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Estimates for the operator of r-th order on simplex

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Abstract. We present general estimates with optimal constants of the degree of approximation by Kirov - Popova operators using weighted K-functionals of first order.

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1. Introduction

In [2] Kirov and Popova associated to each positive linear operator $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$ a new operator $L_r(f,x) = L\left(T_{r,f,\cdot}(x), x\right), r \in \mathbb{N}$ (a generalization of r-th order), where $T_{r,f,y}(x)$ is the Taylor polynomial of degree r for the function f at y. This operator is linear but not necessarily positive. In [7], using same ideea as in [1] we gave a quantitative estimate for the remainder in Taylor's formula using K-functional K_1^{∞} and weighted K-functional $K_{1,\varphi}^{\infty}$ and we obtained estimates for the operator of r-th order. In this paper we extend the results for the generalization of r-th order of a positive linear operator $L : \mathbf{C}(S) \longrightarrow \mathbf{C}(S)$ defined by $L_r(f,\mathbf{x}) = L\left(T_{r,f,\cdot}(\mathbf{x}), \mathbf{x}\right), r \in \mathbb{N}$, with $T_{r,f,\mathbf{y}}(\mathbf{x}) = \sum\limits_{j=0}^{r} \frac{1}{j!} d^{(j)} f(\mathbf{y})(\mathbf{x}-\mathbf{y})^j$, where $S = \left\{\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d | x_1, \cdots, x_d \geq 0,\ x_1 + \cdots + x_d \leq 1\right\}$ is the simplex in $\mathbb{R}^d$, $d \in \mathbb{N}$.

Starting from the weight function used in [5], we consider the function

$$\varphi(\mathbf{x}) = \left[(x_1 + \cdots + x_d)(1-x_1)\cdots(1-x_d)\right]^{\alpha},\ \alpha \in (0,1).$$

We denote by

$$\mathbf{C}_{\varphi}(S) = \left\{ f \in \mathbf{C}(S \setminus \left\{\mathbf{v}_i,\ i = \overline{0,d}\right\}) \,|\, (\exists) \lim_{\mathbf{x}\to\mathbf{v_i}} f(\mathbf{x})\varphi(\mathbf{x}) \in \mathbb{R},\ i = \overline{0,d} \right\}$$

and

$$\mathbf{W}^1_{\mathbf{C}_\varphi}(S)=\left\{f\in\mathbf{C}(\mathbf{S})\mid\frac{\partial f}{\partial x_i}\in\mathbf{C}_\varphi(S),\,i=\overline{1,d}\right\}$$

where $\mathbf{v_i}$, $i=\overline{0,d}$ are simplex vertices. We consider the K-functional

$$K^\infty_{1,\varphi}(f,t)=K^\infty\left(f,t;\mathbf{C}(S),\mathbf{W}^1_{\mathbf{C}_\varphi}(S)\right),\,t>0,$$

defined for the Banach space $(\mathbf{C}(S),\|\cdot\|)$ and the semi-Banach subspace $\left(\mathbf{W}^1_{\mathbf{C}_\varphi}(S),|\cdot|_{W^1_{C_\varphi}}\right)$, $|f|_{W^1_{C_\varphi}}=\|\varphi\nabla f\|_\infty=\max\limits_{i=\overline{1,d}}\left\|\varphi\frac{\partial f}{\partial x_i}\right\|$ by

$$K^\infty_{1,\varphi}(f,t)=\inf_{g\in\mathbf{W}^1_{\mathbf{C}_\varphi}(S)}\max\left\{\|f-g\|,t\|\varphi\nabla g\|_\infty\right\}.$$

We use the notation e_0 for the function $e_0:S\subset\mathbb{R}^d\to\mathbb{R}$, $e_0(\mathbf{x})=1$ and e_1 for the function $e_1:S\subset\mathbb{R}^d\to\mathbb{R}^d$, $e_1(\mathbf{x})=\mathbf{x}$.

2. Estimates with weighted K - functionals $K^\infty_{1,\varphi}$

Lemma 2.1. *If $f\in\mathbf{C^r}(S)$, $r\in\mathbb{N}$, $\mathbf{x}\in S\setminus\left\{\mathbf{v}_i,\,i=\overline{0,d}\right\}$ and $\mathbf{y}\in S$ then $(\forall)t>0$ for the remainder in Taylor's formula of order r we have the following estimate*

$$|R_{r,f,\mathbf{y}}(\mathbf{x})|\le\left(2\frac{\|\mathbf{y}-\mathbf{x}\|_1^r}{r!}+\frac{\|\mathbf{y}-\mathbf{x}\|_1^{r+1}}{\prod\limits_{k=1}^{r+1}(k-\alpha)t\varphi(\mathbf{x})}\right)\max_{r_1+\cdots+r_d=r}K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}},t\right).\tag{2.1}$$

Proof. Let $f\in\mathbf{C^r}(S)$ and $\mathbf{x}\in S\setminus\left\{\mathbf{v}_i,\,i=\overline{0,d}\right\}$, $\mathbf{y}\in S$, $\mathbf{x}\neq\mathbf{y}$. We consider $\psi(u)=(1-u)\mathbf{x}+u\mathbf{y}$, $u\in[0,1]$ and $h(u)=f(\psi(u))$.

