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Flow of Herschel-Bulkley fluid through a two dimensional thin layer

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Abstract. The paper is devoted to the study of asymptotic behaviour of the solution of two dimensional steady flow of Herschel-Bulkley fluid through a thin layer. We prove some convergence results when the thicness tends to zero and we give the mechanical interpretation of the results.

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1. Introduction

The rigid, viscoplastic and incompressible fluid of Herschel-Bulkley has been studied and used by many mathematicians, physicists and engineers, in order to model the flow of metals, plastic solids and a variety of polymers. Due to existence of yield limit, the model can capture phenomena connected with the development of discontinuous stresses. A particularity of Herschel-Bulkley fluid lies in the presence of rigid zones located in the interior of the flow and as the yield limit increases, the rigid zones become larger and may completely block the flow, this phenomenon is known as the blockage property. The literature concerning this topic is extensive; see e.g. [10, 11, 12, 13].

The purpose of this paper is to study the asymptotic behaviour of steady flow of Herschel-Bulkley fluid in a two dimensional thin layer.

The paper is organized as follows. In Section 2 we present the mechanical problem of the steady flow of Herschel-Bulkley fluid in a two dimensional thin layer. We introduce some notations and preliminaries. Moreover, we define some function spaces and we recall the variational formulation. In Section 3, we are interested in the asymptotic behaviour, to this aim we prove some convergence results concerning the velocity and pressure when the thickness tends to zero. In addition, the uniqueness of limit solution has been also established. Finally, we will discuss in Section 4 the mechanical interpretation of the convergence results.

2. Problem statement

Denoting by I the open interval $I =]0,1[$. Introducing the function $h : I \longrightarrow \mathbb{R}_+^*$ such that $h \in \mathcal{C}^1(I)$.

Considering the following domains

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 \mid x \in I \text{ and } 0 < y < h(x)\right\},$$
$$\Omega^\varepsilon = \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in I \text{ and } 0 < x_2 < \varepsilon h(x_1)\right\},$$

where $\varepsilon > 0$.

Remark that if $(x_1,x_2) \in \Omega^\varepsilon$ then $(x,y) = \left(x_1, \dfrac{x_2}{\varepsilon}\right) \in \Omega$. This permits us to define, for every function $\varphi^\varepsilon : \Omega^\varepsilon \longrightarrow \mathbb{R}$, the function $\widehat{\varphi^\varepsilon} : \Omega \longrightarrow \mathbb{R}$ given by

$$\widehat{\varphi^\varepsilon}(x,y) = \varphi^\varepsilon(x_1,x_2).$$

Let $1 < p \leq 2$, p' the conjugate p, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ and $\mathbf{f} \in L^{p'}(\Omega)^2$ a given function. We define the function $\mathbf{f}^\varepsilon \in L^{p'}(\Omega^\varepsilon)$ such that $\widehat{\mathbf{f}^\varepsilon} = \mathbf{f}$.

We consider a mathematical problem modelling the steady flow of a rigid viscoplastic and incompressible Herschel-Bulkley fluid in the domain Ω^ε. We suppose that the consistency and yield limit of the fluid are respectively $\mu\varepsilon^p$, $g\varepsilon$ where μ, $g > 0$ and p represents the power law index. The fluid is acted upon by given volume forces of density $\mathbf{f}^\varepsilon$. On $\partial\Omega^\varepsilon$ we suppose that the velocity is known and equal to zero.

We denote by $\mathbb{S}_2$ the space of symmetric tensors on $\mathbb{R}^2$. We define the inner product and the Euclidean norm on $\mathbb{R}^2$ and $\mathbb{S}_2$, respectively, by

$$u \cdot v = u_i v_i \quad \forall\, u,\ v \in \mathbb{R}^2 \text{ and } \sigma \cdot \tau = \sigma_{ij}\tau_{ij} \quad \forall\, \sigma,\ \tau \in \mathbb{S}_2.$$
$$|u| = (u \cdot u)^{\frac{1}{2}} \quad \forall u \in \mathbb{R}^2 \text{ and } |\sigma| = (\sigma \cdot \sigma)^{\frac{1}{2}} \quad \forall \sigma \in \mathbb{S}_2.$$

Here and below, the indices i and j run from 1 to 2 and the summation convention over repeated indices is used. We denote by $\widetilde{\sigma^\varepsilon}$ the deviator of $\sigma^\varepsilon = \left(\sigma^\varepsilon_{ij}\right)$ given by

$$\widetilde{\sigma^\varepsilon} = \left(\widetilde{\sigma^\varepsilon_{ij}}\right), \quad \widetilde{\sigma^\varepsilon_{ij}} = \sigma^\varepsilon_{ij} + p^\varepsilon \delta_{ij},$$

where p^ε represents the hydrostatic pressure and $\delta = (\delta_{ij})$ denotes the identity tensor. We consider the rate of deformation operator defined for every $\mathbf{v}^\varepsilon \in W^{1,p}(\Omega^\varepsilon)^2$ by

$$D(\mathbf{v}^\varepsilon) = (D_{ij}(\mathbf{v}^\varepsilon)), \quad D_{ij}(\mathbf{v}^\varepsilon) = \frac{1}{2}\left(v^\varepsilon_{i,j} + v^\varepsilon_{j,i}\right).$$

The steady flow of Herschel-Bulkley fluid in the domain Ω^ε is given by the following mechanical problem.

