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On Cheney and Sharma type operators reproducing linear functions

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Abstract. With the help of generating functions, we present general conditions to construct positive linear operators which reproduce linear functions. The results are used to present a modification of the Cheney and Sharma operators and the rate of convergence is studied.

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1. Introduction

For an interval I, let $C(I)$ ($C_B(I)$) be the space of the real (bounded) continuous functions defined on I. As usual, we denote $e_k(x) = x^k$, for $k \in \mathbb{N}_0$.

In [5] Cheney and Sharma introduced a modification of Meyer-König and Zeller operators by defining, for a fixed $t \leq 0$, $f \in C[0,1]$ and $x \in [0,1)$

$$L_{n,t}(f,x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f\left(x_{n,k}\right) L_k^{(n)}(t) x^k, \tag{1.1}$$

where

$$x_{n,k} = \frac{k}{n+k},$$

and the functions $L_k^{(n)}(t)$ are the Laguerre polynomials. It is known that (see [12], p. 101, eq. 5.1.6))

$$L_k^{(n)}(t) = \sum_{j=0}^{k} \binom{n+k}{k-j} \frac{(-t)^j}{j!}. \tag{1.2}$$

Hence $L_k^{(n)}(t) \geq 0$ (for $t \leq 0$) and the operators (1.1) are positive. On the other hand, it follows from the properties of Laguerre polynomials that $L_{n,t}(e_0) = e_0$ (see [12], p. 101, eq. 5.1.9)). It can be proved that $L_{n,t}(e_1) = e_1$ if and only if $t = 0$ (see [1]). This property was asserted in [5], but the proof given there is not correct. When $t = 0$, we obtain what are usually called the (slight modification of the) Meyer-König and Zeller operators (see [11]).

The Meyer-König and Zeller operators have been intensively studied and several modifications have been proposed (for instance, see [1], [6], [9], [10], [13] and the references therein).

In Section 3 of this paper we show that the nodes $x_{n,k}$ in (1.1) can be selected in such a way that the new operators reproduce linear functions, and we also give an estimate of the rate of convergence (in terms of the so called Ditzian-Totik moduli). First, in Section 2, we analyze the problem for general positive linear operators constructed by means of generating functions. Finally, in the last section we provide another example to show that the general approach of Section 2 can be used to modify other known operators.

2. Generating functions

Let us begin with a general approach to construct positive linear operators.

Theorem 2.1. *Fix $a > 0$ and sequence $\{a_k\}$ of positive real numbers such that*

$$\limsup_{k\to\infty}\left(\frac{a_k}{k!}\right)^{1/k} = \frac{1}{a} \tag{2.1}$$

and set

$$g(z) = \sum_{k=0}^{\infty}\frac{a_k}{k!}z^k, \qquad |\, z \,| < a. \tag{2.2}$$

Let $\{y_k\}_{k=0}^{\infty}$ be any increasing sequence of points satisfying $y_k \in [0, a)$.

(i) *If $f : [0, a) \to \mathbb{R}$ is a bounded function and $x \in [0, a)$, then the series*

$$L(f, x) = \frac{1}{g(x)}\sum_{k=0}^{\infty}\frac{a_k}{k!}f\left(y_k\right)x^k, \tag{2.3}$$

defines a function that is continuous on $[0, a)$.

(ii) *The map L defines a positive linear operator in $C_B[0, a)$ which reproduces the constant functions.*

(iii) *One has $L(e_1) = e_1$ if (and only if) $y_0 = 0$ and*

$$y_{k+1} = \frac{(k+1)a_k}{a_{k+1}}, \quad \text{for all} \quad k \geq 0. \tag{2.4}$$

(iv) *Suppose that $y_k \longrightarrow a$ and $g(x) \longrightarrow \infty$ as $x \longrightarrow a$. If $f \in C[0, a]$ and we set $L(f, a) = f(a)$, then $L(f) \in C[0, a]$.*

Proof. (i) It follows from (2.1) that g is an analytic function in the domain $|\, z \,| < a$. If $|\, f(x) \,| \leq C(f)$ for $x \in [0, a)$, then $|\, L(f, x) \,| \leq C(f)g(x)$ and the series converges uniformly on the compact subsets of $[0, a)$.

