Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 2, 199–211

Maximum principles for elliptic systems and the problem of the minimum matrix norm of a characteristic matrix, revisited

Ioan A. Rus

Abstract. In 1968, the existence of a maximum principle for some systems of partial differential equations led us to the following problem (see I.A. Rus, Studia Univ. Babeş-Bolyai, 15(1968), No. 1, 19-26 and Glasnik Matematički, 5(1970), No. 2, 356): Let $A \in \mathbb{R}^{n\times n}$ be a matrix and $\|\cdot\|_2$ the spectral norm on $\mathbb{R}^{n\times n}$. The problem is to determine, $\min\limits_{x\in\mathbb{R}}\|A - xI\|_2$. In this paper we study the evolution of this interesting relation between the theory of partial differential equations and the matrix theory. An application of an elliptic partial differential equation with complex valued coefficients is presented. New maximum principles are given and the case of infinite systems is also studied. Some open problems are formulated.

Mathematics Subject Classification (2010): 35B50, 15A60, 40C05, 35J47, 35K40, 15F60, 65F35, 65J05.

Keywords: strongly elliptic system, maximum principle, matrix norm, spectral matrix norm, equation with complex valued coefficients, infinite matrix, infinite system.

1. Introduction

Some time ago, studying maximum principle for elliptic systems of second order we was conducted to the following problem (see [37]-[41]):

Let $\|\cdot\|_2$ be the spectral norm on $\mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$ be a matrix. The problem is to determine, $\min\limits_{x\in\mathbb{R}}\|A - xI\|_2$.

In 1971, E. Deutsch informed me that H. Heinrich (see [15] and [16]) studied a similar problem in the case of Frobenius norm, $\|\cdot\|_F$, column sum norm, $\|\cdot\|_1$, and row sum norm, $\|\cdot\|_\infty$. The problem corresponding to the spectral norm was studied by A.S. Mureşan ([29]) and by I.C. Chifu ([5] and [6]). In 1975, S. Friedland studied the following problem (see [11]):

Let $A, B \in \mathbb{C}^{n\times n}$ be nonzero matrices such that $A \neq xB$ for any $x \in \mathbb{R}$. Let $d := \min\limits_{x\in\mathbb{R}}\|A - xB\|_2$. The problem is to study the solution set of the equation:

$$\|A - xB\|_2 = d.$$

The aim of the present paper is to revisit the "abstract model" in [37], to give new maximum principles in terms of spectral norms and to consider the case of elliptic equations with complex valued coefficients and the case of an infinite system of elliptic equations. Some open problems are also formulated.

2. Preliminaries

2.1. Vector norms and matrix norms

Let us denote by K, $\mathbb{R}$ or $\mathbb{C}$. If $x \in K^n$, then, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $x^* = (x_1, \dots, x_n)$.

We consider on K^n the following norms:

$$\|x\|_\infty := \max_{1\leq k\leq n} |x_k|$$

and

$$\|x\|_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}, \text{ for } p \geq 1.$$

If, $\|\cdot\|$, is a norm on K^n then we denote by the same symbol the operatorial norm (subordonate norm, or natural norm) on $\mathbb{K}^{n\times n}$ corresponding to the norm, $\|\cdot\|$, on K^n.

So, we have $\|A\|_\infty = \max\limits_{1\leq k\leq n}\left(\sum\limits_{j=1}^{n}|a_{kj}|\right)$, $\|A\|_1 = \max\limits_{1\leq j\leq n}\left(\sum\limits_{k=1}^{n}|a_{kj}|\right)$ and $\|A\|_2 = \left(\rho(A^*A)\right)^{\frac{1}{2}}$ - the spectral norm of A.

We also consider on $\mathbb{K}^{n\times n}$ the Frobenius (or Euclidean) norm defined by

$$\|A\|_F := \left(\sum_{k,j=1}^{n} |a_{kj}|^2\right)^{\frac{1}{2}}.$$

This norm is not induced by any norm on K^n, but is a matrix norm, i.e.,

$$\|A \cdot B\|_F \leq \|A\|_F \cdot \|B\|_F, \ \forall\ A, B \in \mathbb{K}^{n\times n}.$$

For an operator norm on $\mathbb{K}^{n\times n}$, $\|\cdot\|$, we have that

$$\|Ax\| \leq \|A\|\|x\|, \ \forall\ A \in \mathbb{K}^{n\times n} \text{ and } x \in K^n.$$

We also have that, $\|A\|_2 \leq \|A\|_F$, $\forall\ A \in \mathbb{K}^{n\times n}$. For the minimum norm problem we mention the following result

