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On a certain class of analytic functions

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Abstract. In this paper, authors introduce a new class $R(\beta,\alpha,n)$ of Salagean-type analytic functions. We obtain extreme points of $R(\beta,\alpha,n)$ and some sharp bounds for $Re\left\{\frac{D^n f(z)}{z}\right\}$ and $Re\left\{\frac{D^{n-1} f(z)}{z}\right\}$. Relevant connections of the results presented here with various known results are briefly indicated.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$.

Further, let S be the class of functions in A which are univalent in U. For $0 \le \beta < 1$, $\alpha > 0$ and $n \in N_0 = N \cup 0$, we let

$$R(\beta,\alpha,n) = \left\{ f(z) \in A : Re\left\{ \frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} \right\} > \beta,\ z \in U \right\},$$

where D^n stands for Salagean derivative operator introduced by Salagean [9].

By specializing the parameters in the subclass $R(\beta,\alpha,n)$, we obtain the following known subclasses of S studied earlier by various researchers.
(i) $R(\beta,\alpha,1) \equiv R(\beta,\alpha)$ studied by Gao and Zhou [4].
(ii) $R(\beta,1,1) \equiv R(\beta)$ studied by various authors ([2], [3] and [8]), see also ([1], [6], [11]).
(iii) $R(\beta,0,1) \equiv R_\beta$ studied by Hallenbeck [5].

Now, we introduce Alexander operator $I^n f(z) : A \to A,\ n \in N_0$ by

$$I^0 f(z) = f(z)$$
$$I^1 f(z) = \int_0^z \frac{f(t)}{t} dt$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$I^n f(z) = I^1(I^{n-1} f(z)),\ n \in N.$$

Thus

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{1}{k^n} a_k z^k.$$

It can be easily seen that

$$D^n(I^n f(z)) = f(z) = I^n(D^n f(z)).$$

In the present paper, we determine extreme points of $R(\beta, \alpha, n)$ and also to obtain some sharp bounds for $Re\left\{\frac{D^n f(z)}{z}\right\}$ and $Re\left\{\frac{D^{n-1} f(z)}{z}\right\}$.

2. Main results

Theorem 2.1. *A function $f(z)$ is in $R(\beta, \alpha, n)$, if and only if $f(z)$ can be expressed as,*

$$f(z) = \int_{|x|=1} \left[(2\beta - 1)z + 2(1-\beta)\overline{x} \sum_{k=0}^{\infty} \frac{(xz)^{k+1}}{(k+1)^n (k\alpha + 1)}\right] d\mu(x), \tag{2.1}$$

where $\mu(x)$ is the probability measure defined on the $X = \{x : |x| = 1\}$. For fixed α, β, n and $R(\beta, \alpha, n)$ the probability measure μ defined on X are one-to-one by the expression (2.1)*.*

Proof. By the definition of $R(\beta, \alpha, n)$, $f(z) \in R(\beta, \alpha, n)$, if and only if

$$\frac{\frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} - \beta}{1 - \beta} \in P,$$

where P denotes the normalized well-known class of analytic functions which have positive real part. By the aid of Herglotz expression of functions in P, we have

$$\frac{\frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} - \beta}{1 - \beta} = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x),$$

which is equivalent to

$$\frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z} = \int_{|x|=1} \frac{1 + (1-2\beta)xz}{1 - xz} d\mu(x).$$

So we have

$$I^n\left[z\left\{\frac{D^n f(z) + \alpha(D^{n+1} f(z) - D^n f(z))}{z}\right\}\right] = \int_{|x|=1} I^n z \left\{\frac{1 + (1-2\beta)xz}{1 - xz}\right\} d\mu(x),$$

or

$$f(z)+\alpha(zf'(z)-f(z))=\int_{|x|=1}\left\{z+\sum_{k=2}^{\infty}\frac{2(1-\beta)x^{k-1}z^k}{k^n}\right\}d\mu(x),$$

that is,

$$z^{1-\frac{1}{\alpha}}\int_0^z\left\{\frac{1}{\alpha}f(\zeta)+(\zeta f'(\zeta)-f(\zeta))\right\}\zeta^{\frac{1}{\alpha}-2}d\zeta$$

$$=\frac{1}{\alpha}\int_{|x|=1}\left\{z^{1-\frac{1}{\alpha}}\int_0^z\left\{\zeta+2(1-\beta)\sum_{k=2}^{\infty}\frac{x^{k-1}\zeta^k}{k^n}\right\}\zeta^{\frac{1}{\alpha}-2}\right\}d\mu(x).$$

