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Compact composition operators on spaces of Laguerre polynomials kernels

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Abstract. We study the action of the composition operator on the analytic function spaces whose kernels are special cases of Laguerre polynomials. These function spaces become Banach spaces when the kernels are integrated with respect to the complex Borel measures of the unit circle. Necessary and sufficient conditions for the composition operator to be compact are found.

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1. Introduction

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbf{C}$. For $z \in \mathbf{D}$, $t \in \mathbf{R}$ and $a > -1$ the generating function of the associated Laguerre polynomials [7, Formula 5.1.9] is given by

$$G(a,t,z) = (1-z)^{-a-1} \exp\left(\frac{-tz}{1-z}\right) = \sum_{n=0}^{\infty} L_n^{(a)}(t) z^n \tag{1.1}$$

where $L_n^{(a)}(t)$ is the generalized Laguerre polynomial of degree n given by

$$L_n^{(a)}(t) = \sum_{k=0}^{n} \binom{n+a}{n-k} \frac{(-t)^k}{k!}. \tag{1.2}$$

Formula (1.2) can extended for $a \leq -1$ by using the identity in [7, p. 102 , eq. 5.2.1],

$$L_n^{(-a)}(t) = (-t)^a \frac{\Gamma(n-a+1)}{n!} L_{n-a}^{(a)}(t) \quad \text{for } a \geq 1.$$

The first few terms of $L_n^{(a)}(t)$ (see [3, p. 114]) are given by,

$$\begin{array}{ll} L_0^{(a)}(t) & = 1 \\ L_1^{(a)}(t) & = -t + a + 1 \\ L_2^{(a)}(t) & = \frac{1}{2}t^2 - (a+2)t + \frac{(a+2)(a+1)}{2} \\ L_3^{(a)}(t) & = -\frac{1}{6}t^3 + \frac{(a+3)}{2}t^2 - \frac{(a+2)(a+3)}{2}t + \frac{(a+1)(a+2)(a+3)}{6} \end{array} \tag{1.3}$$

and the recurrence relation for the coefficients $L_n^{(a)}(x)$ in [3, p. 114, Eq. 4.5.5] is given by

$$(n+1)\,L_{n+1}^{(a)}(t) = (2n + a + 1 - t)\,L_n^{(a)}(t) - (n+a)\,L_{n-1}^{(a)}(t). \tag{1.4}$$

The Laguerre generating function in (1.1) can be written in terms of the classical Cauchy kernel $K(z) = (1-z)^{-1}$ as follows

$$G(a,t,z) = [K(z)]^{a+1} \exp[t - tK(z)] \tag{1.5}$$

and special cases of the generating function $G(a,t,z)$ give interesting kernels of analytic functions spaces. For instance we have:

$$\begin{array}{rl} G(0,0,z) & = K(z) = (1-z)^{-1} \\ G(\alpha-1,0,z) & = K^{\alpha}(z) = (1-z)^{-\alpha}, \alpha > 0 \\ eG(-1,-1,z) & = K_e(z) = \exp\left[(1-z)^{-1}\right] \end{array} \tag{1.6}$$

where it is known in the litirature that

$$\begin{array}{ll} K(z) = (1-z)^{-1} & \text{is the classical Cauchy kernel,} \\ K^{\alpha}(z) = (1-z)^{-\alpha}, \alpha > 0 & \text{is the fractional Cauchy kernel and} \\ K_e(z) = \exp\left[(1-z)^{-1}\right] & \text{is the exponential Cauchy kernel.} \end{array} \tag{1.7}$$

Using (1.1) the corresponding Taylor series of these kernels are:

$$\begin{aligned} K(z) &= \sum_{n=0}^{\infty} L_n^{(0)}(0) z^n = \sum_{n=0}^{\infty} z^n, \\ K^{\alpha}(z) &= \sum_{n=0}^{\infty} L_n^{(\alpha-1)}(0) z^n = \sum_{n=0}^{\infty} A_n(\alpha)\, z^n, \\ K_e(z) &= e\sum_{n=0}^{\infty} L_n^{(-1)}(-1) z^n = e\sum_{n=0}^{\infty} A_n z^n. \end{aligned} \tag{1.8}$$

The coefficients above have the following properties:

1. $A_n(\alpha) = (-1)^n \binom{-\alpha}{n} = \binom{n+\alpha-1}{n}$
2. $A_n = L_n^{(-1)}(-1) = \sum\limits_{i=0}^{n} \frac{1}{i!}\binom{n-1}{n-i}$ where $A_0 = A_1 = 1$.
3. $(n+1)\,A_{n+1} = (2n+1)\,A_n - (n-1)\,A_{n-1}$.
4. $\dfrac{A_{n+1}}{A_n} > 1$, and $\dfrac{A_{n+1}}{A_n} \to 1$, as $n \to \infty$.
5. The sequence $\left\{\frac{1}{A_n}\right\}_{n=0}^{\infty}$ is convex.