Heinrich's Theorem. *Let $A \in \mathbb{C}^{n\times n}$. Then,*

$$\min_{z\in\mathbb{C}} \|A - zI\|_F = \left(\|A\|_F^2 - \frac{1}{n}|trA|^2 \right)^{\frac{1}{2}}.$$

From this theorem we have

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$. Then,*

$$\min_{x\in\mathbb{R}} \|A - xI\|_F = \left(\|A\|_F^2 - \frac{1}{n}|trA|^2 \right)^{\frac{1}{2}}.$$

For more considerations of the above notions and results see: [17] (especially Chapter 37 by R. Byers and B.N. Dalta), [1], [35], [2], [15], [16], [36], [42], [3], ...

2.2. Elliptic systems of second order

Let $\Omega \subset \mathbb{R}^n$ be an open subset. Let us consider the following second order system of partial differential equations:

$$\sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} + F\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0 \tag{2.1}$$

where $A_{kj} : \Omega \to \mathbb{R}^{m\times m}$, $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}^m$.

There are many points of view in classifying systems of partial differential equations (see for example, [27], [10], [8], [20], [4], [13], [23], [31], [34], ...).

In this paper we need the following notions.

Definition 2.2. *The system* (2.1) *is called elliptic on Ω if*

$$\det\left(\sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}(x)\lambda_k\lambda_j \right) \neq 0$$

for all $x \in \Omega$ and all $\lambda \in \mathbb{R}^n \setminus \{0\}$.

Definition 2.3. *The system* (2.1) *is called strongly elliptic on Ω, if*

$$\sum_{k=1}^{n}\sum_{j=1}^{n} \left(\tau^* A_{kj}(x)\tau\right)\lambda_k\lambda_j > 0, \text{ for all } x \in \Omega,$$

for all $\tau \in \mathbb{R}^m \setminus \{0\}$ and all $\lambda \in \mathbb{R}^n \setminus \{0\}$.

Definition 2.4. *The system* (2.1) *satisfies Somigliana's condition on Ω, if*

$$\sum_{k=1}^{n}\sum_{j=1}^{n} \tau_k^* A_{kj}(x)\tau_j > 0, \text{ for all } x \in \Omega,$$

for all $\tau_k \in \mathbb{R}^m$, $k = \overline{1,n}$ with $\sum_{k=1}^{n}\|\tau_k\| \neq 0$.

3. Basic idea and examples

The basic idea of the paper [37] may be presented as follows.

For a subset $\Omega \subset \mathbb{R}^n$ we denote

$$\mathcal{F}(\Omega, \mathbb{R}^m) := \{u \mid u : \Omega \to \mathbb{R}^m\}.$$

Let $D \subset \mathbb{R}^p$, $1 \le p \le n$ and $X \subset \mathcal{F}(\Omega, \mathbb{R}^m)$ be a linear subspace. By definition

$$\langle \cdot, \cdot \rangle : X \times X \to \mathcal{F}(D, \mathbb{R})$$

is a generalized inner product on X if the following axioms are satisfied:

(i) $\langle u, v\rangle = \langle v, u\rangle$, $\forall\ u, v \in X$;
(ii) $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$, $\forall\ u, v \in X$, $\forall\ \lambda \in \mathbb{R}$;
(iii) $\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$, $\forall\ u_1, u_2, v \in X$;
(iv) $\langle u, u\rangle \ge 0$, $\forall\ u \in X$ and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$.

Let $\langle \cdot, \cdot \rangle$ be a generalized inner product on X, $L : X \to \mathcal{F}(\Omega, \mathbb{R}^m)$ be a linear operator and $Y \subset \mathcal{F}(D, \mathbb{R})$ be a linear subspace. In which conditions for each $u \in X$, there exists a linear operator $T_u : Y \to \mathcal{F}(D, \mathbb{R})$ such that

$$\langle u, L(u)\rangle(x) = \|u\|(x) T_u(\|u\|)(x),$$

for all $x \in D$, with $\|u\|(x) \ne 0$.

Let us consider the equations

$$L(u) = 0 \tag{3.1}$$

and

$$T_u(v) = 0. \tag{3.2}$$

If the pair L, T_u is a solution of the above problem and u is a solution of (3.1) and iff all solution of (3.2) has a property (p), then the norm, $\|u\|$, of u has the property (p).