(ii) It is clear that $L(f)$ is well defined for each $f \in C_B[0, 1]$ and L is a positive linear operator on this space. The assertion $L(e_0) = e_0$ follows from (2.2).

(iii) If $L(e_1) = e_1$, then $L(e_1, 0) = a_0e_1(y_0) = e_1(0) = 0$. Since $a_0 > 0$, we obtain $y_0 = 0$. On the other hand, if $y_0 = 0$, then $L(e_1, 0) = 0 = e_1(0)$.

For $0 < x < a$,

$$L(e_1, x) = \frac{x}{g(x)} \sum_{k=1}^{\infty} \frac{a_k}{k!}\, y_k\, x^{k-1} = \frac{x}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!}\, \frac{a_{k+1} y_{k+1}}{a_k (k+1)}\, x^k$$

$$= x + \frac{x}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{a_{k+1}\, y_{k+1}}{a_k (k+1)} - 1 \right) x^k .$$

Thus, $L(e_1, x) = x$ if and only if

$$\frac{a_{k+1}\, y_{k+1}}{a_k (k+1)} = 1, \quad \text{for all} \quad k \geq 0,$$

and this is equivalent to (2.4).

(iv) Fix $\varepsilon > 0$ and $t > 0$ such that $| f(x) - f(a) | < \varepsilon/2$, whenever $| x - a | < t$. Since $y_k \longrightarrow a$, there exists a natural m such that $| y_k - a | < t$, for all $k > m$. Set

$$C = \sup_{x \in [0,a]} \left| \sum_{k=0}^{m} \frac{a_k}{k!} x^k \right| .$$

On the other hand, there exists $\delta > 0$ such that, if $0 < a - x < \delta$, then

$$\frac{1}{g(x)} < \frac{\varepsilon}{4C(1 + \|f\|)},$$

where we consider the sup norm on $[0, a]$. Therefore, if $0 < a - x < \delta$, then

$$| L(f, x) - f(x) | = \left| \frac{1}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!} \Big(f(y_k) - f(a) \Big) x^k \right|$$

$$\leq \frac{2\|f\|}{g(x)} \sum_{k=0}^{m} \frac{a_k}{k!} x^k + \frac{1}{g(x)} \sum_{k=m+1}^{\infty} \frac{a_k}{k!} \, | f(y_k) - f(a) | \, x^k$$

$$\leq \frac{2\|f\| C}{g(x)} + \frac{\varepsilon}{2} \frac{1}{g(x)} \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k < \varepsilon .$$

This proves the assertion. □

Notice that, in order to use condition (2.4), we also need the inequality

$$a_k < a\, \frac{a_{k+1}}{k+1}, \qquad (k \geq 0),$$

which follows from the conditions $y_k \in [0, a)$.

In the next result we consider the case $a = 1$.

Theorem 2.2. *Suppose that the analytic function h has the expansion*

$$h(z) = \sum_{k=0}^{\infty} b_k z^k, \qquad | z | < 1,$$

with $0 < b_{k-1} < b_k$, for all $k \in \mathbb{N}$. Then the equation

$$L(f, x) = \frac{1}{h(x)} \sum_{k=0}^{\infty} b_k\, f\left(\frac{b_{k-1}}{b_k} \right) x^k, \qquad x \in [0, 1), \tag{2.5}$$

where $b_{-1}=0$, defines a positive linear operator, $L : C_B[0,1) \to C_B[0,1)$ such that $L(e_0)=e_0$ and $L(e_1)=e_1$.

Moreover, if

$$\lim_{x\to 1-} h(x)=\infty, \qquad \lim_{k\to\infty}\frac{b_{k-1}}{b_k}=1 \tag{2.6}$$

and we set $L(f,1)=f(1)$, then $L : C[0,1]\to C[0,1]$.

