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A unified theory of weakly contra-(μ, λ)-continuous functions in generalized topological spaces

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Abstract. We introduce a new notion called weakly contra-(μ, λ)-continuous functions as functions on generalized topological spaces [17]. We obtain some characterizations and several properties of such functions. The functions enable us to formulate a unified theory of several modifications of weak contra-continuity due to Baker [9].

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1. Introduction

In [15]-[25], Á. Császár founded the theory of generalized topological spaces, and studied the elementary character of these classes. Especially he introduced the notion of continuous functions on generalized topological spaces and investigated characterizations of generalized continuous functions (= (μ, λ)-continuous functions in [17]). We recall some notions defined in [17]. Let X be a non-empty set and $\exp X$ the power set of X. We call a class $\mu \subseteq \exp X$ a generalized topology [17] (briefly, GT) if $\phi \in \mu$ and the arbitrary union of elements of μ belongs to μ. A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ).

For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ- closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [17], [22]). Obviously in a topological space (X, τ), if one takes τ as the GT, then c_μ becomes the usual closure operator.

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : \exp X \to \exp X$ is said to be idempotent if $A \subseteq B \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if $\gamma(A) \subseteq \gamma(B)$. It is also well known from [20] and [23] that let μ be a

GT on $X, A \subset X$ and $x \in X$, then (1) $x \in c_\mu(A)$ if and only if $M \cap A \neq \phi$ for every $M \in \mu$ containing x and (2) $c_\mu(X - A) = X - i_\mu(A)$.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

Definition 2.1. *Let (X, τ) be a topological space. A subset A of X is said to be* semi-open [40] *(resp.* preopen [42], α-open [47], β-open [1] *or* semi-preopen [4], b-open [5]*) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$).*

We note that for any topological space (X, τ), the collection of all open (resp. semi-open, preopen, α-open, β-open, b-open) sets in X is denoted by τ (resp. $\mathrm{SO}(X)$ $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ or $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$). These collection form a GT.

Definition 2.2. *The complement of a semi-open (resp. preopen, α-open, β-open, b-open) set is said to be* semi-closed [14] *(resp.* preclosed [42], α-closed [43], β-closed [1] *or* semi-preclosed [4], b-closed [5]*).*

Definition 2.3. *The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, b-closed) sets of X containing A is called the* semi-closure [14] *(resp.* preclosure [32], α-closure [43], β-closure [2] *or* semi-preclosure [4], b-closure [5]*) of A and is denoted by sCl(A) (resp. pCl(A), αCl(A), ${}_\beta$Cl(A) or spCl(A), bCl(A)).*

Definition 2.4. The union of all semi-open (resp. preopen, α-open, β-open, b-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior* or *semi-preinterior*, *b-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$, $\mathrm{bInt}(A)$).

Throughout the present paper, (X, τ) and (Y, σ) denote topological spaces and $f : (X, \tau) \to (Y, \sigma)$ presents a (single valued) function from a topological space (X, τ) into a topological space (Y, σ).

Definition 2.5. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *semi-continuous* [40] (resp. *precontinuous* [42], *α-continuous* [43], *β-continuous* [1], *b-continuous* [31]) if for each point $x \in X$ and each open set V of Y containing $f(x)$, there exists a semi-open (resp. preopen, α-open, β-open , b-open) set U of X containing x such that $f(U) \subset V$.

Definition 2.6. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly continuous* [39] (resp. *weakly quasicontinuous* [59] or *weakly semi-continuous* [7], [13], [38], *almost weakly continuous* [37] or *quasi precontinuous* [55], *weakly α-continuous* [48], *weakly β-continuous* [56], weakly b-continuous [60]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open (resp. semi-open, preopen, α-open, β-open, b-open) set U of X containing x such that $f(U) \subset \mathrm{Cl}(V)$.

A unified theory of weakly continuous functions is investigated in [58].

Definition 2.7. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *slightly continuous* [34] (resp. *slightly semi-continuous* [53], *slightly precontinuous* or *faintly semi-continuous* [54], *slightly β-continuous* [49], *slightly b-continuous* [31]) if for each point $x \in X$ and each clopen set V of Y containing $f(x)$, there exists an open (resp. semi-open, preopen, β-open , b-open) set U of X containing x such that $f(U) \subset V$.

A unified theory of slightly continuous functions is investigated in [57].

Definition 2.8. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly contra-continuous* [9] (resp. weakly contra-precontinuous [11], weakly contra-β-continuous [10]) if for each open set V of Y and each closed set A of Y such that $A \subset V$, $\mathrm{Cl}(f^{-1}(A)) \subset f^{-1}(V)$ (resp. $\mathrm{pCl}(f^{-1}(A)) \subset f^{-1}(V)$, $\mathrm{spCl}(f^{-1}(A)) \subset f^{-1}(V)$).

Definition 2.9. [51] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *weakly contra-semi-continuous* (resp. *weakly contra-α-continuous*, *weakly contra-γ-continuous* or weakly contra-b-continuous) if for each open set V of Y and each closed set A of Y such that $A \subset V$, $\mathrm{sCl}(f^{-1}(A)) \subset f^{-1}(V)$ (resp. $\alpha\mathrm{Cl}(f^{-1}(A)) \subset f^{-1}(V)$, $\mathrm{bCl}(f^{-1}(A)) \subset f^{-1}(V)$).