Example 3.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $X := C^2(\Omega, \mathbb{R}^m)$, $D := \Omega$, $Y := C^2(\Omega, \mathbb{R})$ and

$$\langle u, v\rangle := \sum_{k=1}^{m} u_k v_k.$$

As L, let us take the following operator

$$L(u) := \Delta u + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial x_k} + Cu$$

where $B_k, C \in \mathcal{F}(\Omega, \mathbb{R}^{m\times m})$.

Let $u \in C^2(\Omega, \mathbb{R}^m)$ and $x \in \Omega$ be such that $\|u\|(x) \ne 0$. Then

$$u(x) = \|u\|(x)e(x), \text{ with } \langle e, e\rangle(x) = 1.$$

So, in all point $x \in \Omega$, where $\|u\|(x) \ne 0$, we have

$$\langle e, \frac{\partial e}{\partial x_k}\rangle = 0,\ k = \overline{1, m}, \quad \text{and} \quad \langle \frac{\partial e}{\partial x_k}, \frac{\partial e}{\partial x_k}\rangle + \langle e, \frac{\partial^2 e}{\partial x_k^2}\rangle = 0.$$

This relations imply that

$$\langle u, L(u)\rangle = \|u\|\langle e, L(\|u\|e)\rangle = \|u\|T_u(\|u\|)$$

where

$$T_u(v) = \Delta v + \sum_{k=1}^{m}\langle e, B_k e\rangle\frac{\partial v}{\partial x_k} + \langle e, Le\rangle v.$$

From a well known maximum principle for an elliptic differential equation we have (see [12], [34], [33])

Theorem 3.2. *Let L be such that*

$$\langle e, Le\rangle(x) < 0 \tag{3.3}$$

for all $e \in C^2(\Omega, \mathbb{R}^m)$, with $\|e\| = 1$ and all $x \in \Omega$.

Then the norm of each solution $u \in C^2(\Omega, \mathbb{R})$ of (3.1) *has no positive local maximums in Ω.*

Remark 3.3. For the case when Ω is open and bounded and $u \in C^2(\Omega, \mathbb{R}^m) \cap C(\overline{\Omega}, \mathbb{R}^m)$ see [37] and [38].

Remark 3.4. The problem is in which conditions on B_k and C we have the condition (3.3) ?

First of all we have that

$$\langle e, Le\rangle = -\sum_{k=1}^{n}\Big\langle\frac{\partial e}{\partial x_k}, \frac{\partial e}{\partial x_k}\Big\rangle + \sum_{k=1}^{n}\Big\langle e, B_k\frac{\partial e}{\partial x_k}\Big\rangle + \langle e, Ce\rangle \prec 0 \tag{3.4}$$

(a function $u \prec 0 \Leftrightarrow u(x) < 0,\ \forall\ x \in \Omega$).

On the other hand we remark that

$$\Big\langle e, B_k\frac{\partial e}{\partial x_k}\Big\rangle = \Big\langle e, (B_k - b_k I)\frac{\partial e}{\partial x_k}\Big\rangle,$$

for all $b_k \in \mathcal{F}(\Omega, \mathbb{R})$.

So, we have the condition

$$\Big\langle e, Le - \sum_{k=1}^{n} b_k I\frac{\partial e}{\partial x_k}\Big\rangle \prec 0 \tag{3.5}$$

and we have that, (3.4) $\Leftrightarrow$ (3.5).

Now, let us suppose that

$$\sum_{k=1}^{m}\sum_{j=1}^{m} C_{kj}(x)\lambda_k\lambda_j \le -c(x)\sum_{k=1}^{m}|\lambda_k|^2 \tag{3.6}$$

for all $\lambda \in \mathbb{R}^m \setminus \{0\}$, with $c(x) \in \mathbb{R}_+^*$, $\forall\ x \in \Omega$.

Since

$$\Big|\Big\langle e, (B_k - b_k I)\frac{\partial e}{\partial x_k}\Big\rangle(x)\Big| \le \|B_k - b_k I\|_2(x)\Big\|\frac{\partial e}{\partial x_k}\Big\|,\ \forall\ x \in \Omega$$

from Theorem 3.2 it follows

Theorem 3.5. *We suppose that in Theorem* 3.2 *we put instead the condition* (3.3) *the following:*

(*i*) *the matrix* C *satisfies condition* (3.6) *with* $c = \sum\limits_{k=1}^{n} c_k^2$*;*

(*ii*) *there exist* $b_k \in \mathcal{F}(\Omega, \mathbb{R})$ *such that*

$$\|B_k - b_k\|_2 \le 2c_k, \ \ k = \overline{1,n}.$$

Then we have the conclusions in Theorem 3.2.

