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# A simple proof of the fundamental theorem of calculus for the Lebesgue integral

Rodrigo López Pouso

**Abstract.** This paper contains a new elementary proof of the Fundamental Theorem of Calculus for the Lebesgue integral. The hardest part of our proof simply concerns the convergence in $\mathrm{L}^1$ of a certain sequence of step functions, and we prove it using only basic elements from Lebesgue integration theory.

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## 1. Introduction

Let $f : [a, b] \longrightarrow \mathbb{R}$ be absolutely continuous on $[a, b]$, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{(a_j, b_j)\}_{j=1}^n$ is a family of pairwise disjoint subintervals of $[a, b]$ satisfying

$$\sum_{j=1}^{n} (b_j - a_j) < \delta,$$

then

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon.$$

Classical results ensure that $f$ has a finite derivative almost everywhere in $I = [a, b]$, and that $f' \in \mathrm{L}^1(I)$, see [3] or [9, Corollary 6.83]. These results, which we shall use in this paper, are the first steps in the proof of the main connection between absolute continuity and Lebesgue integration: the Fundamental Theorem of Calculus for the Lebesgue integral.

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**Theorem 1.1.** *If $f : I = [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous on $I$, then*

$$f(b) - f(a) = \int_a^b f'(x)\,dx \quad \textit{in Lebesgue's sense.}$$

In this note we present a new elementary proof to Theorem 1.1 which seems more natural and easy than the existing ones. Indeed, our proof can be sketched simply as follows:

1. We consider a well–known sequence of step functions $\{h_n\}_{n\in\mathbb{N}}$ which tends to $f'$ almost everywhere in $I$ and, moreover,
$$\int_a^b h_n(x)\,dx = f(b) - f(a) \quad \text{for all } n \in \mathbb{N}.$$
2. We prove, by means of elementary arguments, that
$$\lim_{n\to\infty} \int_a^b h_n(x)\,dx = \int_a^b f'(x)\,dx.$$

More precise comparison with the literature on Theorem 1.1 and its several proofs will be given in Section 3.

In the sequel $m$ stands for the Lebesgue measure in $\mathbb{R}$.

## 2. Proof of Theorem 1.1

For each $n \in \mathbb{N}$ we consider the partition of the interval $I = [a,b]$ which divides it into $2^n$ subintervals of length $(b-a)2^{-n}$, namely

$$x_{n,0} < x_{n,1} < x_{n,2} < \cdots < x_{n,2^n},$$

where $x_{n,i} = a + i(b-a)2^{-n}$ for $i = 0, 1, 2, \ldots, 2^n$.

Now we construct a step function $h_n : [a,b) \longrightarrow \mathbb{R}$ as follows: for each $x \in [a,b)$ there is a unique $i \in \{0, 1, 2, \ldots, 2^n - 1\}$ such that

$$x \in [x_{n,i}, x_{n,i+1}),$$

and we define

$$h_n(x) = \frac{f(x_{n,i+1}) - f(x_{n,i})}{x_{n,i+1} - x_{n,i}} = \frac{2^n}{b-a}[f(x_{n,i+1}) - f(x_{n,i})].$$

On the one hand, the construction of $\{h_n\}_{n\in\mathbb{N}}$ implies that

$$\lim_{n\to\infty} h_n(x) = f'(x) \quad \text{for a.a. } x \in [a,b]. \tag{2.1}$$

To prove (2.1), we fix $x \in (a,b)$ such that $f'(x)$ exists and $x \neq x_{n,i}$ for all $n \in \mathbb{N}$ and all $i \in \{1, 2, \ldots, 2^n - 1\}$. Now for each $n \in \mathbb{N}$ we consider the unique index

