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A proof of a covering correspondence

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Abstract. We show that the isomorphism between the Clifford extensions of two Brauer corresponding blocks of normal subgroups induces a defect group preserving bijection which coincides with the Harris-Knörr correspondence between their covering blocks.

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1. Introduction

Clifford extensions for blocks were introduced by E.C. Dade in [5], where he proved that two Brauer correspondent blocks b and b_1 with defect group D of normal subgroups K and $N_K(D)$ of the finite groups H and $N_H(D)$ respectively, have isomorphic Clifford extensions.

Dade [5, Section 3] also gives a bijective correspondence between the blocks of a strongly graded algebra that cover a fixed block b of the identity component and the conjugacy classes of blocks of the twisted group algebra corresponding to the Clifford extension of b.

A generalization of Dade's main result is given in [3], where we prove an isomorphism of Clifford extensions for points of identity components of certain H/K-graded H-interior algebras, without assuming that the ground field is algebraically closed.

The aim of this paper is to establish a link between the above isomorphism of Clifford extensions and the result of M.E. Harris and R. Knörr [6] which states that the Brauer correspondence induces a bijection between the blocks of H covering b and the blocks of $N_H(D)$ covering b_1. Actually, there is some suggestion in [6] that such a connection is possible, but no details are given. Note also that a module-theoretic version of the Harris-Knörr correspondence was given by J. Alperin [1]. Here we prove that the isomorphism of Clifford extensions induces a defect group preserving bijective correspondence between the blocks of H covering b and the blocks of $N_H(D)$ covering b_1, which coincides with the Harris-Knörr correspondence.

We present our general setting in Section 2, while in Section 3 we review the required results on the defect groups of covering blocks. The details on the correspondence induced by the isomorphism of the Clifford extensions are presented in Section 4, following [3]. The last section is devoted to the proof of our main result, stated in Theorem 5.1. The reader is referred to [9] and [7] for general facts on block theory.

2. Preliminaries

2.1. Let $\mathcal{O}$ be a discrete valuation ring having residual field k of characteristic $p \geq 0$. Let K be a normal subgroup of the finite group H, denote $G = H/K$, and consider the group algebra $\mathcal{O}H$ regarded as a strongly G-graded algebra

$$A := \mathcal{O}H = \bigoplus_{\sigma \in G} \mathcal{O}\sigma,$$

which is also an H-algebra under the conjugation action of H. We fix a block b of the identity component $A_1 := \mathcal{O}K$ of A. We denote by D a defect group in K of the block b.

2.2. If H_b denotes the stabilizer of b in H, and G_b is the quotient H_b/K, as in [5] we consider the G_b-graded subalgebra

$$bC := bC_A(A_1) = (b\mathcal{O}H_b)^K = \bigoplus_{\sigma \in G_b} (b\mathcal{O}\sigma)^K = \bigoplus_{\sigma \in G_b} bC_\sigma^K$$

of A. We truncate bC by taking the components indexed by the normal subgroup

$$G[b] = \{\sigma \in G_b \mid bC_\sigma^K \cdot bC_{\sigma^{-1}}^K = bC_1^K\}$$

of G_b; this yields the strongly $G[b]$-graded G_b-algebra, and hence an H_b-algebra

$$C[b] := \bigoplus_{\sigma \in G[b]} bC_\sigma^K.$$

The identity component

$$bC_1^K = b(\mathcal{O}K)^K = bZ(\mathcal{O}K)$$

is a local ring such that the field

$$\hat{k}_1 = bZ(\mathcal{O}K)/J(bZ(\mathcal{O}K))$$

is a finite extension of k.

2.3. Consider also the quotient $C[b]/C[b]J(C[b]_1)$, which is the twisted group algebra of $G[b]$ over the field $\hat{k}_1$, corresponding to the *Clifford extension*

$$1 \to \hat{k}_1^* \to hU(C[b]/C[b]J(C[b]_1)) \to G[b] \to 1 \tag{2.1}$$

of the block b. Where by hU we denoted the homogeneous units of $C[b]/C[b]J(C[b]_1)$. Explicitly, the set of elements that satisfy

$$\bar{a} \in (C[b]/C[b]J(C[b]_1))^* \cap bC_g/bC_gJ(C[b]_1),$$

for some $g \in G[b]$. Since $bC_1 = C[b]_1$ is a H_b-algebra, the H_b-invariance of $J(C[b]_1)$ implies that the canonical map

$$C[b] \to C[b]/C[b]J(C[b]_1)$$

is a homomorphism of H_b-algebras.