Remark 3.6. Since $\|\cdot\|_2 \le \|\cdot\|_F$, by Theorem 2.1 we can take in Theorem 3.5,

$$c_k := \frac{1}{2}\big(\|B_k\|_F^2 - \frac{1}{m}|trB_k|^2\big)^{\frac{1}{2}}.$$

Remark 3.7. In a similar way we have

Theorem 3.8 (see [37]**,** [38]**).** *Let us consider the following second order system*

$$L(u) := \sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial x_k} + Cu = 0. \tag{3.7}$$

We suppose that:

(*i*) $\Omega \subset \mathbb{R}^n$ *is an open subset and* $A_{kj}, B_k, C : \Omega \to \mathbb{R}^{m\times m}$ *are arbitrary matriceal functions;*

(*ii*) *the system* (3.7) *is strongly elliptic;*

(*iii*) $\langle e, Le\rangle \prec 0$, *for all* $e \in C^2(\Omega, \mathbb{R}^m)$ *such that* $\|e\| = 1$.

In these conditions the norm of each solution, $u \in C^2(\Omega, \mathbb{R}^m)$, *of* (3.7) *has no positive local maximums.*

Remark 3.9. For the maximum principles for elliptic equations and systems see [27], [13], [34], [12], [21], [23], [33], [43], [44], [5], [6], [29], [30], ...

Example 3.10. Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^n$ be an open subset, where $\Omega_1 \subset \mathbb{R}^p$, $\Omega_2 \subset \mathbb{R}^{n-p}$, $1 \le p < n$, are domains with smooth boundary. If $x \in \Omega$, then $x = (x', x'')$ with $x' = (x_1, \dots, x_p) \in \Omega_1$, $x'' = (x_{p+1}, \dots, x_n) \in \Omega_2$.

Let $X := C^2(\overline{\Omega}, \mathbb{R}^m)$, $D := \Omega_1$, $Y := C^2(\overline{\Omega}, \mathbb{R})$ and

$$\langle u, v\rangle := \int\limits_{\Omega_2} \left(\sum_{k=1}^{m} u_k v_k\right) dx''.$$

For $x \in \Omega$ such that $\|u\|(x) \ne 0$ we have

$$u(x) = \|u\|(x_1, \dots, x_p)e(x), \ \text{with} \ \langle e, e\rangle = 1;$$

and

$$\langle e, \frac{\partial e}{\partial x_k}\rangle = 0, \ k = \overline{1,p}, \ \langle \frac{\partial e}{\partial x_j}, \frac{\partial e}{\partial x_k}\rangle + \langle e, \frac{\partial^2 e}{\partial x_k \partial x_j}\rangle = 0$$

for $k, j = \overline{1,p}$.

Let us take

$$L(u) := \Delta u + \sum_{k=1}^{n} B_k(x_1, \ldots, x_p)\frac{\partial u}{\partial x_k} + C(x_1, \ldots, x_p)u = 0 \tag{3.8}$$

$$L(\|u\|e) = \sum_{k=1}^{p} \frac{\partial^2 \|u\|}{\partial x_k^2} e + 2\sum_{k=1}^{p} \frac{\partial \|u\|}{\partial x_k}\frac{\partial e}{\partial x_k} + \sum_{k=1}^{p} B_k \frac{\partial \|u\|}{\partial x_k} e + \|u\| Le.$$

So,

$$T_u(v) = \sum_{k=1}^{p} \frac{\partial^2 v}{\partial x_k^2} + \sum_{k=1}^{p} \langle e, B_k e\rangle \frac{\partial v}{\partial x_k} + \langle e, Le\rangle v.$$

From the above considerations, we have

Theorem 3.11. *Let L be such that*

$$\langle e, Le\rangle(x') < 0, \ \forall\ x' \in \Omega_1, \ \textit{and}$$

for all $e \in C^2(\overline{\Omega}, \mathbb{R}^m)$ with $\|e\| = 1$.

Then the norm of each solution $u \in C^2(\overline{\Omega}, \mathbb{R}^m)$ of, $L(u) = 0$, has no positive local maximums in Ω_1.

Remark 3.12. As in the case of condition (3.3), the problem is in which conditions we have

$$\langle e, Le\rangle(x') < 0 \tag{3.9}$$

for all $e \in C^2(\overline{\Omega}, \mathbb{R}^m)$ with $\|e\| = 1$ and all $x' \in \Omega_1$.