$i \in \{0, 1, 2, \ldots, 2^n - 1\}$ such that $x \in (x_{n,i}, x_{n,i+1})$ and we have

$$\begin{aligned}
|f'(x) - h_n(x)| &= \left| f'(x) - \frac{f(x_{n,i+1}) - f(x) + f(x) - f(x_{n,i})}{x_{n,i+1} - x_{n,i}} \right| \\
&= \left| f'(x) \frac{x_{n,i+1} - x + x - x_{n,i}}{x_{n,i+1} - x_{n,i}} \right. \\
&\quad - \frac{f(x_{n,i+1}) - f(x)}{x_{n,i+1} - x} \frac{x_{n,i+1} - x}{x_{n,i+1} - x_{n,i}} \\
&\quad \left. - \frac{f(x) - f(x_{n,i})}{x - x_{n,i}} \frac{x - x_{n,i}}{x_{n,i+1} - x_{n,i}} \right| \\
&\le \left| f'(x) - \frac{f(x_{n,i+1}) - f(x)}{x_{n,i+1} - x} \right| + \left| f'(x) - \frac{f(x) - f(x_{n,i})}{x - x_{n,i}} \right|,
\end{aligned}$$

which yields (2.1).

On the other hand, for each $n \in \mathbb{N}$ we compute

$$\int_a^b h_n(x)dx = \sum_{i=0}^{2^n-1} \int_{x_{n,i}}^{x_{n,i+1}} h_n(x)dx = \sum_{i=0}^{2^n-1} [f(x_{n,i+1}) - f(x_{n,i})] = f(b) - f(a),$$

and therefore it only remains to prove that

$$\lim_{n\to\infty} \int_a^b h_n(x)\,dx = \int_a^b f'(x)\,dx.$$

Let us prove that, in fact, we have convergence in $\mathrm{L}^1(I)$, i.e.,

$$\lim_{n\to\infty} \int_a^b |h_n(x) - f'(x)|\,dx = 0. \tag{2.2}$$

Let $\varepsilon > 0$ be fixed and let $\delta > 0$ be one of the values corresponding to $\varepsilon/4$ in the definition of absolute continuity of $f$.

Since $f' \in \mathrm{L}^1(I)$ we can find $\rho > 0$ such that for any measurable set $E \subset I$ we have

$$\int_E |f'(x)|\,dx < \frac{\varepsilon}{4} \quad \text{whenever } m(E) < \rho. \tag{2.3}$$

The following lemma will give us fine estimates for the integrals when many of the $|h_n|$ are "large". We postpone its proof for better readability.

**Lemma 2.1.** *For each $\varepsilon > 0$ there exist $k, n_k \in \mathbb{N}$ such that*

$$k \cdot m\left(\left\{x \in I \,:\, \sup_{n \ge n_k} |h_n(x)| > k\right\}\right) < \varepsilon.$$

Lemma 2.1 guarantees that there exist $k, n_k \in \mathbb{N}$ such that

$$k \cdot m\left(\left\{x \in I \,:\, \sup_{n \ge n_k} |h_n(x)| > k\right\}\right) < \min\left\{\delta, \frac{\varepsilon}{4}, \rho\right\}. \tag{2.4}$$

Let us denote

$$A = \left\{x \in I \,:\, \sup_{n \ge n_k} |h_n(x)| > k\right\},$$

which, by virtue of (2.4) and (2.3), satisfies the following properties:

$$m(A) < \delta, \tag{2.5}$$

$$k \cdot m(A) < \frac{\varepsilon}{4}, \tag{2.6}$$

$$\int_A |f'(x)|\,dx < \frac{\varepsilon}{4}. \tag{2.7}$$

We are now in a position to prove that the integrals in (2.2) are smaller than $\varepsilon$ for all sufficiently large values of $n \in \mathbb{N}$. We start by noticing that (2.7) guarantees that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_I |h_n(x) - f'(x)|\,dx &= \int_{I\setminus A} |h_n(x) - f'(x)|\,dx + \int_A |h_n(x) - f'(x)|\,dx \\ &< \int_{I\setminus A} |h_n(x) - f'(x)|\,dx + \int_A |h_n(x)|\,dx + \frac{\varepsilon}{4}. \end{aligned} \tag{2.8}$$