Step 1. We prove that

$$|R_{r,f,\mathbf{y}}(\mathbf{x})|=|R_{r,h,1}(0)|\le\left(\frac{2}{r!}+\frac{1}{\prod\limits_{k=1}^{r+1}(k-\alpha)t\varphi(\mathbf{x})}\right)K^\infty_{1,\varphi\circ\psi}\left(h^{(r)},t\right).\tag{2.2}$$

Let $g\in\mathbf{W}^{\mathbf{r+1}}_{\mathbf{C}_{\varphi\circ\psi}}[0,1]$. Let us now make use of the fact that the function

$$u\mapsto\frac{1-u}{\varphi(\psi(u))^{\frac{1}{\alpha}}},\,u\in(0,1)$$

is decreasing [5]. Using the integral form of the remainder we have

$$\begin{aligned}
|R_{r,g,1}(0)| &= \left| \frac{1}{r!} \int_1^0 g^{(r+1)}(u)(0-u)^r du \right| \\
&\le \frac{1}{r!} \int_0^1 \left| (\varphi \circ \psi)(u) g^{(r+1)}(u) \right| \frac{u^r}{(\varphi \circ \psi)(u)} du \\
&\le \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{r!} \int_0^1 u^r \frac{1}{(\varphi \circ \psi)(0)\,(1-u)^\alpha} du \\
&= \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{r! \varphi(\mathbf{x})} \int_0^1 \frac{u^r}{(1-u)^\alpha} du \\
&= \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{r! \varphi(\mathbf{x})} B\left(r+1, 1-\alpha\right) = \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{\prod\limits_{k=1}^{r+1} (k-\alpha) \cdot \varphi(\mathbf{x})}
\end{aligned}$$

where B is Euler beta function.

We have

$$\begin{aligned}
|R_{r,h-g,1}(0)| &= \left| (h-g)(0) - \sum_{k=0}^{r} \frac{(h-g)^{(k)}(1)}{k!} (-1)^k \right| \\
&= \left| R_{r-1,h-g,1}(0) - \frac{(h-g)^{(r)}(1)}{r!} (-1)^r \right| \\
&\le |R_{r-1,h-g,1}(0)| + \frac{\left\| (h-g)^{(r)} \right\|}{r!} \le 2 \frac{\left\| h^{(r)} - g^{(r)} \right\|}{r!}.
\end{aligned}$$

Then

$$\begin{aligned}
&|R_{r,h,1}(0)| \le |R_{r,h-g,1}(0)| + |R_{r,g,1}(0)| \\
&\le \left(2 \frac{1}{r!} \left\| h^{(r)} - g^{(r)} \right\| + \frac{\left\| (\varphi \circ \psi) g^{(r+1)} \right\|}{\prod\limits_{k=1}^{r+1} (k-\alpha) \cdot \varphi(\mathbf{x})} \right) \\
&\le \left(\frac{2}{r!} + \frac{1}{\prod\limits_{k=1}^{r+1} (k-\alpha) \cdot t\varphi(\mathbf{x})} \right) \cdot \max \left\{ \left\| h^{(r)} - g^{(r)} \right\|, t \left\| (\varphi \circ \psi) g^{(r+1)} \right\| \right\}.
\end{aligned}$$

Since g is arbitrary this implies (2.2).

Step 2. From (2.2) it results

$$|R_{r,f,\mathbf{y}}(\mathbf{x})| \leq \left(\frac{2}{r!} + \frac{\|\mathbf{y}-\mathbf{x}\|_1}{\prod\limits_{k=1}^{r+1}(k-\alpha)\cdot t\varphi(\mathbf{x})} \right) K_{1,\varphi\circ\psi}^{\infty}\left(h^{(r)}, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1} \right). \tag{2.3}$$

Let $\varepsilon > 0$. We choose $g_{r_1,\cdots,r_d} \in \mathbf{C^1}(S)$, $r_i \in \mathbb{N}\cup\{0\}$, $i=\overline{1,d} :\ r_1+\cdots+r_d = r$, such that

$$K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t \right) + \varepsilon$$
$$\geq \max\left\{ \left\| \frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d} \right\|, t\left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_\infty \right\}.$$

We consider the function

$$h_0(u) = \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} g_{r_1,\cdots,r_d}\left(\psi(u)\right) \prod_{i=1}^{d}(y_i-x_i)^{r_i}.$$