Results (1) is from [3] while (2)-(5) are in [8].

2. Cauchy type analytic function spaces

Let $\mathbf{T} = \partial\mathbf{D}$ be the boundary of $\mathbf{D}$ and let $H(\mathbf{D})$ denotes the class of holomorphic functions on $\mathbf{D}$. $H(\mathbf{D})$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of $\mathbf{D}$. We denote by $\mathbf{M}$ the set of all complex–valued Borel measures on $\mathbf{T}$ and $\mathbf{M}^*$ the subset of $\mathbf{M}$ consisting of probability measures. An analytic function f is subordinate to g in $\mathbf{D}$, written $f(z) \prec g(z)$, if there exists an analytic self-map φ in $\mathbf{D}$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, satisfying $f(z) = g[\varphi(z)]$. If in particular g is also univalent in $\mathbf{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbf{D}) \subset g(\mathbf{D})$.

Let $z \in \mathbf{D}$ and let $k \in H(\mathbf{D})$ be one of the kernels in (1.6). We define X to be the subspace of $H(\mathbf{D})$ consisting of functions for which there exists a measure $\mu \in \mathbf{M}$ such that

$$f_\mu(z) = \int_{\mathbf{T}} k(xz)d\mu(x). \tag{2.1}$$

where $x = e^{it} \in \mathbf{T}$. The norm on X defined by

$$\|f_\mu\|_X = \inf_{\mu\in\mathbf{M}} \left\{ \|\mu\| : f_\mu(z) = \int_{\mathbf{T}} k(xz)\, d\mu(x) \right\} \tag{2.2}$$

makes X into a Banach space. If the series expansion of the kernel function k is given by,

$$k(z) = \sum_{n=0}^{\infty} a_n z^n$$

then the series of the function is given by

$$f_\mu(z) = \int_{\mathbf{T}} k(xz)d\mu(x) = \sum_{n=0}^{\infty} a_n \mu_n z^n \tag{2.3}$$

where

$$\mu_n = \int_{\mathbf{T}} x^n d\mu(x) = \int_{-\pi}^{+\pi} e^{int} d\mu\left(e^{it}\right).$$

According to the Lebesgue decomposition theorem $\mathbf{M} = \mathbf{M}_a \oplus \mathbf{M}_s$, where $\mathbf{M}_a := \{\mu_a \in \mathbf{M} : \mu_a \ll m\}$ where m is the normalized Lebesgue measure on the unit circle, and $\mathbf{M}_s := \{\mu_s \in \mathbf{M} : \mu_s \perp m\}$. Thus any $\mu \in \mathbf{M}$ can be written as $\mu = \mu_a + \mu_s$ where $\mu_a \in \mathbf{M}_a$, $\mu_s \in \mathbf{M}_s$ and $\|\mu\| = \|\mu_a\| + \|\mu_s\|$. Consequently the Banach space X may be written as $X = (X)_a \oplus (X)_s$, where $(X)_a$ is isomorphic to $L^1/\overline{H_0^1}$ the closed subspace of $\mathbf{M}$ of absolutely continuous measures, and $(X)_s$ is isomorphic to $\mathbf{M}_s$ the subspace of $\mathbf{M}$ of singular measures. If $f \in X_a$, then the singular part is null and the measure μ for which the integral in (2.1) holds reduces to $d\mu(e^{it}) = g(e^{it})dt$ where $g(e^{it}) \in L^1$ and dt is the Lebesgue measure on $\mathbf{T}$. In this case the functions in $(X)_a$ may be then written as,

$$f_\mu(z) = \int_{-\pi}^{\pi} k\left(e^{it}z\right) g(e^{it})dt$$

where if $g(e^{it})$ is nonnegative then $\|f\|_X = \left\|g(e^{it})\right\|_{L^1}$.