Lemma 2.4. *The algebras bC^{H_b} and $C[b]^{H_b}$ have the same primitive idempotents.*

Proof. The proof of this statement is based on results of [5, Paragraph 3], which remain true even if the field k in not algebraically closed. One easily checks that in our setting [5, Lemma 3.3] is valid. So there is a two-sided ideal

$$I = (\bigoplus_{\sigma \in G_b \backslash G[b]} bC_\sigma) \oplus C[b]J(C[b]_1)$$

of bC that is both H_b-invariant and contained in $J(bC)$. This gives the equality

$$bC = C[b] \oplus (\bigoplus_{\sigma \in G_b \backslash G[b]} bC_\sigma) = C[b] + I = C[b] + J(bC);$$

showing that every primitive idempotent of bC belongs to $C[b]$. So, any block, that is a primitive idempotent of $Z(bC) = bC^{H_b}$, lies in $C[b]^{H_b}$. Conversely, any primitive idempotent of $C[b]^{H_b}$ remains primitive in bC^{H_b}, since I is contained in $J(bC)$. □

3. Remarks on defect groups

In this section we discuss the connections between the defect groups of blocks covering the block b of C_1 and the defect groups of primitive idempotents of $C[b]^{H_b}$. Some of the results have already been proven in [5, Paragraph 6 and 7], but for the sake of completeness we present them here. As a definition of a defect group of a block we will use [9, Paragraph 18] or [5, Paragraph 4]. Dade uses the maximal ideal corresponding to a block in order to define the defect group of that block. Nevertheless, one easily shows that both treatments lead to the same definition.

3.1. As it is well known, the blocks of H covering b are the primitive idempotents of $Z(s\mathcal{O}H)$, where

$$s = \mathrm{Tr}_{H_b}^{H}(b).$$

By [5, Proposition 4.9] we have the isomorphism

$$Z(s\mathcal{O}H) \simeq Z(b\mathcal{O}Hb) = Z(b\mathcal{O}H_b) = bC^{H_b}. \tag{3.1}$$

Using this and the results of Section 2 above, we see that the blocks of H that cover b are actually the primitive idempotents of $C[b]^{H_b}$.

3.2. We denote by B a block that covers b and by B' the correspondent of B through the isomorphism (3.1). Then $B = \mathrm{Tr}_{H_b}^{H}(B')$. Let Q denote a defect group in H_b of B'. This means that Q is with the properties $B' \in b\mathcal{O}Hb_Q^{H_b}$ and $B' \not\subseteq \mathrm{Ker}(\mathrm{Br}_Q)$, where Br_Q denotes the Brauer homomorphism with respect to Q. But then, since $B's = B'$ we get $B \in s\mathcal{O}H_Q^H$. For $x \in H \setminus H_b$ we also have $bb^x = 0$. Taking into account that $B' = bB' = bB'$, then obviously $BB' = B'$. This forces $B \not\subseteq \mathrm{Ker}(\mathrm{Br}_Q)$. We have shown that any block that covers b has a defect group in H that is contained in H_b.

3.3. By [8, Proposition 4.2], the block B' has a defect group Q (in H_b) satisfying $Q \cap K = D$. The ending of Paragraph 3.2 assures that Q is also a defect group of B. We can apply [8, Proposition 4.2] to obtain a defect group L of B in H that satisfies the same condition as Q, that is $L \cap K = D$. Thus, there is $y \in H$ with $L^y = Q$, and then $y \in N_H(D)$.

4. Clifford extensions of blocks

We keep the notations of the preceding sections. For the details on the following statements the reader is referred to [3].

4.1. The restriction to $bC = b(\mathcal{O}H_b)^K$ of the Brauer homomorphism

$$\mathrm{Br}_D^H : (\mathcal{O}H)^D \to kC_H(D)$$

gives the epimorphism

$$\mathrm{Br}_D^H : b(\mathcal{O}H_b)^K \to \bar{b}kC_H(D)_{\bar{b}}^{N_K(D)}, \tag{i}$$

where $\bar{b} = \mathrm{Br}_D^H(b)$.