We have

$$K_{1,\varphi\circ\psi}^{\infty}\left(h^{(r)}, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1} \right) \leq \max\left\{ \left\| h^{(r)} - h_0 \right\|, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1} \left\|(\varphi\circ\psi)h_o'\right\| \right\}.$$

Since

$$\left| h^{(r)}(u) - h_0(u) \right| = \left| d^r f\left(\psi(u)\right)(\mathbf{y}-\mathbf{x})^r - h_0(u) \right|$$
$$= \left| \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d} \right)\left(\psi(u)\right) \prod_{i=1}^{d}(y_i-x_i)^{r_i} \right|$$
$$\leq \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \left\| \frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d} \right\| \cdot \prod_{i=1}^{d}|y_i-x_i|^{r_i}.$$

hold for u arbitrary this implies

$$\left\| h^{(r)} - h_0 \right\| \leq \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \left\| \frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g_{r_1,\cdots,r_d} \right\| \cdot \prod_{i=1}^{d}|y_i-x_i|^{r_i}.$$

Also since

$$|\varphi(\psi(u))h_0'(u)|$$
$$= \left| \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \varphi(\psi(u)) d g_{r_1,\cdots,r_d}\left(\psi(u)\right)(\mathbf{y}-\mathbf{x}) \prod_{i=1}^{d}(y_i-x_i)^{r_i} \right|$$
$$\leq \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!} \left\|\varphi\nabla g_{r_1,\cdots,r_d}\right\|_\infty \cdot \|\mathbf{y}-\mathbf{x}\|_1 \cdot \prod_{i=1}^{d}|y_i-x_i|^{r_i}.$$

hold for u arbitrary this implies

$$\|(\varphi\circ\psi)h_0'\| \le \sum_{r_1+\cdots+r_d=r} \frac{r!}{r_1!\cdots r_d!}\, \|\varphi\nabla g_{r_1,\cdots,r_d}\|_\infty \cdot \|\mathbf{y}-\mathbf{x}\|_1 \cdot \prod_{i=1}^{d} |y_i-x_i|^{r_i}.$$

Then

$$\begin{aligned}
&K^\infty_{1,\varphi\circ\psi}\left(h^{(r)}, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\right) \le \max\left\{\left\|h^{(r)}-h_0\right\|, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\,\|(\varphi\circ\psi)h_o'\|\right\}\\
&\le \sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^{d}|y_i-x_i|^{r_i}\cdot\\
&\qquad\cdot\max\left\{\left\|\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}-g_{r_1,\cdots,r_d}\right\|, t\,\|\varphi\nabla g_{r_1,\cdots,r_d}\|_\infty\right\}\\
&\le \sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^{d}|y_i-x_i|^{r_i}\left(K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t\right)+\varepsilon\right).
\end{aligned}$$

Since ε is arbitrary this implies

$$\begin{aligned}
&K^\infty_{1,\varphi\circ\psi}\left(h^{(r)}, \frac{t}{\|\mathbf{y}-\mathbf{x}\|_1}\right)\\
&\qquad\le \sum_{r_1+\cdots+r_d=r}\frac{r!}{r_1!\cdots r_d!}\prod_{i=1}^{d}|y_i-x_i|^{r_i}\, K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t\right)\\
&\qquad\le \|\mathbf{y}-\mathbf{x}\|_1^r \max_{r_1+\cdots+r_d=r} K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t\right).
\end{aligned}$$

Finally, with (2.3) result (2.1). □

Theorem 2.2. *Let* $r\in\mathbb{N}$, $L:\mathbf{C}(S)\longrightarrow\mathbf{C}(S)$ *a positive linear operator and* $f\in\mathbf{C^r}(S)$. *Then* $(\forall)\mathbf{x}\in S\setminus\left\{v_i,\, i=\overline{0,d}\right\}$, $(\forall)t>0$ *we have*

$$\begin{aligned}
&|L_r(f,\mathbf{x})-f(\mathbf{x})| \le |f(\mathbf{x})|\cdot|L(e_0,\mathbf{x})-1| \qquad (2.4)\\
&\quad+\left(\frac{2}{r!}L\left(\|e_1-\mathbf{x}e_0\|_1^r,\mathbf{x}\right)+\frac{L\left(\|e_1-\mathbf{x}e_0\|_1^{r+1},\mathbf{x}\right)}{\prod_{k=1}^{r+1}(k-\alpha)t\varphi(\mathbf{x})}\right)\cdot\\
&\quad\cdot\max_{r_1+\cdots+r_d=r}K^\infty_{1,\varphi}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t\right)
\end{aligned}$$