We obtain

$$f(z)=\int_{|x|=1}\left\{z+2(1-\beta)\sum_{k=2}^{\infty}\frac{x^{k-1}z^k}{k^n(\alpha k+1-\alpha)}\right\}d\mu(x),$$

or equivalently

$$f(z)=\int_{|x|=1}\left\{(2\beta-1)z+2(1-\beta)\overline{x}\sum_{k=0}^{\infty}\frac{(xz)^{k+1}}{(k+1)^n(\alpha k+1)}\right\}d\mu(x).$$

This deductive process can be converse, so we have proved the first part of the theorem. we know that both probability measure μ and class P, class P and $R(\beta,\alpha,n)$ are one-to-one, so the second part of the theorem is true. Thus the proof of Theorem 2.1 is established. □

Corollary 2.2. *The extreme points of the class $R(\beta,\alpha,n)$ are*

$$f_x(z)=(2\beta-1)z+2(1-\beta)\overline{x}\sum_{k=0}^{\infty}\frac{(xz)^{k+1}}{(k+1)^n(\alpha k+1)},\ |x|=1. \tag{2.2}$$

Proof. Using the notation $f_x(z)$ equation (2.1) can be written as

$$f_\mu(z)=\int_{|x|=1}f_x(z)d\mu(x).$$

By Theorem 2.1, the map $\mu\to f_\mu$ is one-to-one so the assertion follows (see [5]). □

Corollary 2.3. *If $f(z)=z+\sum_{k=2}^{\infty}a_kz^k\in R(\beta,\alpha,n)$, then*

$$|a_k(z)|\le\frac{2(1-\beta)}{k^n(\alpha k+1-\alpha)},\ (k\ge 2).$$

The results are sharp.

Proof. The coefficient bounds are maximized at an extreme point. Now from (2.2), $f_x(z)$ can be expressed as

$$f_x(z)=z+2(1-\beta)\sum_{k=2}^{\infty}\frac{x^{k-1}z^k}{k^n(\alpha k+1-\alpha)},\ |x|=1, \tag{2.3}$$

and the result follows. □

Corollary 2.4. *If $f(z) \in R(\beta, \alpha, n)$, then for $|z| = r < 1$*

$$|f(z)| \leq r + 2(1-\beta)\sum_{k=2}^{\infty} \frac{r^k}{k^n(\alpha k + 1 - \alpha)}.$$

The result follows from (2.3).

Next, we determine the sharp lower bound of $Re\left\{\frac{D^n f(z)}{z}\right\}$ and $Re\left\{\frac{D^{n-1} f(z)}{z}\right\}$ for $f(z) \in R(\beta, \alpha, n)$. Since $R(\beta, \alpha, n)$ is rotationally invariant, we may restrict our attention to the extreme point of

$$g(z) = z + 2(1-\beta)\sum_{k=2}^{\infty} \frac{z^k}{k^n(\alpha k + 1 - \alpha)}. \tag{2.4}$$

Theorem 2.5. *If $f(z) \in R(\beta, \alpha, n)$, then for $|z| \leq r < 1$ we have*

$$Re\left\{\frac{D^n f(z)}{z}\right\} \geq 1+2(1-\beta)\sum_{k=2}^{\infty} \frac{(-r)^{k-1}}{\alpha(k-1)+1} > 1+2(1-\beta)\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\alpha(k-1)+1}, \tag{2.5}$$

and

$$Re\left\{\frac{D^n f(z)}{z}\right\} \leq 1+2(1-\beta)\sum_{k=2}^{\infty} \frac{(-r)^{k-1}}{\alpha(k-1)+1} < 1+2(1-\beta)\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\alpha(k-1)+1}. \tag{2.6}$$

These inequalities are both sharp.

