right)\right\}=:\gamma_1.$$

Therefore by (2.59) and (2.101) we derive

$$\int_A|f(y)|\,dy=\int_{S^{N-1}}\left(\int_{R_1}^{R_2}|f(s\omega)|\,s^{N-1}ds\right)d\omega\le$$

$$\gamma_1\int_{S^{N-1}}d\omega=\gamma_1\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}=\left(\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\right)\cdot\tag{2.102}$$

$$\left\{\left\|\frac{\partial_{R_1+}^{\alpha}f}{\partial r^{\alpha}}\right\|_{\infty,\overline{A_1}}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right)+\right.$$

$$\left.\left\|\frac{\partial_{R_2-}^{\alpha}f}{\partial r^{\alpha}}\right\|_{\infty,\overline{A_2}}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right)\right\},$$

proving the claims of the theorem. □

We give also

Theorem 2.24. *Let $f\in C\left(\overline{A}\right)$, not necessarily radial, $\alpha\ge1$, $m=[\alpha]$. For each $\omega\in S^{N-1}$, we assume that $f(\cdot\omega)\in C_{R_1+}^{\alpha}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega)\in C_{R_2-}^{\alpha}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$, with $\frac{\partial^k f(R_1\omega)}{\partial r^k}=\frac{\partial^k f(R_2\omega)}{\partial r^k}=0$, $k=0,1,...,m-1$. We further assume*

$$\left\|\frac{\partial_{R_1+}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)},\ \left\|\frac{\partial_{R_2-}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_1\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\le\Psi_1,\tag{2.103}$$

for every $\omega\in S^{N-1}$, where $\Psi_1>0$.

Then

(i)

$$\int_A|f(y)|\,dy\le\frac{\Psi_1\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-2}}\cdot\tag{2.104}$$

$$\left\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)+\right.$$

$$\left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)\Bigg\},$$

and

(ii)

$$\frac{1}{Vol(A)}\int_A |f(y)|\,dy \le \frac{\Psi_1 N!}{2^{\alpha+N-1}\left(R_2^N-R_1^N\right)}\cdot \tag{2.105}$$

$$\Bigg\{\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)+$$

$$\left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)\Bigg\}.$$

Proof. Similar to Theorem 2.23, using (2.84)-(2.86). □

We finish with

Theorem 2.25. *Let $f \in C\left(\overline{A}\right)$, not necessarily radial, $\alpha > \frac{1}{q}$, where $p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $m = [\alpha]$. For each $\omega \in S^{N-1}$, we assume that $f(\cdot\omega) \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega) \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, with $\frac{\partial^k f(R_1\omega)}{\partial r^k} = \frac{\partial^k f(R_2\omega)}{\partial r^k} = 0$, $k = 0, 1, ..., m-1$. When $\frac{1}{q} < \alpha < 1$ the last boundary conditions is void. We further assume*

$$\left\|\frac{\partial^{\alpha}_{R_1+} f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)}, \quad \left\|\frac{\partial^{\alpha}_{R_2-} f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \le \Psi_2, \tag{2.106}$$

for every $\omega \in S^{N-1}$, where $\Psi_2 > 0$.

Then

(i)

$$\int_A |f(y)|\,dy \le \frac{\Psi_2 \pi^{\frac{N}{2}} (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}}\cdot \tag{2.107}$$

$$\Bigg\{\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+$$

$$\left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)}\right)\Bigg\},$$

and

(ii)

$$\frac{1}{Vol(A)}\int_A |f(y)|\,dy \le \frac{N!\Gamma\left(\alpha+\frac{1}{p}\right)\Psi_2}{2^{\alpha+N-\frac{1}{q}}\left(R_2^N-R_1^N\right)\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}}\cdot \tag{2.108}$$

$$\left\{ \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\}.$$

Proof. Similar to Theorem 2.23, using (2.90)-(2.92). □

References

[1] Anastassiou, G.A., *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.

[2] Anastassiou, G.A., *On Right Fractional Calculus*, Chaos, Solitons and Fractals, **42**(2009), 365-376.

[3] Anastassiou, G.A., *Balanced Canavati type fractional Opial inequalities*, to appear, J. of Applied Functional Analysis, 2014.

[4] Anastassiou, G.A., *Fractional Pólya type integral inequality*, submitted, 2013.

[5] Canavati, J.A., *The Riemann-Liouville Integral*, Nieuw Archief Voor Wiskunde, **5**(1) (1987), 53-75.

[6] El-Sayed, A.M.A., Gaber, M., *On the finite Caputo and finite Riesz derivatives*, Electronic Journal of Theoretical Physics, **3**(2006), no. 12, 81-95.

[7] Frederico, G.S., Torres, D.F.M., *Fractional Optimal Control in the sense of Caputo and the fractional Noether's theorem*, International Mathematical Forum, **3**(2008), no. 10, 479-493.

[8] Gorenflo, R., Mainardi, F., *Essentials of Fractional Calculus*, 2000, Maphysto Center, http://www.maphysto.dk/oldpages/events/LevyCAC2000/ MainardiNotes/fm2k0a.ps.

[9] Pólya, G., *Ein mittelwertsatz für Funktionen mehrerer Veränderlichen*, Tohoku Math. J., **19**(1921), 1-3.

[10] Pólya, G., and Szegő, G., *Aufgaben und Lehrsätze aus der Analysis*, Volume I, Springer-Verlag, Berlin, 1925. (German)

[11] Pólya, G., and Szegő, G., *Problems and Theorems in Analysis*, Volume I, Classics in Mathematics, Springer-Verlag, Berlin, 1972.

[12] Pólya, G., and Szegő, G., *Problems and Theorems in Analysis*, Volume I, Chinese Edition, 1984.

[13] Feng Qi, *Pólya type integral inequalities: origin, variants, proofs, refinements, generalizations, equivalences, and applications*, article no. 20, 16th vol. 2013, RGMIA, Res. Rep. Coll., http://rgmia.org/v16.php.

[14] Rudin, W., *Real and Complex Analysis*, International Student edition, McGraw Hill, London, New York, 1970.

[15] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives, Theory and Applications*, (Gordon and Breach, Amsterdam, 1993) [English translation from the Russian, Integrals and Derivatives of Fractional Order and Some of Their Applications (Nauka i Tekhnika, Minsk, 1987)].

[16] Stroock, D., *A Concise Introduction to the Theory of Integration*, Third Edition, Birkhaüser, Boston, Basel, Berlin, 1999.

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