First of all we have that

$$\begin{aligned}\langle e, Le\rangle = &\int_{\Omega_2} \sum_{k=1}^{p} \Big(-\sum_{j=1}^{m} \big(\frac{\partial e_j}{\partial x_k}\big)^2 \Big) d\xi'' + \int_{\Omega_2} \sum_{k=p}^{n} \sum_{j=1}^{m} e_j \frac{\partial^2 e_j}{\partial x_k} d\xi'' \\ &+ \sum_{k=1}^{p} \int_{\Omega_2} \sum_{j=1}^{m} e_j \big(B_k \frac{\partial e}{\partial x_k}\big)_j d\xi'' + \sum_{k=p+1}^{n} \int_{\Omega_2} \sum_{j=1}^{m} e_j \big(B_k \frac{\partial e}{\partial x_k}\big)_j d\xi'' \\ &+ \int_{\Omega_2} \sum c_{kj} e_k e_j d\xi''.\end{aligned}$$

From this relation and for a well known maximum principle for an elliptic equation, we have

Theorem 3.13. *Let us suppose that*

(*i*) $-\sum_{j=1}^{p} \langle \tau_j, \tau_j\rangle_E + \sum_{k=1}^{p} \langle \eta, (B_k - b_k I)\tau_k\rangle_E + \sum_{k=p+1}^{n} \langle \eta, B_k \tau_k\rangle_E + \langle \eta, c\eta\rangle_e \prec 0$ *for all* $\eta, \tau_k \in \mathbb{R}^m \setminus \{0\}$, *and for some* $b_k \in \mathcal{F}(\Omega_1, \mathbb{R})$, $k = \overline{1, p}$;

(*ii*) $u \in C^2(\overline{\Omega}, \mathbb{R}^m)$ *is a solution of* (3.9) *such that,* $u\big|_{\Omega_1 \times \partial\Omega_2} = 0$.

Then the norm of u has no positive local maximums in Ω_1.

Here, $\langle \cdot, \cdot \rangle_E$ denotes the Euclidean inner product.

Remark 3.14. For the case of $p = 1$ see [21].

Remark 3.15. For the case of a class of systems which satisfy Somiglian's condition see [41].

Remark 3.16. It is clear that, in all of above cases, a solution for the minimum norm problem is very important.

4. An application to an elliptic equation with complex valued coefficients

Let us consider the following elliptic equation

$$\Delta u + \sum_{k=1} p_k \frac{\partial u}{\partial x_k} + qu = 0 \tag{4.1}$$

where $p_k, q : \Omega \to \mathbb{C}$ with $\Omega \subset \mathbb{R}^n$ an open subset.

By a solution of (4.1) we understand a function $u \in C^2(\Omega, \mathbb{C})$ which satisfies the equation (4.1).

The equation (4.1) is equivalent with the following system of elliptic equations

$$\Delta \begin{pmatrix} \mathrm{Re}u \\ \mathrm{Im}u \end{pmatrix} + \sum_{k=1}^{m} \begin{pmatrix} \mathrm{Re}p_k & -\mathrm{Im}p_k \\ \mathrm{Im}p_k & \mathrm{Re}p_k \end{pmatrix} \frac{\partial}{\partial x_k} \begin{pmatrix} \mathrm{Re}u \\ \mathrm{Im}u \end{pmatrix}$$
$$+ \begin{pmatrix} \mathrm{Re}q & -\mathrm{Im}q \\ \mathrm{Im}q & \mathrm{Re}q \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Re}u \\ \mathrm{Im}u \end{pmatrix} = 0.$$

If in Theorem 3.5 we take $b_k := \mathrm{Re}p_k$, we have from this theorem the following result.

Theorem 4.1. *Let us consider the equation* (4.1). *We suppose that*

$$\mathrm{Re}q(x) < -\frac{1}{4} \sum_{k=1}^{n} \big(\mathrm{Im}p_k(x)\big)^2, \ \forall\, x \in \Omega.$$

If $u \in C^2(\Omega, \mathbb{C})$ is a solution of (4.1), *then, $|u|$ has no positive local maximums in Ω.*

Remark 4.2. For a similar results, see [28].

5. Infinite elliptic systems of partial differential equations

We start this section with some words on infinite matrices.

Let $A \in \mathbb{K}^{\mathbb{N}^* \times \mathbb{N}^*}$ be an infinite matrix which is row-column-finite. This matrix induces the linear operator

$$\tilde{A} : l_2(\mathbb{K}) \to \mathbb{K}^{\mathbb{N}^*}.$$

By definition the matrix A is 2-bounded if:

(*i*) $\tilde{A}(l_2(\mathbb{K})) \subset l_2(\mathbb{K})$;

(*ii*) the operator $\tilde{A} : l_2(\mathbb{K}) \to l_2(\mathbb{K})$ is a bounded operator.