Proof. We need only consider $g(z)$ defined by (2.4). We have

$$\frac{D^n g(z)}{z} = 1 + 2(1-\beta)\sum_{k=2}^{\infty} \frac{z^{k-1}}{\alpha(k-1)+1}. \tag{2.7}$$

It can be written as

$$\frac{D^n g(z)}{z} = 1 + 2\frac{(1-\beta)}{\alpha}\int_0^1 t^{\frac{1}{\alpha}} \frac{z}{1-tz} dt. \tag{2.8}$$

So we have

$$Re\left\{\frac{D^n g(z)}{z}\right\} = 1 + 2\frac{(1-\beta)}{\alpha}\int_0^1 t^{\frac{1}{\alpha}} Re\left\{\frac{z}{1-tz}\right\} dt. \tag{2.9}$$

Since $k(z) = \frac{z}{1-tz}$ is convex in U, $k(\overline{z}) = \overline{k(z)}$ and $k(z)$ maps real axis to real axis, we have

$$-\frac{r}{1+tr} \leq Re\left\{\frac{z}{1-tz}\right\} \leq \frac{r}{1-tr}, \ (|z| \leq r).$$

Substituting the last inequalities in (2.9) and expanding the integrand into the power series of t and integrating it, we can obtain the inequalities (2.5) and (2.6).

The sharpness can be seen from (2.7). □

Theorem 2.6. *$D^{n-1}[R(\beta, \alpha, n)] \subset S$ for $\beta \geq \beta_0$ and this result can not be extended to $\beta < \beta_0$, where*

$$\beta_0 = 1 + \frac{1}{2}\left(\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\alpha(k-1)+1}\right)^{-1}.$$

Proof. Let $f(z) \in R(\beta, \alpha, n)$.

Now using (2.5)

$$\left(D^{n-1} f(z)\right)' = 1 + 2(1-\beta) \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\alpha(k-1)+1} \geq 0. \tag{2.10}$$

$D^{n-1}f(z) \in S$, that is, if $\beta \geq \beta_0$, we have $D^{n-1}\left[R(\beta,\alpha,n)\right] \subset S$. The result can not be extended to $\beta < \beta_0$ because $\left(D^{n-1}f(-1)\right)' = 0$ at $\beta = \beta_0$. Thus $\left(D^{n-1}f(-r)\right)' = 0$ for some $r = r(\beta) < 1$ when $\beta < \beta_0$. □

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Theorem 2.7. *If $f(z) \in R(\beta, \alpha, n)$, then for $|z| \leq r < 1$*

$$Re\left\{\frac{D^{n-1}f(z)}{z}\right\} \geq 1 + 2(1-\beta)\sum_{k=2}^{\infty} \frac{(-r)^{k-1}}{k[\alpha(k-1)+1]} \tag{2.11}$$

$$> 1 + 2(1-\beta)\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k[\alpha(k-1)+1]}.$$

The result is sharp.

Proof. According to the same reasoning as in Theorem 2.5, we need only consider $g(z)$ defined by (2.4). We have

$$\frac{D^{n-1}g(z)}{z} = 1 + 2(1-\beta)\sum_{k=2}^{\infty} \frac{z^{k-1}}{k[\alpha(k-1)+1]}$$

$$= 1 + 2\frac{(1-\beta)}{\alpha}\int_0^1 t^{\frac{1}{\alpha}}\left(\int_0^1 \frac{vz}{1-tvz}dv\right)dt.$$

Thus

$$Re\left\{\frac{D^{n-1}g(z)}{z}\right\} = 1 + 2\frac{(1-\beta)}{\alpha}\int_0^1 t^{\frac{1}{\alpha}}\left(\int_0^1 vRe\left\{\frac{z}{1-tvz}\right\}dv\right)dt$$

$$> 1 - 2\frac{(1-\beta)}{\alpha}\int_0^1 t^{\frac{1}{\alpha}}\left(\int_0^1 \frac{vr}{1+tvr}dv\right)dt$$

$$= 1 + 2(1-\beta)\sum_{k=2}^{\infty} \frac{(-r)^{k-1}}{k[\alpha(k-1)+1]}$$

$$> 1 + 2(1-\beta)\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k[\alpha(k-1)+1]}.$$

The sharpness can be seen from (2.4). □

Remark 2.8. If we put $n = 1$ in Theorem 2.1, 2.5 and 2.7 then we obtain the corresponding results due to Gao and Zhou [4].

Remark 2.9. If we put $n = 1, \alpha = 1$ in Theorem 2.1, 2.5, 2.7 then we obtain the corresponding results due to Silverman [10].

Remark 2.10. If we put $n = 1, \alpha = 0$ in Theorem 2.1, 2.5, 2.7 then we obtain the corresponding results due to Hallenbeck [5].

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