If the kernel function in (2.1) is replaced by $K(z) = (1-z)^{-1}$, $K^{\alpha}(z) = (1-z)^{-\alpha}$ or $K_e(z) = \exp[K(z)]$ respectively then the corresponding analytic function spaces are the classical Cauchy transform space $\mathbf{K}$ [5], the fractional Cauchy transform spaces F_{α} [6] and the exponential Cauchy transform space $\mathbf{K}_e$ introduced in [8], thus using (2.1) and replacing a_n by the appropriate coefficients from (1.8) in (2.3) we get the following:

$$\begin{aligned}
\mathbf{K} &= \left\{ f_{\mu} \in H\left(\mathbf{D}\right) : f_{\mu}\left(z\right) = \int_{\mathbf{T}} K(xz) d\mu(x) = \sum_{n=0}^{\infty} \mu_n z^n \right\} \\
\mathbf{K}_{\alpha} &= \left\{ f_{\mu} \in H\left(\mathbf{D}\right) : f_{\mu}\left(z\right) = \int_{\mathbf{T}} K^{\alpha}(xz) d\mu(x) = \sum_{n=0}^{\infty} A_n\left(\alpha\right) \mu_n z^n \right\} \qquad (2.4) \\
\mathbf{K}_{e} &= \left\{ f_{\mu} \in H\left(\mathbf{D}\right) : f_{\mu}\left(z\right) = \int_{\mathbf{T}} K_e(xz) d\mu(x) = \sum_{n=0}^{\infty} eA_n \mu_n z^n \right\}
\end{aligned}$$

where

$$\begin{aligned}
\mu_n &= \int_{\mathbf{T}} x^n d\mu\left(x\right) = \int_{\mathbf{T}} e^{int} d\mu\left(e^{it}\right) \\
A_n\left(\alpha\right) &= \left(-1\right)^n \binom{-\alpha}{n} = \binom{n+\alpha-1}{n} \qquad (2.5) \\
A_n &= L_n^{(-1)}(-1) = \sum_{i=0}^{n} \frac{1}{i!} \binom{n-1}{n-i}.
\end{aligned}$$

Clearly $\mathbf{K}$ is a special case of $\mathbf{K}_{\alpha}$ when $\alpha = 1$. It is also known that $\mathbf{K}_{\alpha} \subset \mathbf{K}_{\beta}$ for $0 < \alpha < \beta$. It was also shown in [8] that $\mathbf{K} \subset \left(\mathbf{K}_e\right)_a$ and if $f \in \mathbf{K}$ then $\|f\|_{\mathbf{K}_e} < \|h\|_{L^1} \|f\|_{\mathbf{K}}$ where $h \in L^1$.

The next result gives us examples of elements of $\mathbf{K}_e$.

Lemma 2.1. *Suppose that* $|w| \leq 1$ *and let* $f_w\left(z\right) = K_e\left(wz\right) = \exp\left[\left(1-wz\right)^{-1}\right]$ *for* $|z| < 1$. *Then* $f_w\left(z\right) \in \mathbf{K}_e$ *and there exists a probability measure* $\mu \in \mathbf{M}^*$ *such that*

$$f_w\left(z\right) = \int_{\mathbf{T}} K_e\left(xz\right) d\mu(x) \quad \textit{and} \quad \|f_w\|_{\mathbf{K}_e} = \|\mu\| = 1.$$

Proof. For $|w| \leq 1$ and $|z| < 1$ we have $\operatorname{Re}\left\{K\left(wz\right)\right\} = \operatorname{Re}\left\{\left(1-wz\right)^{-1}\right\} > \frac{1}{2}$. The Riesz-Herglotz formula implies that there exists a probability measure $\mu = \mu_w \in \mathbf{M}^*$ such that

$$K\left(wz\right) = \left(1-wz\right)^{-1} = \int_{\mathbf{T}} K(xz) d\mu(x) = \int_{\mathbf{T}} \left(1-xz\right)^{-1} d\mu(x). \qquad (2.6)$$

The left hand side of the above equation is $\left(1-wz\right)^{-1} = \sum_{n=0}^{\infty} w^n z^n$ and right hand is $\sum\limits_{n=0}^{\infty} \mu_n z^n$. Equating coefficients of the power series of both sides of (2.6) we get that