Problem P_ε. Find the velocity field $\mathbf{u}^\varepsilon = (u^\varepsilon_i) : \Omega^\varepsilon \longrightarrow \mathbb{R}^2$, the stress field $\sigma^\varepsilon = \left(\sigma^\varepsilon_{ij}\right) : \Omega^\varepsilon \longrightarrow \mathbb{S}_2$ and the pressure $p^\varepsilon : \Omega^\varepsilon \longrightarrow \mathbb{R}$ such that

$$\text{div}\sigma^\varepsilon + \mathbf{f}^\varepsilon = 0 \text{ in } \Omega^\varepsilon. \tag{2.1}$$

$$\left\{\begin{array}{ll} \widetilde{\sigma^\varepsilon} = \mu\varepsilon^p \left|D(\mathbf{u}^\varepsilon)\right|^{p-2} D(\mathbf{u}^\varepsilon) + g\varepsilon \dfrac{D(\mathbf{u}^\varepsilon)}{|D(\mathbf{u}^\varepsilon)|} & \text{if } |D(\mathbf{u}^\varepsilon)| \neq 0 \\ \left|\widetilde{\sigma^\varepsilon}\right| \leq g\varepsilon & \text{if } |D(\mathbf{u}^\varepsilon)| = 0 \end{array}\right. \text{ in } \Omega^\varepsilon. \tag{2.2}$$

$$\text{div}\mathbf{u}^\varepsilon = 0 \text{ in } \Omega^\varepsilon. \tag{2.3}$$

$$\mathbf{u}^{\varepsilon}=0 \text{ on } \partial\Omega^{\varepsilon}. \tag{2.4}$$

Here, the flow is given by the equation (2.1). Equation (2.2) represents the constitutive law of Herschel-Bulkley fluid. (2.3) represents the incompressibility condition. Equality (2.4) gives the velocity on the boundary $\partial\Omega^{\varepsilon}$.

Let us define now the following Banach spaces

$$W_{\text{div}}^{p,\varepsilon}=\left\{\mathbf{v}\in W_0^{1,p}\left(\Omega^{\varepsilon}\right)^2 : div\left(\mathbf{v}\right)=0 \text{ in } \Omega^{\varepsilon}\right\}, \tag{2.5}$$

$$W_{\text{div}}^{p}=\left\{\mathbf{v}\in W_0^{1,p}\left(\Omega\right)^2 : div\left(\mathbf{v}\right)=0 \text{ in } \Omega\right\}, \tag{2.6}$$

$$W_p=\left\{\varphi\in L^p\left(\Omega\right) : \frac{\partial\varphi}{\partial y}\in L^p\left(\Omega\right)\right\}. \tag{2.7}$$

$$L_0^p\left(\Omega^{\varepsilon}\right)=\left\{\varphi^{\varepsilon}\in L^p\left(\Omega^{\varepsilon}\right) : \int_{\Omega^{\varepsilon}}\varphi^{\varepsilon}\left(x_1,x_2\right)dx_1dx_2=0\right\}, \tag{2.8}$$

$$L_0^p\left(\Omega\right)=\left\{\varphi\in L^p\left(\Omega\right) : \int_{\Omega}\varphi\left(x,y\right)dxdy=0\right\}, \tag{2.9}$$

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem.

The use of Green's formula permits us to derive the following variational formulation of the mechanical problem (P_{ε}), see [13].

Problem $P\text{V}_{\varepsilon}$. For prescribed data $\mathbf{f}^{\varepsilon}\in L^{p'}\left(\Omega^{\varepsilon}\right)^2$. Find $\left(\mathbf{u}^{\varepsilon},p^{\varepsilon}\right)\in W_{\text{div}}^{p,\varepsilon}\times L_0^{p'}\left(\Omega^{\varepsilon}\right)$ satisfying the variational inequality

$$\begin{aligned}
&\mu\varepsilon^p\int_{\Omega^{\varepsilon}}\left|D\left(\mathbf{u}^{\varepsilon}\right)\right|^{p-2}D\left(\mathbf{u}^{\varepsilon}\right)\cdot D\left(\mathbf{v}-\mathbf{u}^{\varepsilon}\right)dx_1dx_2+\\
&g\varepsilon\int_{\Omega^{\varepsilon}}\left|D\left(\mathbf{v}\right)\right|dx_1dx_2-g\varepsilon\int_{\Omega^{\varepsilon}}\left|D\left(\mathbf{u}^{\varepsilon}\right)\right|dx_1dx_2\\
\geq&\int_{\Omega^{\varepsilon}}\mathbf{f}^{\varepsilon}\cdot\left(\mathbf{v}-\mathbf{u}^{\varepsilon}\right)dx_1dx_2+\int_{\Omega^{\varepsilon}}p^{\varepsilon}\text{div}\left(\mathbf{v}-\mathbf{u}^{\varepsilon}\right)dx_1dx_2 \quad \forall\mathbf{v}\in W_0^{1,p}\left(\Omega^{\varepsilon}\right)^2.
\end{aligned} \tag{2.10}$$

It is known that this variational problem has a unique solution $\left(\mathbf{u}^{\varepsilon},p^{\varepsilon}\right)\in W_{\text{div}}^{p,\varepsilon}\times L_0^{p'}\left(\Omega^{\varepsilon}\right)$, see for more details [10, 13].

3. Asymptotic behaviour

In this section we establish some results concerning the asymptotic behaviour of the solution when ε tends to zero.

