Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 2, 171–180

Neighborhood and partial sums results on the class of starlike functions involving Dziok-Srivastava operator

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Abstract. In this paper, we introduce a new subclasses of univalent functions defined in the open unit disc involving Dziok-Srivastava Operator. The results on partial sums, integral means and neighborhood results are discussed.

Mathematics Subject Classification (2010): 30C45.

Keywords: Univalent, starlike and convex functions, coefficient bounds, neighborhood results, partial sums.

1. Introduction

Denote by $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are normalized by $f(0) = 0 = f'(0) - 1$ and univalent in $\mathcal{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha (0 \leq \alpha < 1)$ if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha (0 \leq \alpha < 1)$ if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ in $\mathcal{U}$. It readily follows that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

Denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \;\; a_n \geq 0, \;\; z \in \mathcal{U} \tag{1.2}$$

studied extensively by Silverman [11].

For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots;\ j = 1, 2, \ldots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \qquad (1.3) \\ (l \leq m+1;\ l, m \in N_0 &:= N \cup \{0\}; z \in U) \end{aligned}$$

where N denotes the set of all positive integers and $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n \in N. \end{cases} \qquad (1.4)$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial and others; for example see [3].

Let $\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A} \to \mathcal{A}$ be a linear operator defined by

$$\begin{aligned} \mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) f(z) &:= z\ {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Gamma_n\ a_n\ z^n \qquad (1.5) \end{aligned}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{1}{(n-1)!}, \qquad (1.6)$$

unless otherwise stated and $*$ the Hadamard product (or convolution) of two functions $f, g \in \mathcal{A}$ where $f(z)$ of the form (1.1) and $g(z)$ be given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ then $f(z) * g(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}.$

For simplicity, we can use a shorter notation $\mathcal{H}_m^l[\alpha_1]$ for $\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel. The linear operator $\mathcal{H}_m^l[\alpha_1]$ is called Dziok-Srivastava operator [3] (see [6, 8]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [2], Ruscheweyh [9] and Owa-Srivastava [7]. Motivated by earlier works of Aouf et al.,[1] and Dziok and Raina[4] we define the following new subclass of $\mathcal{T}$ involving hypergeometric functions.

For $0 \leq \lambda \leq 1, 0 < \beta \leq 1, -1 \leq B < A \leq 1, 0 \leq \gamma \leq 1$,we let $\mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ denote the subclass of $\mathcal{T}$ consisting of functions $f(z)$ of the form (1.2) satisfying the analytic condition

$$\left| \frac{\frac{zF_\lambda'(z)}{F_\lambda(z)} - 1}{(B-A)\gamma \left[\frac{zF_\lambda'(z)}{F_\lambda(z)} - \alpha \right] - B \left[\frac{zF_\lambda'(z)}{F_\lambda(z)} - 1 \right]} \right| < \beta, \qquad z \in U \qquad (1.7)$$

where

$$\frac{zF_\lambda'(z)}{F_\lambda(z)} = \frac{z\mathcal{H}f'(z) + \lambda z^2 \mathcal{H}f''(z)}{(1-\lambda)\mathcal{H}f(z) + \lambda z \mathcal{H}f'(z)}, \qquad 0 \leq \lambda \leq 1 \qquad (1.8)$$

and

$$\mathcal{H}f(z) = z + \sum_{n=2}^{\infty} a_n \Gamma_n z^n \tag{1.9}$$

where Γ_n is given by (1.6)

The main object of the present paper is to investigate (n, δ)- neighborhoods of functions $f(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$. Furthermore, we obtain Partial sums $f_k(z)$ and Integral means inequality of functions $f(z)$ in the class $\mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$.

We state the following Lemma , due to Vijaya and Deppa [15] which provide the necessary and sufficient conditions for functions $f(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$.

Lemma 1.1. *A function $f(z) \in \mathcal{T}$ is in the class $\mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$ if and only if*

$$\sum_{n=2}^{\infty} \phi_B^A(n, \lambda, \alpha, \beta, \gamma) a_k \leq 1 \tag{1.10}$$

where

$$\phi_B^A(n, \lambda, \alpha, \beta, \gamma) = \frac{(1 + n\lambda - \lambda)[(n-1)(1 - \beta B) + \beta\gamma(B - A)(n - \alpha)]\Gamma_n}{\beta\gamma(B - A)(1 - \alpha)}. \tag{1.11}$$

2. Neighborhood results

In this section we discuss neighborhood results of the class $\mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$ due to Goodman [5] and Ruscheweyh [10] . We define the $\delta-$ neighborhood of function $f(z) \in \mathcal{T}$.