$w^n = \mu_n = \int_{\mathbf{T}} x^n d\mu(x)$ for $n = 0, 1, 2, \ldots$ and thus

$$\begin{aligned} f_w(z) = K_e(wz) &= e\sum_{n=0}^{\infty} A_n w^n z^n \\ &= e\sum_{n=0}^{\infty} A_n \left(\int_{\mathbf{T}} x^n d\mu(x)\right) z^n \\ &= \int_{\mathbf{T}} \left(e\sum_{n=0}^{\infty} A_n x^n z^n\right) d\mu(x) \\ &= \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \end{aligned}$$

Hence $f_w \in \mathbf{K}_e$ and since $\mu \in \mathbf{M}^*$, we have $\|f_w\|_{\mathbf{K}_e} = \|\mu\| = 1$. □

Corollary 2.2. *A special case of the previous result is*

$$K_e(xz) \in \mathbf{K}_e \textit{ for all } x \in \mathbf{T} \textit{ and } \|K_e(xz)\|_{\mathbf{K}_e} = 1.$$

Lemma 2.3. *Suppose* $\{f_{\mu_n}\}_{n=1}^{\infty}$ *is a sequence of functions in* $\mathbf{K}_e$ *such that there is a constant* A *for which which* $\|f_{\mu_n}\|_{\mathbf{K}_e} \le A$ *for* $n = 1, 2, \ldots..$ *If* $f_\mu(z) = \lim_{n\to\infty} f_{\mu_n}(z)$ *exists for* $|z| < 1$, *then* $f \in \mathbf{K}_e$ *and* $\|f\|_{\mathbf{K}_e} \le A$.

Proof. Let $z \in \mathbf{D}$ suppose $f_{\mu_n} \in \mathbf{K}_e$ for $n = 1, 2, \ldots$ then by definition we have,

$$f_{\mu_n}(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu_n(x) \text{ and } \mu_n \in \mathbf{M}, \|f_{\mu_n}(z)\|_{\mathbf{K}_e} = \|\mu_n\| \le A$$

The Banach-Alaoglu theorem yields a subsequence $\{\mu_{n_k}\}$ for $k = 1, 2, \ldots, \|\mu_{n_k}\| \le A$ and $\mu \in \mathbf{M}, \|\mu\| \le A$ such that $\mu_{n_k} \to \mu \in \mathbf{M}$ as $k \to \infty$ in the weak* topology. Hence we get,

$$\int_{\mathbf{T}} K_e(xz)\, d\mu_{n_k}(x) \longrightarrow \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \text{ as } k \to \infty.$$

Since we also have that $f_\mu(z) = \lim_{k\to\infty} f_{\mu_{n_k}}(z)$ then

$$f_\mu(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x) \in \mathbf{K}_e \text{ and } \|f_\mu\| \le A.$$ □

3. The composition operator on $\mathbf{K}_e$

If φ is an analytic self map of the unit disc $\mathbf{D}$, we say that φ induces a bounded composition operator C_φ on X if there exists a positive constant A such that for all $f \in X$, $\|C_\varphi(f)\|_X = \|(f \circ \varphi)\|_X \le A\|f\|_X$. A bounded operator C_φ will be a compact operator if the image of every bounded set of X is relatively compact (i.e. has compact closure) in X. Equivalently C_φ is a compact operator on X if and only if for every bounded sequence $\{f_n\}$ of X, $\{C_\varphi(f_n)\}$ has a convergent subsequence in X.

The composition operator C_φ has been thoroughly studied on the Cauchy space $\mathbf{K}$ such as in [4, 5] and on the fractional Cauchy spaces $\mathbf{K}_\alpha$ such as in [2, 6]. In particular it is known that;