Proof. With the notation given above, one has $k!b_k = a_k$ and $a\geq 1$. Thus, in this case, equation (2.4) can be written as

$$y_{k+1}=\frac{(k+1)a_k}{a_{k+1}}=\frac{b_k}{b_{k+1}}<1, \quad \text{for all} \quad k\geq 0. \qquad \square$$

3. A variation of Cheney and Sharma operators

Theorem 3.1. *Fix $t\leq 0$ and let the numbers $L_k^{(n)}(t)$ ($n\in\mathbb{N}$, $k\geq 0$) be defined by (1.2) and set $L_{-1}^{(n)}=0$. Then the equation*

$$S_{n,t}(f,x) \;=\; (1-x)^{n+1}\exp\left(\frac{tx}{1-x}\right)\sum_{k=0}^{\infty} L_k^{(n)}(t)\; f\left(\frac{L_{k-1}^{(n)}(t)}{L_k^{(n)}(t)}\right)x^k, \tag{3.1}$$

where $L_{-1}^{(n)}(t)=0$ and $x\in[0,1)$, defines a positive linear operator on $C_B[0,1)$ such that

$$S_{n,t}(e_0,x)=1, \qquad S_{n,t}(e_1,x)=x \tag{3.2}$$

and

$$0\;\leq\; S_{n,t}(e_2,x)-x^2 \;\leq\; x^2+\frac{2x(1-x)(1-tx)^2}{n}\,. \tag{3.3}$$

Moreover, if for $f\in C[0,1]$ we set $S_{n,t}(f,1)=f(1)$, then $S_{n,t} : C[0,1]\to C[0,1]$.

Proof. Since t will be fixed, in order to simplify, we write $L_k^{(n)}$ instead of $L_k^{(n)}(t)$.

It is known that (see [12], p. 102, Eq. (5.1.14) and (5.1.13))

$$\frac{L_k^{(n)}}{L_{k+1}^{(n)}}=\frac{k+1}{n+k+1}+\frac{t}{n+k+1}\,\frac{L_k^{(n+1)}}{L_{k+1}^{(n)}} \tag{3.4}$$

and

$$\frac{L_k^{(n)}}{L_{k+1}^{(n)}}=1-\frac{L_{k+1}^{(n-1)}}{L_{k+1}^{(n)}}. \tag{3.5}$$

From the last equation we know that $L_k^{(n)}<L_{k+1}^{(n)}$. Thus the operators (3.1) are well defined and it follows from the first part of Theorem 2.2 that (3.2) holds.

Let us verify (3.3). Since S_{n-t} is a positive linear operator, from (3.2) we know that $0\leq S_{n,t}((e_1-x)^2,x)=S_{n,t}(e_2,x)-x^2$.

From (1.2) we know that

$$L_k^{(n+1)}=\sum_{j=0}^{k}\frac{k+1-j}{n+1+j}\binom{n+k+1}{k+1-j}\frac{(-t)^j}{j!}<\frac{k+1}{n+1}L_{k+1}^{(n)}. \tag{3.6}$$

and from (3.4) and (3.5) we obtain (recall that $t \le 0$)

$$
\begin{aligned}
L_k^{(n)}\,\frac{L_k^{(n)}}{L_{k+1}^{(n)}} &= L_k^{(n)} - L_{k+1}^{(n-1)}\frac{L_k^{(n)}}{L_{k+1}^{(n)}} \\
&= L_k^{(n)} - L_{k+1}^{(n-1)}\left(\frac{k+1}{n+k+1} + \frac{t}{n+k+1}\frac{L_k^{(n+1)}}{L_{k+1}^{(n)}}\right) \\
&\le L_k^{(n)} - L_{k+1}^{(n-1)}\frac{k+1}{n+k+1} - L_{k+1}^{(n-1)}\frac{t(k+1)}{(n+1)(n+k+1)}\,.
\end{aligned}
$$

Let us set $g_n(x) = (1-x)^{n+1}\exp\left(tx/(1-x)\right)$. From the last estimate, taking into account that $L_{-1}^{(n)}(t) = 0$, for $n > 2$ and $x \in (0,1)$ we obtain