Definition 2.1. *For $f \in \mathcal{T}$ of the form (1.2) and $\delta \geq 0$. We define a $\delta-$neighbourhood of a function $f(z)$ by*

$$\mathcal{N}_{\delta}(f) = \left\{ g : g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} c_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - c_n| \leq \delta \right\}. \tag{2.1}$$

In particular, for the identity function $e(z) = z$, we immediately have

$$\mathcal{N}_{\delta}(e) = \left\{ g : g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} c_n z^n \text{ and } \sum_{n=2}^{\infty} n|c_n| \leq \delta \right\}. \tag{2.2}$$

Theorem 2.2. *If $\delta = \frac{2}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)}$ then $\mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B) \subset \mathcal{N}_{\delta}(e)$, where*

$$\phi_B^A(2, \lambda, \alpha, \beta, \gamma) = \frac{(1 + \lambda)[(1 - \beta B) + \beta\gamma(B - A)(2 - \alpha)]\Gamma_2}{\beta\gamma(B - A)(1 - \alpha)}. \tag{2.3}$$

Proof. For a function $f(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$ of the form (1.2), Lemma 1.1 immediately yields,

$$\sum_{n=2}^{\infty} (1 + n\lambda - \lambda)[(n-1)(1 - \beta B) + \beta\gamma(B - A)(n - \alpha)]\Gamma_n a_n \leq \beta\gamma(B - A)(1 - \alpha)$$

$$(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)]\Gamma_2 \sum_{n=2}^{\infty} a_n \leq \beta\gamma(B-A)(1-\alpha)$$

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta\gamma(B-A)(1-\alpha)}{(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)]\Gamma_2} = \frac{1}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)}. \qquad (2.4)$$

On the other hand, we find from (1.10) and (2.4) that

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n a_n \leq \beta\gamma(B-A)(1-\alpha).$$

That is

$$\sum_{n=2}^{\infty} n a_n \leq \frac{\beta\gamma(B-A)(1-\alpha)[1-\beta B(1+\lambda-\lambda+1)]}{(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)](1-\beta B)}.$$

Hence

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)} = \delta.$$

□

A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{HF}_{\gamma}^{\lambda}(\rho,\alpha,\beta,A,B)$ if there exists a function $h \in \mathcal{HF}_{\gamma}^{\lambda}(\rho,\alpha,\beta,A,B)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right| < 1-\rho, \quad (z \in U,\ 0 \leq \rho < 1). \qquad (2.5)$$

Now we determine the neighborhood for the class $\mathcal{HF}_{\gamma}^{\lambda}(\rho,\alpha,\beta,A,B)$.

Theorem 2.3. *If $h \in \mathcal{HF}_{\gamma}^{\lambda}(\rho,\alpha,\beta,A,B)$ and*

$$\rho = 1 - \frac{\delta\phi_B^A(2,\lambda,\alpha,\beta,\gamma)}{2[\phi_B^A(2,\lambda,\alpha,\beta,\gamma)-1]} \qquad (2.6)$$

then $N_\delta(h) \subset \mathcal{HF}_{\gamma}^{\lambda}(\rho,\alpha,\beta,A,B)$ where $\phi_B^A(2,\lambda,\alpha,\beta,\gamma)$ is defined in (2.3).

Proof. Suppose that $f \in N_\delta(h)$ we then find from (2.1) that

$$\sum_{n=2}^{\infty} n|a_n-d_n| \leq \delta$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n-d_n| \leq \frac{\delta}{2}.$$