We begin by recalling the following lemmas, see [1, 3, 7].

Lemma 3.1. *1. Poincaré's inequality. For every* $\mathbf{v}\in W_0^{1,p}\left(\Omega^{\varepsilon}\right)^2$ *we have*

$$\left\|\mathbf{v}^{\varepsilon}\right\|_{L^p\left(\Omega^{\varepsilon}\right)^2}\leq\varepsilon\left\|\frac{\partial\mathbf{v}^{\varepsilon}}{\partial x_2}\right\|_{L^p\left(\Omega^{\varepsilon}\right)^2}. \tag{3.1}$$

2. Korn's inequality. For every $\mathbf{v} \in W_0^{1,p}(\Omega^\varepsilon)^2$ *there exists a positive constant* C_0 *independent on* ε, *such that*

$$\|\nabla \mathbf{v}^\varepsilon\|_{L^p(\Omega^\varepsilon)^4} \leq C_0 \|D(\mathbf{v}^\varepsilon)\|_{L^p(\Omega^\varepsilon)^4}. \tag{3.2}$$

Lemma 3.2 (Minty). *Let* E *be a Banach spaces,* $A : E \longrightarrow E'$ *a monotone and hemicontinuous operator,* $J : E \longrightarrow]-\infty, +\infty]$ *a proper and convex functional. Let* $u \in E$ *and* $f \in E'$. *Then the following assertions are equivalent:*

1. $\langle Au; v-u\rangle_{E'\times E} + J(v) - J(u) \geq \langle f; v-u\rangle_{E'\times E} \quad \forall v \in E.$

2. $\langle Av; v-u\rangle_{E'\times E} + J(v) - J(u) \geq \langle f; v-u\rangle_{E'\times E} \quad \forall v \in E.$

The main results of this section are stated by the following proposition.

Proposition 3.3. *Let* $(\mathbf{u}^\varepsilon, p^\varepsilon) \in W_{\mathrm{div}}^{p,\varepsilon} \times L_0^{p'}(\Omega^\varepsilon)$ *be the solution of variational problem* (PV_ε). *Then, there exists* $(\widehat{\mathbf{u}}, \widehat{p}) \in W_p^2 \times L_0^{p'}(\Omega)$ *such that*

$$\widehat{\mathbf{u}^\varepsilon} \longrightarrow \widehat{\mathbf{u}} \text{ in } W_p^2 \text{ weakly,} \tag{3.3}$$

$$\frac{\partial \widehat{u_2^\varepsilon}}{\partial y} \longrightarrow 0 \text{ in } L^p(\Omega) \text{ weakly,} \tag{3.4}$$

$$\widehat{p^\varepsilon} \longrightarrow \widehat{p} \text{ in } L_0^{p'}(\Omega) \text{ weakly.} \tag{3.5}$$

Proof. Choosing $\mathbf{v} = 0$ as test function in inequality (2.10), we deduce that

$$\mu\varepsilon^p \|D(\mathbf{u}^\varepsilon)\|_{L^p(\Omega^\varepsilon)^4}^p \leq \int_{\Omega^\varepsilon} \mathbf{f}^\varepsilon \cdot \mathbf{u}^\varepsilon dx_1 dx_2.$$

This permits us to obtain, making use of Poincaré's and Korn's inequalities and by passage to variables x and y

$$\|\widehat{\mathbf{u}^\varepsilon}\|_{L^p(\Omega)^2} \leq c, \tag{3.6}$$

$$\left\|\frac{\partial \widehat{\mathbf{u}^\varepsilon}}{\partial y}\right\|_{L^p(\Omega)^2} \leq c, \tag{3.7}$$

$$\left\|\frac{\partial \widehat{\mathbf{u}^\varepsilon}}{\partial x}\right\|_{L^p(\Omega)^2} \leq \frac{c}{\varepsilon}. \tag{3.8}$$

Moreover, we get using the incompressibility condition (2.3) and Green's formula, for any function $\varphi^\varepsilon \in W_0^{1,p'}(\Omega^\varepsilon)$

$$\int_\Omega \frac{\partial \widehat{u_2^\varepsilon}}{\partial y} \widehat{\varphi^\varepsilon} dxdy = \varepsilon \int_\Omega u_1^\varepsilon \frac{\partial \widehat{\varphi^\varepsilon}}{\partial x} dxdy.$$

Which gives, making use (2.6)

$$\left\|\frac{\partial \widehat{u_2^\varepsilon}}{\partial y}\right\|_{W^{-1,p}(\Omega)} \leq c\varepsilon. \tag{3.9}$$

We can then extract a subsequence still denoted by $\widehat{\mathbf{u}^{\varepsilon}}$, such that

$$\widehat{\mathbf{u}^{\varepsilon}} \longrightarrow \widehat{\mathbf{u}} \text{ in } L^{p}(\Omega)^{2} \text{ weakly,} \tag{3.10}$$

$$\frac{\partial \widehat{\mathbf{u}^{\varepsilon}}}{\partial y} \longrightarrow \frac{\partial \widehat{\mathbf{u}}}{\partial y} \text{ in } L^{p}(\Omega)^{2} \text{ weakly,} \tag{3.11}$$

$$\frac{\partial \widehat{u_2^{\varepsilon}}}{\partial y} \longrightarrow 0 \text{ in } L^{p}(\Omega) \text{ weakly,} \tag{3.12}$$