By definition, the 2-norm of A is the norm of $\tilde{A}$, i.e.,

$$\|A\|_2 := \sup\left\{\|\tilde{A}x\|_2 \mid x \in l_2(\mathbb{K}) \text{ with } \|x\|_2 = 1\right\}.$$

It is clear that we have

$$\|Ax\|_2 \le \|A\|_2\|x\|_2,$$

for all $x \in l_2(\mathbb{K})$ and all 2-bounded matrices A.

Example 5.1. Let $\Omega \subset \mathbb{R}^n$ be an open subset. We consider

$$X := \Big\{u : \Omega \to l^2(\mathbb{R}) \;\Big|\; u \in C^2(\Omega, l^2(\mathbb{R})),\ \frac{\partial u}{\partial x_k} \in C^1(\Omega, l^2(\mathbb{R})) \\ \text{and } \frac{\partial^2 u}{\partial x_k^2} \in C(\Omega, l^2(\mathbb{R})\Big\},$$

and

$$\langle u, v\rangle := \sum_{k=1}^{\infty} u_k v_k.$$

Now, let us consider the following infinite system

$$L(u) := \Delta u + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial x_k} + Cu = 0 \tag{5.1}$$

where $B_k, C : \Omega \to \mathbb{R}^{\mathbb{N}^* \times \mathbb{N}^*}$ are row-column-finite matrices. We have

Theorem 5.2. *We suppose that:*

(*i*) *the matrices $B_k(x)$, $C(x)$ are 2-bounded for all $x \in \Omega$;*
(*ii*)
$$\langle e, Le\rangle \prec 0,\ \forall\ e \in X \text{ with } \langle e, e\rangle = 1. \tag{5.2}$$

If $u \in X$ is a solution of (5.1)*, then $\|u\|_2$ has no positive local maximums in Ω.*

The proof is similar with that of Theorem 3.2.

Remark 5.3. As in the case of Theorem 3.2, the problem is to study in which conditions on B_k and C we have (5.2). We have

Theorem 5.4. *We suppose that:*

(*i*) *the matrices $B_k(x)$, $C(x)$ are 2-bounded for all $x \in \Omega$;*
(*ii*) *there exist $c_k, b_k \in \mathcal{F}(\Omega, \mathbb{R})$, $k = \overline{1, m}$ such that:*
  (*a*) $\|B_k - b_k\|_2 \le 2c_k$, $k = \overline{1, m}$;
  (*b*) $\langle \xi, C(x)\xi\rangle < -\sum_{k=1}^{m} c_k^2$, $\forall\ \xi \in l_2(\mathbb{R})$, *with* $\langle \xi, \xi\rangle = 1$.

If $u \in X$, with $\|u\|_2$, $\|\frac{\partial u}{\partial x_k}\|_2$ and $\|\frac{\partial^2 u}{\partial x_k^2}\|_2$, $k = \overline{1, m}$, uniformly convergent on each compact in Ω, is a solution of (5.1)*, then $\|u\|_2$ has no positive local maximums in Ω.*

Remark 5.5. For more informations on infinite matrices see: [35], [7], [14], [25], [18], [19], [22], [24], [26], ...

6. Research directions and open problems

The above considerations give rise to the following questions.

Problem 6.1. Use the above technique to study some maximum principles for the following elliptic system in an open subset $\Omega \subset \mathbb{R}^m$:

$$\sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{k=1}^{n} B_k \frac{\partial u}{\partial x_k} + Cu = 0$$

where $A_{kj}, B_k, C : \Omega \to \mathbb{C}^{m\times m}$ and $u \in C^2(\Omega, \mathbb{C}^n)$.

References: [28], [27], [31], [10], ...

Problem 6.2. Let $\Omega \subset \mathbb{R}^m$ be an open subset. Use the above technique to study maximum principles for the following parabolic system:

$$\sum_{k=1}^{n}\sum_{j=1}^{n} A_{kj}(x,t)\frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{k=1}^{n} B_k(x,t) \frac{\partial u}{\partial x_k} + C(x,t)u - \frac{\partial u}{\partial t} = 0$$

for $(x,t) \in \Omega\times]0,T[$. Here $A_{kj}, B_k, C : \Omega\times]0,T[\to \mathbb{R}^{m\times m}$.

A similar problem holds for the case of complex valued matrices A_{kj}, B_k and C.

References: [37], [38], [6], ...

Problem 6.3. Use the maximum principles in this paper to study the uniqueness of the solution of Dirichlet problem for elliptic systems.