$$
\begin{aligned}
S_{n,t}(e_2,x) \;&=\; g_n(x)\sum_{k=1}^{\infty} L_{k-1}^{(n)}\,\left(\frac{L_{k-1}^{(n)}}{L_k^{(n)}}\right)x^k \;=\; x\,g_n(x)\sum_{k=0}^{\infty} L_k^{(n)}\,\left(\frac{L_k^{(n)}}{L_{k+1}^{(n)}}\right)x^k \\
&\le\; x\,g_n(x)\sum_{k=0}^{\infty}\left(L_k^{(n)} - \frac{n+1+t}{n+1}\,\frac{k+1}{n+k+1}L_{k+1}^{(n-1)}\right)x^k \\
&= x - \frac{(n+1+t)}{n+1}g_n(x)\sum_{k=0}^{\infty}\left(\frac{k+1}{n+k+1}L_{k+1}^{(n-1)}\right)x^{k+1} \\
&= x - \frac{(n+1+t)}{n+1}g_n(x)\sum_{k=0}^{\infty}\left(L_k^{(n-1)}\frac{k}{n+k}\right)x^k \\
&= x - \frac{(n+1+t)}{n+1}g_n(x)\sum_{k=0}^{\infty}\left(L_k^{(n-1)}\frac{k}{n-1+k}\left(1-\frac{1}{n+k}\right)\right)x^k \\
&= x - \frac{(n+1+t)(1-x)}{n+1}L_{n-1,t}(e_1,x) \\
&\qquad + \frac{n+1+t}{n+1}g_n(x)\sum_{k=0}^{\infty}\left(L_k^{(n-1)}\frac{k}{(n-1+k)(n+k)}\right)x^k \\
&= x^2 - \frac{tx(1-x)}{n+1}\Big(L_{n-1,t}(e_1,x) - x\Big) - \frac{tx(1-x)}{n+1} \\
&\quad + g_n(x)\sum_{k=1}^{\infty}\left(L_k^{(n-1)}\frac{k}{(n-1+k)(n+k)}\right)x^k \\
&\le x^2 - \frac{tx(1-x)}{n+1}\Big(L_{n-1,t}(e_1,x) - x\Big) - \frac{tx(1-x)}{n+1} \\
&\qquad + \frac{1}{n}g_n(x)\sum_{k=1}^{\infty}\left(L_k^{(n-1)}\frac{k}{n-2+k}\right)x^k \\
&= x^2 - \frac{tx(1-x)}{n+1}\Big(L_{n-1,t}(e_1,x) - x\Big) - \frac{tx(1-x)}{n+1} + \frac{1-x}{n}L_{n-1,t}(e_1,x)\,.
\end{aligned}
$$

Since (see Theorem 1 of [1])

$$L_{n,t}(e_1,x) \le x - \frac{tx}{n+1},$$

one has

$$\begin{aligned} S_{n,t}(e_2,x) &\le x^2 + \frac{(tx)^2(1-x)}{(n+1)^2} - \frac{tx(1-x)}{n+1} + \frac{1-x}{n}\left(x - \frac{tx}{n}\right) \\ &\le x^2 + \frac{x(1-x)}{n} - \frac{2tx(1-x)(2-tx)}{n}. \end{aligned}$$

For the last assertion, first notice that, since $t \le 0$,

$$\lim_{x\to 1-} \frac{1}{(1-x)^{n+1}} \exp\left(\frac{-tx}{1-x}\right) = \infty.$$

In order to finish, we only need to verify the second equality in (2.6). But it is a known result (for instance, see [3]). □

Theorem 3.2. *Fix $\alpha \in (0,1/2]$ and set $\varphi(x) = (x(1-x))^\alpha$. For $t \le 0$, let the operators $S_{n,t}$ be defined as in Theorem 3.1. For $f \in C[0,1]$, $x \in [0,1]$, and $n > 2$ one has*

$$|f(x) - S_{n,t}(f,x)| \le \Big(\frac{3}{2} + 3(1-tx)^2\Big)\omega_2^\varphi\left(f, \sqrt{\frac{(x(1-x))^{1-2\alpha}}{n}}\right),$$

where

$$\omega_2^\varphi(f,h) = \sup_{0\le s\le h} \quad \sup_{x\pm s\varphi(x)\in[0,1]} \left|\Delta^2_{h\varphi(x)} f(x)\right|$$

and $\Delta^2_{h\varphi(x)} f(x) = f(x - s\varphi(x))) - 2f(x) + f(x + s\varphi(x))$.

Proof. The result follows from (3.3) and Theorem 11 of [4]. □

4. Another example

Fix $r \in \mathbb{Z}$ and, for $n \in \mathbb{N}$, consider the identity

$$\frac{1}{(1-z)^{n+r}} = \sum_{k=0}^\infty b_{n,k} z^k \qquad |\, z \,| < 1,$$

where

$$b_{n,k} = \binom{n+r+k-1}{k}.$$

We also set $b_{n,-1} = 0$. Notice that, for $k \ge 1$,

$$\frac{b_{n,k-1}}{b_{n,k}} = \frac{k}{n+r+k-1}.$$

Thus we have all the conditions of Theorem 2.2.