Let now $\mathbf{v}^{\varepsilon} \in W_0^{1,p}(\Omega^{\varepsilon})^2$, we obtain by setting $\mathbf{u}^{\varepsilon} - \mathbf{v}^{\varepsilon}$ as test function in inequality (2.10), using the incompressibility condition (2.3), Green's formula and Hölder's inequality

$$\int_{\Omega^{\varepsilon}} \nabla p^{\varepsilon} \cdot \mathbf{v}^{\varepsilon} dx_1 dx_2 \leq \mu \varepsilon^{p} \left(\int_{\Omega^{\varepsilon}} |D(\mathbf{u}^{\varepsilon})|^{p} dx_1 dx_2 \right)^{\frac{1}{p'}} \left(\int_{\Omega^{\varepsilon}} |D(\mathbf{v}^{\varepsilon})|^{p} dx_1 dx_2 \right)^{\frac{1}{p}}$$
$$+ \varepsilon^{\frac{1}{p'}+1} g\,(\text{meas}\,(\Omega))^{\frac{1}{p'}} \left(\int_{\Omega^{\varepsilon}} |D(\mathbf{v}^{\varepsilon})|^{p} dx_1 dx_2 \right)^{\frac{1}{p}} + \varepsilon \left\| \widehat{\mathbf{f}^{\varepsilon}} \right\|_{L^{p'}(\Omega)^2} \left\| \widehat{\mathbf{v}^{\varepsilon}} \right\|_{W_0^{1,p}(\Omega)^2} . \tag{3.13}$$

On the other hand, it is easy to check that after some algebraic manipulations we find

$$\left(\int_{\Omega^{\varepsilon}} |D(\mathbf{v}^{\varepsilon})|^{p} dx_1 dx_2 \right)^{\frac{1}{p}} \leq \varepsilon^{\frac{1}{p}-1} \left\| \widehat{\mathbf{v}^{\varepsilon}} \right\|_{W_0^{1,p}(\Omega)^2} . \tag{3.14}$$

Thus, from (3.7), (3.8), (3.13) and (3.14) we get

$$\int_{\Omega^{\varepsilon}} \nabla p^{\varepsilon} \cdot \mathbf{v}^{\varepsilon} dx_1 dx_2 \leq c\varepsilon \left\| \widehat{\mathbf{v}^{\varepsilon}} \right\|_{W_0^{1,p}(\Omega)^2} . \tag{3.15}$$

Passing to the variables x and y in (3.15) we find the following estimates

$$\left\| \widehat{p^{\varepsilon}} \right\|_{L_0^{p'}(\Omega)} \leq c \,, \tag{3.16}$$

$$\left\| \frac{\partial \widehat{p^{\varepsilon}}}{\partial x} \right\|_{W^{-1,p'}(\Omega)} \leq c \,, \tag{3.17}$$

$$\left\| \frac{\partial \widehat{p^{\varepsilon}}}{\partial y} \right\|_{W^{-1,p'}(\Omega)} \leq c\varepsilon. \tag{3.18}$$

Consequently, we can extract a subsequence still denoted by $\widehat{p^{\varepsilon}}$ such that

$$\widehat{p^{\varepsilon}} \longrightarrow \widehat{p} \text{ in } L_0^{p'}(\Omega) \text{ weakly,} \tag{3.19}$$

which achieves the proof. □

This proof permits also to deduce that the limit pressure verify $\widehat{p}(x,y) = \widehat{p}(x)$.

Proposition 3.4. *The velocity limit given by (3.3) verifies*

$$\int_0^{h(x)} \widehat{u_1}(x,y)\,dy = 0 \quad \forall x \in I. \tag{3.20}$$

Proof. We know from the incompressibility condition (2.3) that

$$\int_{\Omega^\varepsilon} \operatorname{div}\mathbf{u}^\varepsilon (x_1, x_2)\, \varphi(x_1)\, dx_1 dx_2 = 0 \text{ for all } \varphi \in \mathcal{D}(I).$$

This implies, using Green's formula

$$\int_{\Omega^\varepsilon} u_1^\varepsilon (x_1, x_2) \frac{d\varphi}{dx_1}(x_1)\, dx_1 dx_2 = \int_{\Omega^\varepsilon} \frac{\partial u_2^\varepsilon}{\partial x_2}(x_1, x_2)\, \varphi(x_1)\, dx_1 dx_2.$$

Hence, by passage to the variables x and y and using Fubini's theorem and Green's formula, we can infer

$$-\int_0^1 \varphi(x) \left(\frac{d}{dx} \int_0^{h(x)} \widehat{u_1^\varepsilon}(x, y)\, dy \right) dx = 0 \quad \forall \varphi \in \mathcal{D}(I).$$

Then

$$\frac{d}{dx} \int_0^{h(x)} \widehat{u_1^\varepsilon}(x, y)\, dy = 0.$$

Moreover, the fact that $\widehat{u_1^\varepsilon} \in L^p(\Omega)$ and $h \in \mathcal{C}^1(I)$ gives, using the Sobolev embedding $W^{1,p}(I) \subset \mathcal{C}^0\left(\overline{I}\right)$

$$\int_0^{h(x)} \widehat{u_1^\varepsilon}(x, y)\, dy \in \mathcal{C}^0\left(\overline{I}\right).$$

Thus, by passage to the limit when ε tends to zero, taking into account the boundary condition (2.4), the assertion (3.20) can be deduced.