Next, since $h \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha,\beta,A,B)$, we have

$$\sum_{n=2}^{\infty} d_n = \frac{1}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{h(z)}-1\right| &< \frac{\sum\limits_{n=2}^{\infty}|a_n-d_n|}{1-\sum\limits_{n=2}^{\infty}d_n}\\ &\leq \frac{\delta}{2}\times\frac{\phi_B^A(2,\lambda,\alpha\beta,\gamma)}{\phi_B^A(2,\lambda,\alpha,\beta,\gamma)-1}\\ &\leq \frac{\delta\phi_B^A(2,\lambda,\alpha\beta,\gamma)}{2(\phi_B^A(2,\lambda,\alpha\beta,\gamma)-1)}\\ &= 1-\rho\end{aligned}$$

provided that ρ is given precisely by (2.6). Thus by definition, $f\in\mathcal{HF}_\gamma^\lambda(\rho,\alpha,\beta,A,B)$ for ρ given by (2.6), which completes the proof. □

3. Partial sums

Silverman [14] determined the sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions , viz., $\Re\{f(z)/f_k(z)\}$, $\Re\{f_k(z)/f(z)\}$, $\Re\{f'(z)/f'_k(z)\}$ and $Re\left\{f'_k(z)/f'(z)\right\}$ for their sequences of partial sums $f_k(z)=z+\sum\limits_{n=2}^{k}a_nz^n$ of the analytic function $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$. In the following theorems we discuss results on partial sums for functions $f(z)\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$.

Theorem 3.1. *If f of the form (1.2) satisfies the condition (1.10), then*

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\}\geq\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)} \tag{3.1}$$

and

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\}\geq\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)+1} \tag{3.2}$$

where $\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)$ is given by (1.11). The results are sharp for every k, with the extremal function given by

$$f(z)=z-\frac{1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}z^{n+1}. \tag{3.3}$$

Proof. In order to prove (1.10), it is sufficient to show that

$$\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\left[\frac{f(z)}{f_k(z)}-\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]\prec\frac{1+z}{1-z}\quad(z\in\mathcal{U})$$

we can write

$$\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\left[\frac{1-\sum\limits_{n=2}^{\infty}a_nz^{n-1}}{1-\sum\limits_{n=2}^{k}a_nz^{n-1}}-\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]$$

$$=\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\left[\frac{1-\sum\limits_{n=2}^{k}a_nz^{n-1}-\sum\limits_{n=k+1}^{\infty}a_nz^{n-1}}{1-\sum_{n=2}^{k}a_nz^{n-1}}-\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]$$

$$=\frac{1+w(z)}{1-w(z)}.$$

Then

$$w(z)=\frac{-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_nz^{n-1}}{2-2\sum_{n=2}^{k}a_nz^{n-1}-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_nz^{n-1}}.$$

Obviously $w(0)=0$ and $|w(z)|\le\frac{-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_n}{2-2\sum_{n=2}^{k}a_n-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_n}$. Now,$|w(z)|\le 1$ if and only if

$$2\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_n\le 2-2\sum_{n=2}^{k}a_n$$

which is equivalent to

$$\sum_{n=2}^{k}a_n+\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)\sum_{n=k+1}^{\infty}a_n\le 1. \tag{3.4}$$

In view of (1.10), this is equivalent to showing that

$$\sum_{n=2}^{k}(\phi_B^A(n,\lambda,\alpha,\beta,\gamma)-1)a_n+\sum_{n=k+1}^{\infty}(\phi_B^A(n,\lambda,\alpha,\beta,\gamma)-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma))a_n\ge 0.$$

To see that the function f given by (3.3) gives the sharp results, we observe for $z=re^{\frac{2\pi i}{n}}$ that

$$\frac{f(z)}{f_k(z)}=1-\frac{1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}z^n\to 1-\frac{1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}$$

where $r\to 1^-$. Thus, we have completed the proof of (3.1).
The proof of (3.2) is similar to (3.1) and will be omitted. □

Theorem 3.2. *If $f(z)$ of the from (1.2) satisfies (1.10) then*

$$\Re\left\{\frac{f'(z)}{f_k'(z)}\right\}\ge\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-k-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)} \tag{3.5}$$

and

$$\Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)+k+1} \tag{3.6}$$

where $\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)$ *is given by* (1.11)*. The results are sharp for every* k*, with the extremal function given by* (3.3)*.*