Conversely, if $(\exists)A$, B, $C \geq 0$ *such that*

$$|L_r(f,\mathbf{x}) - f(\mathbf{x})| \leq A \cdot |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| \tag{2.5}$$

$$+ \left(B \cdot L\left(\|e_1 - \mathbf{x}e_0\|_1^r, \mathbf{x}\right) + C \cdot \frac{L\left(\|e_1 - \mathbf{x}e_0\|_1^{r+1}, \mathbf{x}\right)}{t\varphi(\mathbf{x})} \right) \cdot$$

$$\cdot \max_{r_1+\cdots+r_d=r} K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t \right)$$

holds for all positive linear operator $L : \mathbf{C}(S) \longrightarrow \mathbf{C}(S)$*, any* $f \in \mathbf{C^r}(S)$*, any* $\mathbf{x} \in S$ *and any* $t > 0$ *then* $A \geq 1$*,* $B \geq \frac{2}{r!}$ *and* $C \geq \frac{1}{\prod_{k=1}^{r+1}(k-\alpha)}$*.*

Proof. From Lemma 2.1 we have

$$\begin{aligned} |L_r(f,\mathbf{x}) - f(\mathbf{x})| &= |L(T_{r,f,\cdot}(\mathbf{x}),\mathbf{x}) - f(\mathbf{x})| \\ &= |L(f(\mathbf{x})e_0 - R_{r,f,\cdot}(\mathbf{x}),\mathbf{x}) - f(\mathbf{x})| \\ &\leq |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| + |L(R_{r,f,\cdot}(\mathbf{x}),\mathbf{x})| \\ &\leq |f(\mathbf{x})| \cdot |L(e_0,\mathbf{x}) - 1| \\ &+ \left(\frac{2}{r!} L\left(\|e_1 - \mathbf{x}e_0\|_1^r, \mathbf{x}\right) + \frac{L\left(\|e_1 - \mathbf{x}e_0\|_1^{r+1}, \mathbf{x}\right)}{\prod_{k=1}^{r+1}(k-\alpha)t\varphi(\mathbf{x})} \right) \cdot \\ &\quad \cdot \max_{r_1+\cdots+r_d=r} K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t \right). \end{aligned}$$

which is (2.4).

We prove now the converse part. If we choose $L(h,\mathbf{x}) = 0$ and $f = e_0$ and replace in (2.5) we obtain $A \geq 1$.

To show that $B \geq \frac{2}{r!}$ we choose $L(h,\mathbf{x}) = h(0)$ and $f(\mathbf{x}) = 2(x_1 + \cdots + x_d)^{r+a}$ with $a > 0$. For $g = (r+a)\cdot(r+a-1)\cdots(a+1)\,e_0$ we have

$$\begin{aligned} K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t \right) &\leq \max\left\{ \left\| \frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g \right\|, t\,\|\varphi\nabla g\| \right\} \\ = \left\| \frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} - g \right\| &= (r+a)\cdot(r+a-1)\cdots(a+1)\,. \end{aligned}$$

From (2.5) we obtain

$$\begin{aligned} &2(x_1 + \cdots + x_d)^{r+a} \\ &\leq \left(B(x_1+\cdots+x_d)^r + C\frac{(x_1+\cdots+x_d)^{r+1}}{t\varphi(\mathbf{x})} \right)(r+a)\cdot(r+a-1)\cdots(a+1)\,. \end{aligned}$$

Passing to the limit $t \to \infty$, $a \to 0$ we obtain $B \geq \frac{2}{r!}$.

To show that $C \geq \dfrac{1}{\prod_{k=1}^{r+1}(k-\alpha)}$ we choose

$$L(h,\mathbf{x}) = h(0) \text{ and } f(\mathbf{x}) = (x_1 + \cdots + x_d)^{r-\alpha+1}.$$

We have

$$K_{1,\varphi}^{\infty}\left(\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}}, t\right) \leq t \left\| \varphi \frac{\partial^{r+1} f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} \right\| = t \prod_{k=1}^{r+1}(k-\alpha).$$

From (2.5) we obtain

$$(x_1 + \cdots + x_d)^{r-\alpha+1} \leq B(x_1 + \cdots + x_d)^r t \prod_{k=1}^{r+1}(k-\alpha)$$

$$+C\frac{(x_1 + \cdots + x_d)^{r+1}}{\varphi(\mathbf{x})} \cdot \prod_{k=1}^{r+1}(k-\alpha).$$

Passing to the limit $t \to 0$ we obtain

$$C \geq [(1-x_1)\cdots(1-x_d)]^{\alpha} \cdot \frac{1}{\prod_{k=1}^{r+1}(k-\alpha)}$$

and passing to the limit $\mathbf{x} \to \mathbf{0}$ we obtain $C \geq \dfrac{1}{\prod_{k=1}^{r+1}(k-\alpha)}$. □

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