1. If $\alpha > 0$ and φ is conformal automorphism of $\mathbf{D}$, then $C_{\varphi}(f) = f \circ \varphi \in \mathbf{K}_{\alpha}$ for every $f \in \mathbf{K}_{\alpha}$.
2. If $\alpha \geq 1$ and φ is an analytic self map of the unit disc $\mathbf{D}$, then $C_{\varphi}(f) = f \circ \varphi \in \mathbf{K}_{\alpha}$ for every $f \in \mathbf{K}_{\alpha}$.
3. Let G_{α} denote the set of functions that are subordinate to $K^{\alpha}(z) = (1-z)^{-\alpha}$ in $\mathbf{D}$. If $\alpha \geq 1$ then a function f belongs to the closed convex hull of G_{α} if and only if there is a probability measure $\mu \in \mathbf{M}^{*}$ such that $f(z) = \int_{\mathbf{T}} K^{\alpha}(xz) d\mu(x)$.
4. C_{φ} is compact on $\mathbf{K}$ if and only if $C_{\varphi}(\mathbf{K}) \subset (\mathbf{K})_{a}$.
5. If $\alpha \geq 1$ then C_{φ} is compact on $\mathbf{K}_{\alpha}$ if and only if $C_{\varphi}[K^{\alpha}(xz)] \in (\mathbf{K}_{\alpha})_{a}$ for all $|x| = 1$.

Results (1)-(3) are in [6], result (4) it is known from [4] and was extended to result (5) in [2]. The operator C_{φ} is also bounded and Möbius invariant on $\mathbf{K}_{e}$. There is no loss of generality in assuming that $\varphi(0) = 0$, and we will assume so throughout the article. Our focus then is only on when the composition operator is compact on $\mathbf{K}_{e}$.

We need the following interesting two results due to Y. Abu Muhanna and D. Hallenbeck in [1] .

Theorem 3.1. *Let Δ be a bounded convex body, with $0 \in \Delta$ and let H be a covering function mapping the unit disk onto the exterior of the bounded convex body $\Omega = c\Delta$. Suppose that $\log H$ is univalent and also maps the unit disk onto the compliment of a convex set. Then any analytic function f subordinate to H can be expressed as*

$$f(z) = \int_{\mathbf{T}} H(xz) d\mu(x), \tag{3.1}$$

for some positive Borel measure μ on the unit circle with $||\mu|| = 1$.

The previous theorem includes the following special case.

Theorem 3.2. *If φ is analytic self map of the unit disc $\mathbf{D}$, with $\varphi(0) = 0$ then there exist probability measures $\mu, \nu \in \mathbf{M}^{*}$ such that*

$$C_{\varphi}[K_{e}(z)] = K_{e}(\varphi(z)) = \int \exp(K(xz))\, d\mu(x) = \exp\left(\int_{\mathbf{T}} K(xz) d\nu(x)\right).$$

Then λK_{e}, with $|\lambda| = 1$ are all of the universal coverings of $c\mathbf{D}$.

Lemma 3.3. *Suppose $g_{x}(e^{it})$ is a nonnegative $L^{1}-$continuous function of x and let $\{\mu_{n}\}$ be a sequence of nonnegative Borel measures that are weak* convergent to μ. Define*

$$w_{n}(t) = \int_{\mathbf{T}} g_{x}(e^{it})\, d\mu_{n}(x) \;\; \text{and} \;\; w(t) = \int_{\mathbf{T}} g_{x}(e^{it})\, d\mu(x)$$

then $\|w_{n} - w\|_{L^{1}} \longrightarrow 0$.

Proof. Suppose $g_x\left(e^{it}\right)$ is a nonnegative L^1continuous function of x and for $z \in \mathbf{D}$ let,

$$\begin{aligned} g_x(z) &= \int \operatorname{Re}\left(\frac{1+e^{it}z}{1-e^{it}z}\right) g_x\left(e^{it}\right) d(t)\ , \\ w_n(z) &= \int g_x(z)\, d\mu_n(x) \text{ and} \\ w(z) &= \int g_x(z)\, d\mu(x)\,. \end{aligned}$$

Notice that all functions are positive and harmonic in $\mathbf{D}$ and that the radial limits of $w_n(z)$ and $w(z)$ are $w_n(t)$ and $w(t)$ respectively. Then, for $|z| \le \rho < 1$,

$$|g_x(z) - g_y(z)| \le \frac{1}{1-\rho} \left\| g_x\left(e^{it}\right) - g_y\left(e^{it}\right) \right\|_{L^1}$$

The continuity condition implies that $g_x(z)$ is uniformly continuous in x for all $|z| \le \rho < 1$. Weak star convergence, implies that $w_n(z) \to w(z)$ uniformly on $|z| \le \rho < 1$ and consequently the convergence is locally uniformly on $\mathbf{D}$. In addition, we have

$$\left\| w_n(\rho e^{it}) \right\|_{L^1} \to \left\| w(\rho e^{it}) \right\|_{L^1}.$$