Next let b_1 denote the Brauer correspondent of b, seen as a block of $\mathcal{O}N_K(D)$, also having defect group D. Repeating the construction of Section 2 for $N_H(D)$, $N_K(D)$ and b_1 in place of H, K and b respectively we easily obtain another Clifford extension

$$1 \to \hat{k}_2^* \to hU(C'[b_1]/C'[b_1]J(C'[b_1]_1)) \to G'[b_1] \to 1. \tag{4.1}$$

Here we used the $N_H(D)/N_K(D)$-graded centralizer

$$b_1C' := C_{\mathcal{O}N_H(D)}(\mathcal{O}N_K(D)) = \mathcal{O}N_H(D)^{N_K(D)}.$$

In extension (4.1) $C'[b_1]$ and $G'[b_1]$ stand for the analogous notation of $C[b]$ and of the group $G[b]$ respectively. Moreover, $\hat{k}_2$ is the field given by the quotient

$$C[b_1]_1/J(C[b_1]_1) = Z(b_1\mathcal{O}N_K(D))/J(b_1\mathcal{O}N_K(D)).$$

4.2. There is another epimorphism induced by the Brauer map

$$\mathrm{Br}_D^{N_H(D)} : b_1(\mathcal{O}N_H(D)_{b_1})^{N_K(D)} \to \bar{b}_1kC_H(D)_{\bar{b}_1}^{N_K(D)}, \tag{ii}$$

where $\bar{b}_1 = \mathrm{Br}_D^{N_H(D)}(b_1)$. As far as $\bar{b} = \bar{b}_1$ and

$$N_H(D)_b = N_H(D)_{b_1} = N_H(D)_{\bar{b}},$$

applied twice, [3, Theorem 4.1] gives the isomorphism

$$C[b]/C[b]J(C[b_1]_1) \simeq C'[b_1]/C'[b_1]J(C'[b_1]_1). \tag{4.2}$$

Note that the two quotients above are isomorphic as $N_H(D)_b/N_K(D) \simeq H_b/K$-algebras. In fact we have

$$(C[b]/C[b]J(C[b]_1))^{H_b} = (C[b]/C[b]J(C[b]_1))^{H_b/K} \tag{4.3}$$
$$\simeq (C'[b_1]/C'[b_1]J(C'[b_1]_1))^{N_H(D)_b/N_K(D)} = (C'[b_1]/C'[b_1]J(C'[b_1]_1)^{N_H(D)_b}.$$

Proposition 4.3. *There is a bijection between the primitive idempotents of $C[b]^{H_b}$ and the primitive idempotents of $C'[b_1]^{N_H(D)_b}$.*

Proof. The subalgebra of H_b-fixed elements of $C[b]$ lies in the center of $C[b]$, and the subalgebra of $N_H(D)_b$-fixed elements of $C'[b_1]$ lies in the center of $C'[b_1]$. Isomorphisms (4.2), (4.3) and [5, Lemma 3.1] give the desired bijection. □

Remark 4.4. Isomorphism (4.2), Proposition 4.3 and Lemma 2.4 give a bijection between the primitive idempotents of bC^{H_b} and the primitive idempotents of $b_1(C')^{N_H(D)_b}$. If $s' = \mathrm{Tr}_{N_H(D)_b}^{N_H(D)}(b_1)$, isomorphism (3.1) and its analogous isomorphism give a bijection between the blocks of $s\mathcal{O}H$ and the blocks of $s'\mathcal{O}N_H(D)$. Thus, we obtained a correspondence between the blocks of H that cover b and the blocks of $N_H(D)$ that cover b_1. We call this the *Clifford-Dade correspondence.*

5. The Harris - Knörr correspondence

With the above results and notations we have:

Theorem 5.1. *The isomorphic Clifford extensions of b and of b_1 define a defect group preserving bijective correspondence between blocks of $\mathcal{O}H$ covering b and blocks of $\mathcal{O}N_H(D)$ covering b_1. Moreover the Clifford-Dade correspondence between the blocks covering b and b_1 coincides with the Brauer correspondence.*

Proof. Remark 4.4 already gives a bijection between the blocks of H that cover b and the blocks of $N_H(D)$ that cover b_1. We prove that this bijection preserves the defect groups.