We derive in the proposition below the strong equation verified by the limit solution $(\widehat{\mathbf{u}}, \widehat{p}) \in W_p^2 \times L_0^{p'}(\Omega)$. □

Proposition 3.5. *If* $\dfrac{\partial \widehat{u_1}}{\partial y} \neq 0$, *then the limit point* $(\widehat{u_1}, \widehat{p})$ *given by (*3.3*) and (*3.5*) verify the limit problem*

$$-\frac{\partial}{\partial y}\left(\frac{\mu}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u_1}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u_1}}{\partial y} + \frac{\sqrt{2}}{2} g\, sign\left(\frac{\partial \widehat{u_1}}{\partial y} \right) \right) = \widehat{f}_1 - \frac{d\widehat{p}}{dx} \quad in \quad W^{-1,p\prime}(\Omega). \tag{3.21}$$

Proof. Introducing the operator A defined as follows

$$A : W_0^{1,p}(\Omega^\varepsilon)^2 \longrightarrow W^{-1,p\prime}(\Omega^\varepsilon)^2,$$

$$\langle A\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon \rangle_{W^{-1,p\prime}(\Omega^\varepsilon)^2 \times W_0^{1,p}(\Omega^\varepsilon)^2} = \mu \varepsilon^p \int_{\Omega^\varepsilon} |D(\mathbf{v}^\varepsilon)|^{p-2} D(\mathbf{v}^\varepsilon) \cdot D(\mathbf{v}^\varepsilon)\, dx_1 dx_2.$$

It is easy to verify that A is monotone and hemi-continuous (see for more details the reference [13]). Moreover, we know that the functional

$$\mathbf{v}^\varepsilon \in W_0^{1,p}(\Omega^\varepsilon)^2 \longrightarrow g\varepsilon \int_{\Omega^\varepsilon} |D(\mathbf{v}^\varepsilon)|\, dx_1 dx_2$$

is proper and convex. Then, the use of Minty's lemma permits us to affirm that (2.10) is equivalent to the following inequality

$$\begin{aligned} &\mu\varepsilon^p \int_{\Omega^\varepsilon} |D(\mathbf{v}^\varepsilon)|^{p-2} D(\mathbf{v}^\varepsilon)\cdot D(\mathbf{v}^\varepsilon-\mathbf{u}^\varepsilon)\,dx_1dx_2+ \\ &g\varepsilon\int_{\Omega^\varepsilon}|D(\mathbf{v}^\varepsilon)|\,dx_1dx_2 - g\varepsilon\int_{\Omega^\varepsilon}|D(\mathbf{u}^\varepsilon)|\,dx_1dx_2 \\ \geq &\int_{\Omega^\varepsilon}\mathbf{f}^\varepsilon\cdot(\mathbf{v}^\varepsilon-\mathbf{u}^\varepsilon)\,dx_1dx_2+\int_{\Omega^\varepsilon}p^\varepsilon\mathrm{div}(\mathbf{v}^\varepsilon-\mathbf{u}^\varepsilon)\,dx_1dx_2 \quad \forall \mathbf{v}^\varepsilon\in W_0^{1,p}(\Omega^\varepsilon)^2. \end{aligned}$$

Our goal now is to pass to the limit when ε tends to zero. To this aim, we use Proposition 3.4 and the weak lower semi-continuity of the convex and continuous functional $\mathbf{v}^\varepsilon\in W_0^{1,p}(\Omega^\varepsilon)\longrightarrow g\varepsilon\int_{\Omega^\varepsilon}|D(\mathbf{v}^\varepsilon)|\,dx_1dx_2$. We find the following limit inequality

$$\begin{aligned} &\mu\int_\Omega \frac{1}{2^{\frac{p-2}{2}}}\left[\left|\frac{\partial\widehat{v_1}}{\partial y}\right|^2+\left|\frac{\partial\widehat{v_2}}{\partial y}\right|^2\right]^{\frac{p-2}{2}}\left[\frac{1}{2}\frac{\partial\widehat{v_1}}{\partial y}\frac{\partial(\widehat{v_1}-\widehat{u_1})}{\partial y}+\frac{\partial\widehat{v_2}}{\partial y}\frac{\partial(\widehat{v_2}-\widehat{u_2})}{\partial y}\right]dxdy \\ &+g\int_\Omega\left[\frac{1}{2}\left|\frac{\partial\widehat{v_1}}{\partial y}\right|^2+\left|\frac{\partial\widehat{v_2}}{\partial y}\right|^2\right]^{\frac{1}{2}}dxdy-g\int_\Omega\left[\frac{1}{2}\left(\frac{\partial\widehat{u_1}}{\partial y}\right)^2+\left(\frac{\partial\widehat{u_2}}{\partial y}\right)^2\right]^{\frac{1}{2}}dxdy \\ &\geq\int_\Omega\widehat{\mathbf{f}}\cdot(\widehat{\mathbf{v}}-\widehat{\mathbf{u}})\,dxdy+\int_\Omega\widehat{p}\mathrm{div}(\widehat{\mathbf{v}}-\widehat{\mathbf{u}})\,dxdy \quad \forall\mathbf{v}^\varepsilon\in W_0^{1,p}(\Omega^\varepsilon). \end{aligned} \tag{3.22}$$

Furthermore, from (3.3) and (3.4) we find

$$\frac{\partial\widehat{u_2}}{\partial y}=0 \text{ in } \Omega.$$

It follows, keeping in mind (3.20), that

$$\widehat{\mathbf{u}}(x,y)=(\widehat{u_1}(x,y),0).$$

This permits also to choose $\widehat{\mathbf{v}_2}=0$ in (3.22).