Hence we conclude that

$$\left\| w_n(\rho e^{it}) - w(\rho e^{it}) \right\|_{L^1} \longrightarrow 0 \text{ as } n \to \infty.$$

Now using Fatou's Lemma we conclude that

$$\left\| w_n(e^{it}) - w(e^{it}) \right\|_{L^1} \longrightarrow 0. \qquad \square$$

Lemma 3.4. *Let $g_x\left(e^{it}\right)$ be a nonnegative L^1 continuous function of x such that $\|g_x\|_{L^1} \le a < \infty$ and $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$.*
Let $f(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x)$, and let L be the operator given by

$$L[f(z)] = \iint g_x\left(e^{it}\right) K_e\left(e^{it}z\right) dt d\mu(x)$$

then L is compact operator on $\mathbf{K}_e$.

Proof. First note that the condition that $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$ implies that the L operator is a well defined function on $\mathbf{K}_\alpha$. Let $\{f_n(z)\}$ be a bounded sequence in $\mathbf{K}_\alpha$ and let $\{\mu_n\}$ be the corresponding norm bounded sequence of measures in $\mathbf{M}$. Since every norm bounded sequence of measures in $\mathbf{M}$ has a weak star convergent subsequence, let $\{\mu_n\}$ be such subsequence that is convergent to $\mu \in \mathbf{M}$. We want to show that $\{L(f_n)\}$ has a convergent subsequence in $\mathbf{K}_\alpha$. First, let us assume that $d\mu_n(x) >> 0$ for all n, and let $w_n(t) = \int g_x\left(e^{it}\right) d\mu_n(x)$ and $w(t) = \int g_x\left(e^{it}\right) d\mu(x)$,then we know from the Lemma 3.3 that $w_n(t), w(t) \in L^1$ for all n, and $w_n(t) \to w(t)$ in L^1. Now since $g_x\left(e^{it}\right)$ is a nonnegative continuous function in x and $\{\mu_n\}$ is weak star convergent to μ, then

$$L(f_n(z)) = \iint K_e\left(e^{it}z\right) g_x\left(e^{it}\right) d(t) d\mu_n(x) = \int K_e\left(e^{it}z\right) w_n(t) dt$$

$$L(f(z)) = \iint K_e\left(e^{it}z\right) g_x\left(e^{it}\right) d(t) d\mu(x) = \int K_e\left(e^{it}z\right) w(t)\, dt$$

Furthermore because $w_n(t)$ is nonnegative then

$$\|L(f_n)\|_{\mathbf{K}_e} = \|w_n\|_{L^1}$$
$$\|L(f)\|_{\mathbf{K}_e} = \|w\|_{L^1}$$

Now since$\|w_n - w\|_{L^1} \to 0$ then $\|L(f_n) - L(f)\|_{\mathbf{K}_e} \to 0$ which shows that $\{L(f_n)\}$ has a convergent subsequence in $\mathbf{K}_e$ and thus L is a compact operator for the case where μ is a positive measure.
In the case where μ is complex measure we write

$$d\mu_n(x) = (d\mu_n^1(x) - d\mu_n^2(x)) + i(d\mu_n^3(x) - d\mu_n^4(x))$$

where each $d\mu_n^j(x) >> 0$ and define $w_n^j(t) = \int g_x\left(e^{it}\right) d\mu_n^j(x)$ then

$$w_n(t) = \int g_x\left(e^{it}\right) d\mu_n(x) = \left(w_n^1(t) - w_n^2(t)\right) + i\left(w_n^3(t) - w_n^4(t)\right).$$

Using an argument similar to the one above we get that

$$w_n^j(t), w^j(t) \in L^1, \quad \text{and} \quad \left\|w_n^j - w^j\right\|_{L^1} \longrightarrow 0.$$

Consequently, $\|w_n - w\|_{L^1} \longrightarrow 0$, where

$$w(t) = \left(w^1(t) - w^2(t)\right) + i\left(w^3(t) - w^4(t)\right) = \int g_x\left(e^{it}\right) d\mu(x).$$

Hence,

$$\|L(f_n) - L(f)\|_{F_\alpha} \leq \|w_n - w\|_{L^1} \longrightarrow 0.$$

Finally, we conclude that the operator L is compact. □

Now we are ready to prove our main theorem which characterizes compact composition operators on $\mathbf{K}_e$.