First of all let us emphasize that isomorphism (4.2) holds because the two Brauer homomorphisms introduced in (i) and (ii) verify

$$\mathrm{Br}_D^H(C[b]) = \mathrm{Br}_D^{N_H(D)}(C'[b_1]).$$

This last equality holds because both $C[b]$ and $C'[b_1]$ are crossed products. Taking a closer look at the proof of [3, Theorem 4.1] we observe that $C[b]/C[b]J(C[b]_1)$ as well as $C[b_1]/C[b_1]J(C'[b_1]_1)$ are both isomorphic to the twisted group algebra associated to the Clifford extension of $\bar{b} = \bar{b}_1$. It follows that the correspondence obtained in Proposition 4.3 connects the central idempotents B', which is primitive in $C[b]^{H_b}$, and B'_1, which is primitive in $C'[b_1]^{N_H(D)_b}$, that verify

$$\mathrm{Br}_D^H(B') = \mathrm{Br}_D^{N_H(D)}(B'_1). \tag{5.1}$$

Let B be the block covering b corresponding to B' through isomorphism (3.1). Note that it suffices to choose L, a defect group of B in H, such that $L \cap K = D$. Then, according to 3.3 there is $y \in H$ such that $Q := L^y$ is a defect group of B and of B' that is contained in H_b and satisfies $Q \cap K = D$; moreover $y \in N_H(D)$. Mackey decomposition and the equalities $Q \cap K = D$, $H_b = N_H(D)_b K$ prove

$$\mathrm{Br}_D^H((b\mathcal{O}H_b)_Q^{H_b}) = \mathrm{Br}_D^{N_H(D)}((b_1\mathcal{O}N_H(D)_b)_Q^{N_H(D)_b}) := \mathcal{I}'.$$

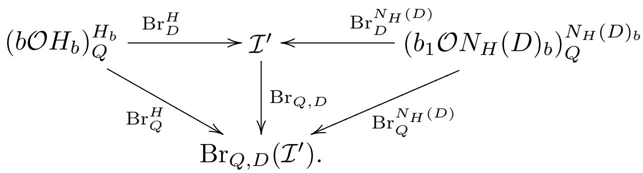
Indeed, since

$$\mathrm{Br}_D^H(b\mathcal{O}H_b^Q) = \mathrm{Br}_D^{N_H(D)}(b_1\mathcal{O}N_H(D)_b^Q),$$

if $\mathrm{Tr}_Q^{H_b}(a) \in (b\mathcal{O}H_b)_Q^{H_b}$ we have

$$\begin{aligned}\mathrm{Br}_D^H(\mathrm{Tr}_Q^{H_b}(a)) &= \mathrm{Br}_D^H(\sum_{x\in D\backslash H_b/Q} \mathrm{Tr}_{D\cap Q^x}^D(a^x)) \\ &= \mathrm{Br}_D^H(\sum_{x\in D\backslash N_H(D)_b/Q} a^x) = \mathrm{Tr}_Q^{N_H(D)_b}(\mathrm{Br}_D^H(a)) \\ &= \mathrm{Tr}_Q^{N_H(D)_b}(\mathrm{Br}_D^{N_H(D)}(a')) = \mathrm{Br}_D^{N_H(D)}(\mathrm{Tr}_Q^{N_H(D)_b}(a')).\end{aligned}$$

At this step [2, Proposition 1.5] gives the commutativity of the diagram



This diagram and [9, Proposition 18.5 (d)] prove that there is a unique correspondent block $\tilde{B}_1' \in (b_1\mathcal{O}N_H(D)_b)^{N_H(D)_b}$ of B' with the same defect group Q in $N_H(D)_b$ as B'. Now we can clearly see, by the commutativity of the above diagram and equality (5.1), that

$$\mathrm{Br}_D^H(B') = \mathrm{Br}_D^{N_H(D)}(B_1') = \mathrm{Br}_D^{N_H(D)}(\tilde{B}_1') \neq 0.$$

This means $B_1' = \tilde{B}_1'$, and moreover that B_1 has defect group L. Hence, the Clifford-Dade correspondence preserves the defect groups. Furthermore, since the Clifford-Dade correspondence is given by the Brauer morphisms (i) and (ii) it is quite clear that it coincides with the Brauer correspondence. □

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