Considering now the operator A such that

$$A: W_p\longrightarrow W_p',$$

$$\langle A\widehat{u_1},\widehat{v_1}\rangle_{W_p'\times W_p}=\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega^\varepsilon}\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\frac{\partial\widehat{v_1}}{\partial y}dxdy.$$

It is clear that the operator A is monotone and hemi-continuous and the functional $\widehat{v_1}\in W_p\longrightarrow\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial\widehat{v_1}}{\partial y}\right|dxdy$ is proper and convex. Hence, we deduce using again Minty's lemma

$$\begin{aligned} &\frac{\mu}{2^{\frac{p}{2}}}\int_\Omega\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\frac{\partial(\widehat{v_1}-\widehat{u_1})}{\partial y}dxdy+\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial\widehat{v_1}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial\widehat{u_1}}{\partial y}\right|dxdy \\ &\geq\int_\Omega\widehat{f_1}(\widehat{v_1}-\widehat{u_1})\,dxdy-\int_\Omega\frac{d\widehat{p}}{dx}(\widehat{v_1}-\widehat{u_1})\,dxdy \quad \forall\widehat{v_1}\in W_p. \end{aligned}\tag{3.23}$$

This yields, via Green's formula

$$-\frac{\mu}{2^{\frac{p}{2}}}\int_\Omega \frac{\partial}{\partial y}\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial \widehat{u_1}}{\partial y}\right)(\widehat{v_1}-\widehat{u_1})\,dxdy+$$
$$\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial \widehat{v_1}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial \widehat{u_1}}{\partial y}\right|dxdy$$
$$\geq \int_\Omega \widehat{f}_1(\widehat{v_1}-\widehat{u_1})\,dxdy-\int_\Omega\frac{d\widehat{p}}{dx}(\widehat{v_1}-\widehat{u_1})\,dxdy \quad \forall \widehat{v_1}\in W_p. \tag{3.24}$$

Due to the fact that $W_0^{1,p}(\Omega)$ is dense in W_p, see [1], we can take $\widehat{v_1}=\widehat{u_1}\pm\varphi$ in (3.24), where $\varphi\in W_0^{1,p}(\Omega)$ to obtain the following inequalities

$$-\frac{\mu}{2^{\frac{p}{2}}}\int_\Omega \frac{\partial}{\partial y}\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial \widehat{u_1}}{\partial y}\right)\varphi dxdy+\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial(\widehat{u_1}+\varphi)}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial \widehat{u_1}}{\partial y}\right|dxdy$$
$$\geq \int_\Omega \widehat{f}_1\varphi dxdy-\int_\Omega\frac{d\widehat{p}}{dx}\varphi dxdy \quad \forall\varphi\in W_0^{1,p}(\Omega).$$

and

$$\frac{\mu}{2^{\frac{p}{2}}}\int_\Omega \frac{\partial}{\partial y}\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial \widehat{u_1}}{\partial y}\right)\varphi dxdy+\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial(\widehat{u_1}-\varphi)}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_\Omega\left|\frac{\partial \widehat{u_1}}{\partial y}\right|dxdy$$
$$\geq -\int_\Omega \widehat{f}_1\varphi dxdy+\int_\Omega\frac{d\widehat{p}}{dx}\varphi dxdy \quad \forall\varphi\in W_0^{1,p}(\Omega).$$

Replacing in these two inequalities the test function φ by $\lambda\varphi$, $\lambda>0$, dividing the obtained inequalities by λ. The passage to the limit when λ tends to 0 implies, under the hypothesis $\frac{\partial \widehat{u_1}}{\partial y}\neq 0$, that

$$-\frac{\mu}{2^{\frac{p}{2}}}\int_\Omega \frac{\partial}{\partial y}\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial \widehat{u_1}}{\partial y}\right)\varphi dxdy+\frac{\sqrt{2}}{2}g\int_\Omega \operatorname{sign}\left(\frac{\partial \widehat{u_1}}{\partial y}\right)\frac{\partial\varphi}{\partial y}dxdy$$
$$\geq \int_\Omega \widehat{f}_1\varphi dxdy-\int_\Omega\frac{d\widehat{p}}{dx}\varphi dxdy \quad \forall\varphi\in W_0^{1,p}(\Omega).$$

and

$$\frac{\mu}{2^{\frac{p}{2}}}\int_\Omega \frac{\partial}{\partial y}\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial \widehat{u_1}}{\partial y}\right)\varphi dxdy-\frac{\sqrt{2}}{2}g\int_\Omega \operatorname{sign}\left(\frac{\partial \widehat{u_1}}{\partial y}\right)\frac{\partial\varphi}{\partial y}dxdy$$
$$\geq -\int_\Omega \widehat{f}_1\varphi dxdy+\int_\Omega\frac{d\widehat{p}}{dx}\varphi dxdy \quad \forall\varphi\in W_0^{1,p}(\Omega).$$

Consequently, we get combining these two inequalities and using a simple integration by parts

$$-\int_{\Omega}\frac{\partial}{\partial y}\left[\frac{\mu}{2^{\frac{p}{2}}}\left(\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right)+\frac{\sqrt{2}}{2}g\text{sign}\left(\frac{\partial\widehat{u_1}}{\partial y}\right)\right]\varphi dxdy$$
$$=\int_{\Omega}\left(\widehat{f}_1-\frac{d\widehat{p}}{dx}\varphi\right)dxdy\quad\forall\varphi\in W_0^{1,p}(\Omega).$$

Which eventually gives (3.21). □

From now on we will denote by $(\widehat{u},\widehat{p})\in W_p\times L_0^{p'}(\Omega)$ the solution of the limit problem (3.21).