Theorem 3.5. *If φ is analytic self map of the unit disc $\mathbf{D}$, with $\varphi(0) = 0$ then the operator C_φ is compact in $\mathbf{K}_e$ if and only if $C_\varphi[K_e(xz)] \in (\mathbf{K}_e)_a$ for all x such that $|x| = 1$.*

Proof. Assume that C_φ is compact on $\mathbf{K}_e$ and let $\{f_j(z)\}_{j=1}^{\infty}$ be the bounded sequence of functions defined as

$$f_j(z) = K_e(\rho_j xz) = \exp\left(\frac{1}{1-\rho_j xz}\right) = \exp\left[K(\rho_j xz)\right],$$

where $0 < \rho_j < 1$ and $\lim_{j\to\infty} \rho_j = 1$. Clearly, $f_j \in H^\infty \cap \mathbf{K}_e$ and there exist probability measures $\mu_j \in \mathbf{M}^*$ such that

$$f_j(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu_j(x)$$

where $\|f_j\|_{\mathbf{K}_e} = \|\mu_j\| = 1$. Since C_φ is compact on $\mathbf{K}_e$, then $C_\varphi(f_j) \in \mathbf{K}_e$ and $\|C_\varphi(f_j)\| \leq \|C_\varphi\| \|f_j\|_{\mathbf{K}_e} = \|C_\varphi\|$ for all j. Furthermore $C_\varphi(f_j) \in H^\infty \cap \mathbf{K}_e \subset (\mathbf{K}_e)_a$ for every j and thus there exists a nonnegative L^1 function $g_j(x)$ such that $d\mu_j(x) = g_j(x)\,dt$ and

$$C_\varphi[f_j(z)] = \int_{\mathbf{T}} K_e(xz)\, g_j(x)\, dt.$$

Since the operator C_φ is compact then the sequence $\{C_\varphi(f_j)\}_{j=1}^{\infty}$ has a convergent subsequence that converges to $C_\varphi[K_e(z)] \in (\mathbf{K}_e)_a$ because of Lemma 2.3 and the fact that $(\mathbf{K}_e)_a$ is a closed subspace of $\mathbf{K}_e$.

For the converse let $f \in \mathbf{K}_e$ then there exists a measure in $\mathbf{M}$ such that

$$f(z) = \int_{\mathbf{T}} K_e(xz)\, d\mu(x).$$

Then

$$(f \circ \varphi)(z) = C_\varphi[f(z)] = \int_{\mathbf{T}} K_e[x\varphi(z)]\, d\mu(x) = \int_{\mathbf{T}} C_\varphi[K_e(xz)]\, d\mu(x)$$

where by assumption $C_\varphi[K_e(xz)] \in (\mathbf{K}_e)_a$ and thus can be written as

$$C_\varphi[K_e(xz)] = \int_{\mathbf{T}} g_x\left(e^{it}\right) K_e\left(e^{it}z\right) dt$$

where $g_x\left(e^{it}\right)$ is a positive L^1−continuous function of x. Hence

$$\begin{aligned} C_\varphi(f)(z) &= \int_{\mathbf{T}} C_\varphi[K_e(xz)]\, d\mu(x) \\ &= \int_{\mathbf{T}}\int_{\mathbf{T}} g_x\left(e^{it}\right) K_e\left(e^{it}z\right) dt\, d\mu(x) \end{aligned}$$

which was proven to be compact in $\mathbf{K}_e$ in the the previous Lemma 3.4. □

Corollary 3.6. *We have the following.*

1. *The operator C_φ is compact in $\mathbf{K}_e$ if and only if $C_\varphi(\mathbf{K}_e) \subset (\mathbf{K}_e)_a$.*
2. *Let $\varphi \in H(\mathbf{D})$, with $\|\varphi\|_\infty < 1$. Then C_φ is compact on $\mathbf{K}_e$.*

Proof. $C_\varphi[K_e(xz)] = K_e[x\varphi(z)] \in H^\infty \cap \mathbf{K}_e \subset (\mathbf{K}_e)_a$ and is subordinate to $K_e(z)$ hence

$$C_\varphi[K_e(xz)] = \int_{\mathbf{T}} K_e(e^{it}z) g_x\left(e^{it}\right) dt \in (\mathbf{K}_e)_a$$

where $g_x\left(e^{it}\right)$ is a nonnegative L^1 function. □

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