The definition of the set $A$ implies that for all $n \in \mathbb{N}$, $n \geq n_k$, we have

$$|h_n(x) - f'(x)| \leq k + |f'(x)| \quad \text{for almost all } x \in I \setminus A,$$

so the Dominated Convergence Theorem yields

$$\lim_{n\to\infty} \int_{I\setminus A} |h_n(x) - f'(x)|\,dx = 0. \tag{2.9}$$

From (2.8) and (2.9) we deduce that there exists $n_\varepsilon \in \mathbb{N}$, $n_\varepsilon \geq n_k$, such that for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$, we have

$$\int_I |h_n(x) - f'(x)|\,dx < \frac{\varepsilon}{2} + \int_A |h_n(x)|\,dx. \tag{2.10}$$

Finally, we estimate $\int_A |h_n|$ for each fixed $n \in \mathbb{N}$, $n \geq n_\varepsilon$. First, we decompose $A = B \cup C$, where

$$B = \{x \in A \,:\, |h_n(x)| \leq k\} \quad \text{and} \quad C = A \setminus B.$$

We immediately have

$$\int_B |h_n(x)|\,dx \leq k \cdot m(B) \leq k \cdot m(A) < \frac{\varepsilon}{4} \tag{2.11}$$

by (2.6).

Obviously, $\int_C |h_n| < \varepsilon/4$ when $C = \varnothing$. Let us see that this inequality holds true when $C \neq \varnothing$. For every $x \in C = \{x \in A \,:\, |h_n(x)| > k\}$ there is a unique index $i \in \{0, 1, 2, \ldots, 2^n - 1\}$ such that $x \in [x_{n,i}, x_{n,i+1})$. Since $|h_n|$ is constant on $[x_{n,i}, x_{n,i+1})$ we deduce that $[x_{n,i}, x_{n,i+1}) \subset C$. Thus there exist indices $i_l \in \{0, 1, 2, \ldots, 2^n - 1\}$, with $l = 1, 2, \ldots, p$ and $i_l \neq i_{\tilde{l}}$ if $l \neq \tilde{l}$, such that

$$C = \bigcup_{l=1}^{p} [x_{n,i_l}, x_{n,i_l+1}).$$

Therefore
$$\sum_{l=1}^{p}(x_{n,i_l+1} - x_{n,i_l}) = m(C) \leq m(A) < \delta \quad \text{by (2.5)},$$
and then the absolute continuity of $f$ finally comes into action:
$$\begin{aligned}\int_C |h_n(x)|\,dx &= \sum_{l=1}^{p} \int_{x_{n,i_l}}^{x_{n,i_l+1}} |h_n(x)|\,dx \\ &= \sum_{l=1}^{p} |f(x_{n,i_l+1}) - f(x_{n,i_l})| < \frac{\varepsilon}{4}.\end{aligned}$$

This inequality, along with (2.10) and (2.11), guarantee that for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$, we have
$$\int_I |h_n(x) - f'(x)|\,dx < \varepsilon,$$
thus proving (2.2) because $\varepsilon$ was arbitrary. The proof of Theorem 1.1 is complete.

Now we go back to Lemma 2.1. A more general version, which seems interesting in its own right, will be established instead. We split it into two parts for better readability.

A first result, elementary but absent from many textbooks, complements Tchebyshev's inequality for integrable functions. It is however an old result which we can already find in Hobson's book [6, page 526]. A proof is included for the convenience of readers.

**Proposition 2.2.** *Let $A \subset \mathbb{R}$ be a measurable set. If $g \in \mathrm{L}^1(A)$ then*
$$\lim_{k\to\infty} k \cdot m\left(\{x \in A \,:\, |g(x)| \geq k\}\right) = 0. \tag{2.12}$$

*Proof.* Standard results guarantee that
$$\begin{aligned}\sum_{n=0}^{\infty} n \cdot m(\{x \in A \,:\, n \leq |g(x)| < n+1\}) &\leq \sum_{n=0}^{\infty} \int_{\{x\,:\,n\leq|g(x)|<n+1\}} |g(x)|\,dx \\ &= \int_{\cup_{n=0}^{\infty}\{x\,:\,n\leq|g(x)|<n+1\}} |g(x)|\,dx \\ &= \int_A |g(x)|\,dx,\end{aligned}$$
hence the series on the left-hand side is convergent.