Proof. In order to prove (3.5) it is sufficient to show that

$$\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{k+1}\left[\frac{f'(z)}{f_k'(z)} - \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-n-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right] \prec \frac{1+z}{1-z} \quad (z\in\mathcal{U})$$

we can write

$$\begin{aligned}&\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{k+1}\left[\frac{f'(z)}{f_k'(z)} - \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-n-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]\\ &= \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{k+1}\left[\frac{1-\sum_{n=2}^{\infty} na_nz^{n-1}}{1-\sum_{n=2}^{k} na_nz^{n-1}} - \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)-k-1}{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}\right]\\ &= \frac{1+w(z)}{1-w(z)}.\end{aligned}$$

Then

$$w(z) = \frac{-\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)(k+1)^{-1}\sum_{n=k+1}^{\infty} na_nz^{n-1}}{2-2\sum_{n=2}^{k} na_nz^{n-1} - \phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)(k+1)^{-1}\sum_{n=k+1}^{\infty} na_nz^{n-1}}.$$

Obviously $w(0)=0$ and

$$|w(z)| \leq \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)(n+1)^{-1}\sum_{n=k+1}^{\infty} na_n}{2-2\sum_{n=2}^{k} na_n - \phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)(n+1)^{-1}\sum_{n=k+1}^{\infty} na_n}.$$

Now,$|w(z)|\leq 1$ if and only if

$$2\frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{(k+1)}\sum_{n=k+1}^{\infty} na_n \leq 2-2\sum_{n=2}^{k} na_n$$

which is equivalent to

$$\sum_{n=2}^{k} na_n + \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{(k+1)}\sum_{n=k+1}^{\infty} na_n \leq 1. \tag{3.7}$$

In view of (1.10), this is equivalent to showing that

$$\sum_{n=2}^{k}(\phi_B^A(n,\lambda,\alpha,\beta,\gamma)-n)a_n + \sum_{n=k+1}^{\infty}\left(\phi_B^A(n,\lambda,\alpha,\beta,\gamma) - \frac{\phi_B^A(k+2,\lambda,\alpha,\beta,\gamma)}{(k+1)n}\right)a_n \geq 0$$

which completes the proof of (3.5).

The proof of (3.6) is similar to (3.5) and hence we omit proof . □

4. Integral means inequality

In 1975, Silverman [13] found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$ and applied this function to resolve his integral means inequality, conjectured in Silverman [12] that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. and settled in Silverman (1997), also proved his conjecture for the subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of $\mathcal{T}$.

Lemma 4.1. *If $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $f(z) \prec g(z)$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^\mu d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\mu d\theta$$

where $\mu \geq 0, z = re^{i\theta}$ and $0 < r < 1$.

Application of Lemma 4.1 to function of $f(z)$ in the class $\mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ gives the following result.

Theorem 4.2. *Let $\mu > 0$. If $f(z) \in \mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ is given by (1.2) and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)} z^2 \tag{4.1}$$

where $\phi_B^A(2, \lambda, \alpha, \beta, \gamma)$ is defined by (2.3).Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \tag{4.2}$$

Proof. For functions f of the form (1.2) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^\infty a_n z^{n-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{1}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)} z\right|^\eta d\theta.$$

By Lemma 4.1, it suffices to show that

$$1 - \sum_{n=2}^\infty a_n z^{n-1} \prec 1 - \frac{1}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)} z.$$

Setting

$$1 - \sum_{n=2}^\infty a_n z^{n-1} = 1 - \frac{1}{\phi_B^A(2, \lambda, \alpha, \beta, \gamma)} w(z), \tag{4.3}$$

and using (1.10), we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{n=2}^{\infty} \phi_B^A(n,\lambda,\alpha,\beta,\gamma)a_n z^{n-1}\right| \\ &\le |z|\sum_{n=2}^{\infty} \phi_B^A(n,\lambda,\alpha,\beta,\gamma)a_n \\ &\le |z|,\end{aligned}$$

where $\phi_B^A(n,\lambda,\alpha,\beta,\gamma)$ is given by (1.11) which completes the proof. □

Remark 4.3. We observe that for $\lambda = 0$, if $\mu = 0$ the various results presented in this chapter would provide interesting extensions and generalizations of those considered earlier for simpler and familiar function classes studied in the literature .The details involved in the derivations of such specializations of the results presented in this chapter are fairly straight- forward.

Acknowledgements. The authors record their sincere thanks to Prof. G. Murugusundaramoorthy, VIT University, Vellore-632014, India and anonymous referees for their valuable suggestions.

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