For example, let us consider the following uniformly elliptic operator in an open and bounded $\Omega \subset \mathbb{R}^n$

$$L = L_0 + c(x) := -\sum_{k=1}^{n}\sum_{j=1}^{n} a_{kj}\frac{\partial^2}{\partial x_k \partial x_j} + \sum_{k=1}^{n} b_k(x) + c(x)$$

with smooth coefficients and smooth boundary Γ of Ω ($u \in C^2(\Omega) \cap C(\overline{\Omega})$)

$$L(u) = f \tag{6.1}$$

$$u\big|_{\Gamma} = g \tag{6.2}$$

The following result is given in [32]:

Theorem of equivalent statements. *We suppose that we have uniqueness for the problem* (6.1) + (6.2). *Then the following statements are equivalent:*

(i) *there exists $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $v(x) > 0$ for $x \in \overline{\Omega}$ and $L(v) \geq 0$ in Ω;*

(ii) *for all smooth $c_1 \geq c$ we have uniqueness for $L_1 = L_0 + c_1$ and Ω;*

(iii) *for all smooth open $\Omega_1 \subset \Omega$ we have uniqueness for L and Ω_1;*

(iv) *$f \geq 0$ in Ω, $g = 0$ on Γ imply $u \geq 0$ in Ω;*

(iv') *the corresponding Green function for L and Ω, $G(x,y) \geq 0$ for all $x, y \in \Omega$;*

(v) *$f = 0$ on Ω, $g \geq 0$ on Γ imply $u \geq 0$ in Ω;*

(v') *$\frac{\partial G(x,y)}{\partial \nu_y} \geq 0$ for all $x \in \Omega$ and $y \in \Gamma$, where ν_y is the inner conormal at $y \in \Gamma$.*

The problem is to give a similar result for a strongly elliptic system of second order.

Problem 6.4. Let $A \in \mathbb{R}^{n\times n}$. Determine some upper estimations for

$$\min_{x\in\mathbb{R}} \|A - xI\|_2.$$

A similar problem for $A \in \mathbb{R}^{\mathbb{N}^*\times\mathbb{N}^*}$.

References: [1], [5], [6], [17], [7], [14], [22], [24], [26], ...

Problem 6.5. Let $A \in \mathbb{C}^{n\times n}$. Determine some upper estimations for

$$\min_{z\in\mathbb{C}} \|A - zI\|_2.$$

A similar problem for $A \in \mathbb{C}^{\mathbb{N}^*\times\mathbb{N}^*}$.

References: [1], [7], [14], [15], [16], [24], ...

References

[1] Allaire, G. and Kaber, S.M., *Numerical Linear Algebra*, Springer, 2008.

[2] Belitskii, G.R. and Lyubich, Yu.I., *Matrix Norms and their Applications*, Birkhäuser, 1988.

[3] Bhatia, R., *Positive Definite Matrices*, Princeton Univ. Press, 2007.

[4] Bitzadze, A.V., *On elliptic systems of partial differential equations of second order*, Dokl. Akad. Nauk, **112**(1957), 983-986.

[5] Chifu, I.C., *Minimum value of a matrix norm with applications to maximum principle for second order elliptic systems*, Studia Univ. Babeş-Bolyai, **48**(2003), No. 4, 9-15.

[6] Chifu, I.C., *Aspecte calitative în teoria ecuaţiilor cu derivate parţiale*, Editura Alma Mater, Cluj-Napoca, 2006.

[7] Cooke, R.G., *Infinite Matrices and Sequences Spaces*, Macmillan, London, 1950.

[8] Cosner, C., *On the definition of ellipticity for systems of partial differential equations*, J. Math. Anal. Appl., **158**(1991), 80-93.

[9] Deutsch, E., *Private communication*, 1971.

[10] Douglis, A. and Nirenberg, L., *Interior estimates for elliptic systems of partial differential equations*, Comm. Pure Appl. Math., **8**(1955), 503-538.

[11] Friedland, S., *On matrix approximation*, Proc. Amer. Math. Soc., **51**(1975), No. 1, 41-43.

[12] Gilborg, D. and Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order*, Springer, 1983.

[13] Gills, P.P., *Propriétés et existence des solutions de certaines classes d'équations du type elliptique*, Bull. Soc. Math. France, **86**(1958), 283-297.

[14] Gohberg, I., Goldberg, S. and Kaashoek, M.A., *Basic Classes of Linear Operators*, Birkhäuser, 2003.

[15] Heinrich, H., *Die Eigenwertingrenzung nach Gerschgorin als natüriche Verbesserung einer Eingrenzung mittels matrix-normen*, Z. Angew. Math. Mech., **46**(1966), 399-400.