Theorem 4.1. *Fix $r \in \mathbb{N}$. For $n \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1)$ set*

$$M_{n,r}(f,x) = (1-x)^{n+r} \sum_{k=0}^{\infty} \binom{n+r+k-1}{k} x^k f\left(\frac{k}{n+r+k-1}\right),$$

and

$$M_{n,r}(f,1) \;=\; f(1)\,.$$

(i) *For each $n \in \mathbb{N}$, $M_{n,r} : C[0,1] \to C[0,1]$ is a positive linear operator such that*

$$M_{n,r}(e_0,x) = 1, \qquad M_{n,r}(e_1,x) = x$$

and, for $n > 2$

$$\frac{x(1-x)^2}{2(n+r-2)} \;\leq\; M_{n,r}(e_2,x) - x^2 \;\leq\; \frac{x(1-x)^2}{n+r-2}\,. \tag{4.1}$$

(ii) *Fix $\alpha \in (0,1/2]$ and set $\varphi(x) = (x(1-x)^2)^\alpha$. For $f \in C[0,1]$, $x \in [0,1]$, and $n > 2$ one has*

$$|f(x) - M_{n,r}(f,x)| \leq 3\omega_2^{\varphi}\left(f, \sqrt{\frac{\varphi^{1-2\alpha}(x)}{n+r-2}}\right).$$

Proof. We only need to verify (4.1). With the notation given above, one has

$$\begin{aligned}
M_{n,r}(e_2,x) - x^2 &= (1-x)^{n+r} \sum_{k=1}^{\infty} b_{n,k-1} \frac{b_{n,k-1}}{b_{n,k}} x^k - x^2 \\
&= x\left((1-x)^{n+r} \sum_{k=1}^{\infty} b_{n,k-1} \frac{b_{n,k-1}}{b_{n,k}} x^{k-1} - x\right) \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n,k} \left(\frac{b_{n,k}}{b_{n,k+1}} - \frac{b_{n,k-1}}{b_{n,k}}\right) x^k \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n,k} \left(\frac{k+1}{n+r+k} - \frac{k}{r+n+k-1}\right) x^k \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n,k} \left(\frac{n+r-1}{(n+r+k)(r+n+k-1)}\right) x^k \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n-1,k} \left(\frac{1}{n+r+k}\right) x^k \\
&= x(1-x)^{n+r} \sum_{k=0}^{\infty} \frac{(n+r+k-2)(n+r+k-3)!}{(n+r-2)k!(n+r-3)!} \left(\frac{1}{n+r+k}\right) x^k \\
&= \frac{x(1-x)^{n+r}}{n+r-2} \sum_{k=0}^{\infty} b_{n-2,k} \left(\frac{n+r+k-2}{n+r+k}\right) x^k.
\end{aligned}$$

Therefore, for $n > 2$,

$$\frac{x(1-x)^2}{2(n+r-2)} = x(1-x)^{n+r} \sum_{k=0}^{\infty} b_{n-2,k} \frac{1}{2(n+r-2)} x^k$$

$$
\begin{aligned}
&\leq \frac{x(1-x)^{n+r}}{n+r-2}\sum_{k=0}^{\infty} b_{n-2,k}\left(\frac{n+r+k-2}{n+r+k}\right)x^k \\
&= M_{n,r}(e_2,x)-x^2 \\
&\leq \frac{x(1-x)^2}{n+r-2}(1-x)^{n-2+r}\sum_{k=0}^{\infty} b_{n-2,k}\,x^k \;=\frac{x(1-x)^2}{n+r-2}\,. \qquad \square
\end{aligned}
$$

Remark 4.2. In [8] (p.17), Götz introduced the operators

$$M^*_{n,r}(f,x)=(1-x)^{n+r}\sum_{k=0}^{\infty}\binom{n+r+k-1}{k}x^k f\left(\frac{k}{n+k}\right).$$

They are similar to the Meyer-König and Zeller operators, but they do not reproduce linear functions (see also [2], p. 126).

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