The following proposition shows the uniqueness of the limit solution $(\widehat{u},\widehat{p})$.

Proposition 3.6. *The limit strong problem (*3.21*) has a unique, solution* $(\widehat{u},\widehat{p})$ *in* $W_p\times L_0^{p'}(\Omega)$ *with the condition (*3.20*).*

Proof. Suppose that the limit problem (3.21) has at least two solutions $(\widehat{u_1},\widehat{p_1})$, $(\widehat{u_2},\widehat{p_2})\in W_p\times L_0^{p'}(\Omega)$. In particular, $(\widehat{u_1},\widehat{p_1}),(\widehat{u_2},\widehat{p_2})$ are solutions of the weak formulation (3.23). Then

$$\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\frac{\partial(\widehat{v}-\widehat{u_1})}{\partial y}dxdy+\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{v}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{u_1}}{\partial y}\right|dxdy$$
$$\geq\int_{\Omega}\widehat{f}_1(\widehat{v}-\widehat{u_1})dxdy-\int_{\Omega}\frac{d\widehat{p_1}}{dx}(\widehat{v}-\widehat{u_1})dxdy\quad\forall\widehat{v}\in W_p,\tag{3.25}$$

and

$$\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\left|\frac{\partial\widehat{u_2}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_2}}{\partial y}\frac{\partial(\widehat{v}-\widehat{u_2})}{\partial y}dxdy+\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{v}}{\partial y}\right|dxdy-\frac{\sqrt{2}}{2}g\int_{\Omega}\left|\frac{\partial\widehat{u_2}}{\partial y}\right|dxdy$$
$$\geq\int_{\Omega}\widehat{f}_1(\widehat{v}-\widehat{u_2})dxdy-\int_{\Omega}\frac{d\widehat{p_2}}{dx}(\widehat{v}-\widehat{u_2})dxdy\quad\forall\widehat{v}\in W_p.\tag{3.26}$$

Setting $\widehat{v}=\widehat{u_2}$, $\widehat{v}=\widehat{u_1}$ as test functions in (3.25) and (3.26), respectively. Subtracting the two obtained inequalities, we can infer

$$\frac{\mu}{2^{\frac{p}{2}}}\int_{\Omega}\left[\left|\frac{\partial\widehat{u_2}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_2}}{\partial y}-\left|\frac{\partial\widehat{u_1}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u_1}}{\partial y}\right]\frac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}dxdy$$
$$\leq\int_{\Omega}\frac{d(\widehat{p_1}-\widehat{p_2})}{dx}(\widehat{u_2}-\widehat{u_1})dxdy\quad\forall\widehat{v}\in W_p.\tag{3.27}$$

Observe that for every $x,y\in\mathbb{R}^n$,

$$\left(|x|^{p-2}x-|y|^{p-2}y\right)\cdot(x-y)\geq(p-1)\frac{|x-y|^2}{(|x|+|y|)^{2-p}},\quad 1<p\leq 2.$$

This leads, making use (3.27), to

$$\frac{\mu(p-1)}{2^{\frac{p}{2}}}\int_\Omega \frac{\left|\frac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}\right|^2}{\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|+\left|\frac{\partial \widehat{u_2}}{\partial y}\right|\right)^{2-p}}dxdy \leq \int_\Omega \frac{d(\widehat{p_1}-\widehat{p_2})}{dx}(\widehat{u_2}-\widehat{u_1})\,dxdy$$

$$=\int_0^1\left(\frac{d(\widehat{p_1}-\widehat{p_2})}{dx}\int_0^{h(x)}(\widehat{u_2}-\widehat{u_1})\,dy\right)dx.$$

The use of (3.20) gives

$$\int_\Omega \frac{\left|\frac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}\right|^2}{\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|+\left|\frac{\partial \widehat{u_2}}{\partial y}\right|\right)^{2-p}}dxdy = 0. \tag{3.28}$$

On the other hand, the application of Hölder's inequality leads to

$$\int_\Omega \left|\frac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}\right|^p dxdy$$

$$\leq c\left(\int_\Omega \frac{\left|\frac{\partial \widehat{u_2}}{\partial y}-\frac{\partial \widehat{u_1}}{\partial y}\right|^2}{\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|+\left|\frac{\partial \widehat{u_2}}{\partial y}\right|\right)^{2-p}}dxdy\right)^{\frac{p}{2}}\left(\int_\Omega\left(\left|\frac{\partial \widehat{u_1}}{\partial y}\right|+\left|\frac{\partial \widehat{u_2}}{\partial y}\right|\right)^p dxdy\right)^{\frac{2-p}{2}}.$$

Which gives, keeping in mind (3.28)

$$\frac{\partial(\widehat{u_2}-\widehat{u_1})}{\partial y}=0.$$

Since $\widehat{u_2}(x,h(x))=\widehat{u_1}(x,h(x))=0$, we deduce that $\widehat{u_2}=\widehat{u_1}$ a.e. in Ω.