For all $k \in \mathbb{N}$ we have
$$\begin{aligned}k \cdot m\left(\{x \in A \,:\, |g(x)| \geq k\}\right) &= k \cdot \sum_{n=k}^{\infty} m\left(\{x \in A \,:\, n \leq |g(x)| < n+1\}\right) \\ &\leq \sum_{n=k}^{\infty} n \cdot m(\{x \in A \,:\, n \leq |g(x)| < n+1\}),\end{aligned}$$
and then (2.12) obtains. □

Now we proceed with our extended version of Lemma 2.1.

**Proposition 2.3.** *Let $A \subset \mathbb{R}$ be a measurable set with $m(A) < \infty$. Assume that $g_n : A \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is measurable for each $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} g_n(x) = g(x) \in \mathbb{R} \quad \text{for almost all } x \in A. \tag{2.13}$$

*Then for every $k \in \mathbb{N}$ we have*

$$\lim_{j\to\infty} m\left(\left\{x \in A \,:\, \sup_{n\geq j} |g_n(x)| \geq k\right\}\right) = m\left(\{x \in A \,:\, |g(x)| \geq k\}\right), \tag{2.14}$$

*and, if $g \in \mathrm{L}^1(A)$ then*

$$\lim_{k\to\infty} \lim_{j\to\infty} k \cdot m\left(\left\{x \in A \,:\, \sup_{n\geq j} |g_n(x)| \geq k\right\}\right) = 0, \tag{2.15}$$

*which implies the result in Lemma* 2.1 *for $g_n = h_n$ and $g = f'$.*

*Proof.* Let $N \subset A$ be a null–measure set such that (2.13) holds for all $x \in A \setminus N$ and let $k \in \mathbb{N}$ be fixed.

We define a family of measurable sets

$$E_j = \left\{x \in A \setminus N \,:\, \sup_{n\geq j} |g_n(x)| \geq k\right\} \quad (j \in \mathbb{N}).$$

Notice that $E_{j+1} \subset E_j$ for every $j \in \mathbb{N}$, and $m(E_1) \leq m(A) < \infty$, hence

$$\lim_{j\to\infty} m(E_j) = m\left(\bigcap_{j=1}^{\infty} E_j\right) = m\left(\{x \in A \setminus N \,:\, |g(x)| \geq k\}\right),$$

so (2.14) is proven.

Now (2.14) and (2.12) yield (2.15). □

## 3. Final remarks

The sequence $\{h_n\}_{n\in\mathbb{N}}$ is used in other proofs of Theorem 1.1, see [1] or [11]. The novelty in this paper is our elementary and self–contained proof of (2.2).

Our proof avoids somewhat technical results often invoked to prove Theorem 1.1. For instance, we do not use any sophisticated estimate for the measure of image sets such as [4, Theorem 7.20], [9, Lemma 6.88] or [11, Proposition 1.2], see also [7]. We do not use the following standard lemma either (usually proven by means of Vitali's Covering Theorem): an absolutely continuous function having zero derivative almost everywhere is constant, see [4, Theorem 7.16] or [9, Lemma 6.89]. It is worth having a look at [5] for a proof of that lemma using tagged partitions; see also [2] for a proof based on full covers [10].

Concise proofs of Theorem 1.1 follow from the Radon–Nikodym Theorem, see [1], [4] or [8], but this is far from being elementary.

Finally, it is interesting to note that (2.2) is an almost trivial consequence of Lebesgue's Dominated Convergence Theorem when $f$ is Lipschitz continuous on $I$.

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Rodrigo López Pouso
Departament of Mathematical Analysis
Faculty of Mathematics
University of Santiago de Compostela
15782 Santiago de Compostela, Spain
e-mail: `rodrigo.lopez@usc.es`