[16] Heinrich, H., *Zur Eingrenzung der charakteristichen Zahlenbeliebigen Matrix*, Wiss. Th. Dresden, **6**(1957), 211-216.

[17] Hogben, L.(Ed.), *Handbook of Linear Algebra*, Chapman and Hall/CRC, Boca Ratom, 2007.

[18] Ionescu Tulcea, C.T., *Spaţii Hilbert*, Ed. Academiei, Bucureşti, 1956.

[19] Kato, T., *On positive eigenvectors of positive infinite matrices*, Comm. on Pure and Appl. Math., **11**(1958), 573-586.

[20] Krupchyk, K., Seiler, W.M. and Tuomela, J., *Overdetermined elliptic systems*, Found. Comput. Math., 2006, 309-351.

[21] Lavrentiev, M.M., *On maximum principles of solutions of a second order elliptic system*, Dokl. Akad. Nauk., **116**(1957), 175-176.

[22] Lax, P.D., *Differential equations, difference equations and matrix theory*, Comm. on Pure and Appl. Math., **11**(1958), 175-194.

[23] Leung, A.W., *Systems of Nonlinear Partial Differential Equations*, Kluwer, 1989.

[24] Lindner, M., *Infinite Matrices and their Finite Sections*, 2006.

[25] Maddox, I.J., *Infinite Matrices of Operators*, Lecture Notes in Math., Springer, **786**(1980).

[26] Malafosse, B.De and Malkowsky, E., *Sequence spaces and inverse of an infinite matrix*, Rend. Circ. Mat. Palermo, **51**(2002), no. 2, 277-294.

[27] Miranda, C., *Equazioni alle derivate parziali di tipo elliptico*, Springer, 1955.

[28] Miranda, C., *Sul teorema del massimo modulo per una classe di sistemi ellittici di equazioni del secundo ordine e per le equazioni a coefficienti complessi*, Ist. Lombardo (Rend. Sc.), **A104**(1970), No. 4, 736-745.

[29] Mureşan, A.S., *Principii de maxim pentru ecuaţii cu derivate parţiale*, Teză de doctorat, Univ. Babeş-Bolyai, Cluj-Napoca, 1975.

[30] Mureşan, A.S. and Rus, I.A., *Maximum principles for some first-order elliptical systems*, Preprint Nr. 3(1986), Babeş-Bolyai Univ., 253-258.

[31] Nirenberg, L., *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa, **13**(1959), 115-162.

[32] Peetre, J. and Rus, I.A., *Sur la positivité de la fonction de Green*, Math. Scand., **21**(1967), 80-89.

[33] Precup, R., *Linear and Semilinear Partial Differential Equations*, De Gruyter, 2013.

[34] Protter, M.H. and Weinberger, H.F., *Maximum Principles in Differential Equations*, Prentice-Hall, New Jersey, 1967.

[35] Riesz, F., *Les systèmes d'équations linéaires a une infinité d'inconnues*, Gauthier-Villars, Paris, 1913.

[36] Robert, F., *Matrices nonnégative et normes vectorielles*, Univ. Sci. et Médicale de Grenoble, 1973/74.

[37] Rus, I.A., *Sur les propriétés des normes des solutions d'un systéme d'équations différentielles du second ordre*, Studia Univ. Babeş-Bolyai, Math., **15**(1968), No. 1, 19-26.

[38] Rus, I.A., *Un principe du maximum pour les solutions d'une systéme fortement elliptique*, Glasnik Matematički, **4**(1969), No. 1, 75-78.

[39] Rus, I.A., *Open problem No. 9*, Glasnik Matematički, **5**(1970), No. 2, 356.

[40] Rus, I.A., *Maximum principle for strongly elliptic systems: a conjecture*, Preprint **8**(1988), Univ. Babeş-Bolyai, 43-46.

[41] Rus, I.A., *Maximum principles for elliptic systems*, International Series of Numerical Math., **107**(1992), 37-45.

[42] Strong, G., *Linear Algebra and its Applications*, Acad. Press., 1980.

[43] Stys, T., *Hopf's theorem for certain elliptic system of the second order linear differential equations*, Prace Mat., **8**(1964), 143-146.

[44] Wasowski, J., *Maximum principle for a certain elliptic system of second order linear equations*, Demonstratio Math., **4**(1972), no. 3, 133-143.

Ioan A. Rus
"Babeş-Bolyai" University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street
400084 Cluj-Napoca, Romania
e-mail: `iarus@math.ubbcluj.ro`