Finally, to prove the uniqueness of the pressure, we use equation (3.21) with the two pressures $\widehat{p_1}$ and $\widehat{p_2}$. We find

$$\frac{d(\widehat{p_1}-\widehat{p_2})}{dx}=0.$$

Then, due to fact that $\widehat{p_1},\,\widehat{p_2}\in L_0^{p'}(\Omega)$, the result can be easily deduced. □

4. Mechanical interpretation

Suppose that $\frac{\partial \widehat{u}}{\partial y}\neq 0$ and let σ^ε be the stress tensor associated to $\mathbf{u}^\varepsilon$. Then using the constitutive law of Herschel-Bulkley fluid, we can infer

$$\int_{\Omega^\varepsilon}\left|\widetilde{\sigma^\varepsilon}\right|^{p'}dx_1dx_2 \leq \int_{\Omega^\varepsilon}\left(\mu\varepsilon^p\left|D(\mathbf{u}^\varepsilon)\right|^{p-2}D(\mathbf{u}^\varepsilon)+g\varepsilon\right)^{p'}dx_1dx_2.$$

We can then easily prove, by passage to the variables x and y, that

$$\frac{1}{\varepsilon}\left\|\widehat{\widetilde{\sigma^{\varepsilon}}}\right\|_{L^{p'}(\Omega)^4}\leq c.$$

Thus, we can extract a subsequence still denoted by σ^{ε} such that

$$\frac{1}{\varepsilon}\widehat{\widetilde{\sigma^{\varepsilon}}}\longrightarrow\widehat{\sigma}\text{ in }L^{p'}(\Omega)^4\text{ weakly.}$$

On the other hand, we know from the flow equation (2.1) that

$$\sum_{j=1}^{2}\frac{\partial\widetilde{\sigma_{ij}^{\varepsilon}}}{\partial x_j}=\frac{\partial p^{\varepsilon}}{\partial x_i}-\widetilde{f_i^{\varepsilon}},\quad i=1,2\text{ in }\Omega^{\varepsilon}.$$

By passage to the variables x and y, taking into account the fact that $\widehat{p}(x,y)=\widehat{p}(x)$, we obtain the following equations

$$\left\{\begin{array}{l}\dfrac{\partial\widehat{\widetilde{\sigma_{11}^{\varepsilon}}}}{\partial x}+\dfrac{1}{\varepsilon}\dfrac{\partial\widehat{\widetilde{\sigma_{12}^{\varepsilon}}}}{\partial y}=\dfrac{d\widehat{p^{\varepsilon}}}{dx}-\widehat{\widetilde{f_1^{\varepsilon}}}\\ \dfrac{\partial\widehat{\widetilde{\sigma_{21}^{\varepsilon}}}}{\partial x}+\dfrac{1}{\varepsilon}\dfrac{\partial\widehat{\widetilde{\sigma_{22}^{\varepsilon}}}}{\partial y}=-\widehat{\widetilde{f_2^{\varepsilon}}}\end{array}\right.\quad\text{in }\Omega.\tag{4.1}$$

The passage to the limit leads

$$\frac{\partial\widehat{\sigma_{21}}}{\partial y}=\frac{d\widehat{p}}{dx}-\widehat{f_1}.\tag{4.2}$$

By comparison with equation (3.21), we find

$$\widehat{\sigma_{21}}=\frac{\mu}{2^{\frac{p}{2}}}\left|\frac{\partial\widehat{u}}{\partial y}\right|^{p-2}\frac{\partial\widehat{u}}{\partial y}+\frac{\sqrt{2}}{2}g\mathrm{sign}\left(\frac{\partial\widehat{u}}{\partial y}\right).$$

Which means that if $\dfrac{\partial\widehat{u}}{\partial y}\neq 0$ then $|\widehat{\sigma_{21}}|>\frac{\sqrt{2}}{2}g$. Hence, if $|\widehat{\sigma_{21}}|\leq\frac{\sqrt{2}}{2}g$ we get

$$\frac{\partial\widehat{u}}{\partial y}=0.$$

This permits us to deduce that at the limit the flow can be described by the following one dimensional constitutive law

$$\left\{\begin{array}{ll}\widehat{\tau}=\dfrac{\mu}{2^{\frac{p}{2}}}\left|\dfrac{\partial\widehat{u}}{\partial y}\right|^{p-2}\dfrac{\partial\widehat{u}}{\partial y}+\dfrac{\sqrt{2}}{2}g\mathrm{sign}\left(\dfrac{\partial\widehat{u}}{\partial y}\right) & \text{if }\dfrac{\partial\widehat{u}}{\partial y}\neq 0\\ |\widehat{\tau}|\leq\frac{\sqrt{2}}{2}g & \text{if }\dfrac{\partial\widehat{u}}{\partial y}=0\end{array}\right.\quad\text{in }\Omega,\tag{4.3}$$

where $\widehat{\tau}$ is the stress of the limit model. Such constitutive law has been studied by many engineers for the particular case of Bingham fluid i.e. $p=2$, see for example [9]. Indeed, the case $p=2$ corresponds to the Bingham flow. For $\mu=2\mu^*$, $g=\sqrt{2}g^*$ and $p=2$ the result in [4] are recovered.

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