3. Weakly contra-(μ, λ)-continuous functions

Definition 3.1. *Let $f : (X, \mu) \to (Y, \lambda)$ be a function on generalized topological spaces. Then the function f is said to be*

1. *(μ, λ)-continuous* [17] *if $G \in \lambda$ implies that $f^{-1}(G) \in \mu$.*
2. *weakly (μ, λ)-continuous* [44] *if for each $x \in X$ and each λ-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq c_\lambda(V)$.*
3. *almost (μ, λ)-continuous* [45] *if for each $x \in X$ and each λ-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq i_\lambda(c_\lambda(V))$.*
4. *almost (μ, λ)-open if $f(U) \subseteq i_\lambda(c_\lambda(f(U)))$ for every $U \in \mu$.*
5. *contra-(μ, λ)-continuous* [3] *if $f^{-1}(V)$ is μ-closed in X for each λ-open set of Y.*

Definition 3.2. *Let (X, μ) and (Y, λ) be GTS's. Then a function $f : X \to Y$ is said to be* weakly contra-(μ, λ)-continuous *if for each λ-open set V of Y and each λ-closed set A of Y such that $A \subset V$, $c_\mu(f^{-1}(A)) \subset f^{-1}(V)$.*

Remark 3.3. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, $\mu = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$) and $\lambda = \sigma$. If $f : (X, \mu) \to (Y, \lambda)$ is weakly contra-(μ, λ)-continuous, then f is weakly contra-continuous [51] (resp. weakly contra-semi-continuous, weakly contra-precontinuous, weakly contra-α-continuous, weakly contra-β-continuous, weakly contra-b-continuous).

Theorem 3.4. *If a function $f : (X, \mu) \to (Y, \lambda)$ is contra-(μ, λ)-continuous, then f is weakly contra-(μ, λ)-continuous.*

Proof. Let V be any λ-open set of Y and A a λ-closed set of Y such that $A \subset V$. Since f is contra-(μ, λ)-continuous, $f^{-1}(V) = c_\mu(f^{-1}(V))$. Therefore, we have $c_\mu(f^{-1}(A)) \subset c_\mu(f^{-1}(V)) = f^{-1}(V)$. This shows that f is weakly contra-(μ, λ)-continuous. $\square$

Remark 3.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ a GT on X. If $\mu = \tau$ (resp. $\mathrm{PO}(X)$, $\beta(X)$), then by Theorem 3.4 we obtain the results established in Theorem 3.1 of [9] (resp. Theorem 3.3 of [11], Theorem 3.3 of [10]).

Theorem 3.6. *If a function $f : (X, \mu) \to (Y, \lambda)$ is (μ, λ)-continuous, then f is weakly contra-(μ, λ)-continuous.*

Proof. Let V be any λ-open set of Y and A a λ-closed set of Y such that $A \subset V$. Since f is (μ, λ)-continuous, then we have $c_\mu(f^{-1}(A)) = f^{-1}(A) \subset f^{-1}(V)$. Therefore f is weakly contra-(μ, λ)-continuous. □

Remark 3.7. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ a GT on X. If $\mu = \tau$ (resp. $\mathrm{PO}(X)$, $\beta(X)$), then by Theorem 3.6 we obtain the results established in Theorem 3.4 of [9] (resp. Theorem 3.2 of [11], Theorem 3.2 of [10]).

Definition 3.8. *A function $f : (X, \mu) \to (Y, \lambda)$ is said to be* slightly (μ, λ)-continuous *if for each point $x \in X$ and each λ-clopen set V of Y containing $f(x)$, there exists $U \in \mu$ containing x such that $f(U) \subset V$.*

Theorem 3.9. *Let (X, μ) and (Y, λ) be GTSs. For a function $f : (X, \mu) \to (Y, \lambda)$, the following statements are equivalent:*

1. *f is slightly (μ, λ)-continuous;*
2. *for every λ-clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ-open;*
3. *for every λ-clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ-closed;*
4. *for every λ-clopen set $V \subseteq Y$, $f^{-1}(V)$ is μ-clopen.*

Proof. (1) ⇒ (2): Let V be a λ-clopen subset of Y and let $x \in f^{-1}(V)$. Since f is slightly (μ, λ)-continuous, there exists a μ-open set U_x in X containing x such that $f(U_x) \subseteq V$; hence $U_x \subseteq f^{-1}(V)$. We obtain that $f^{-1}(V) = \cup\{U_x | x \in f^{-1}(V)\}$. Thus $f^{-1}(V)$ is μ-open.
(2) ⇒ (3): Let V be a λ-clopen subset of Y. Then $Y \backslash V$ is λ-clopen. By (2) $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is μ-open. Thus $f^{-1}(V)$ is μ-closed.
(3) ⇒ (4): It can be shown easily.
(4) ⇒ (1): Let V be a λ-clopen subset in Y containing $f(x)$. By (4), $f^{-1}(V)$ is μ-clopen. Take $U = f^{-1}(V)$. Then, $f(U) \subseteq V$. Hence, f is slightly (μ, λ)-continuous. □

Theorem 3.10. *If a function $f : (X, \mu) \to (Y, \lambda)$ is weakly contra-(μ, λ)-continuous, then f is slightly (μ, λ)-continuous.*

Proof. Let V be a λ-clopen set of Y. If we put $A = V$, then by the weak contra-(μ, λ)-continuity we have $c_\mu(f^{-1}(V)) \subset f^{-1}(V)$ and hence $c_\mu(f^{-1}(V)) = f^{-1}(V)$. It follows from Theorem 3.9 that f is slightly (μ, λ)-continuous. □

Remark 3.11. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ be a GT on X. If $\mu = \tau$ (resp. $\mathrm{PO}(X)$, $\beta(X)$), then by Theorem 3.10 we obtain the results established in Theorem 3.7 of [9] (resp. Theorem 3.6 of [11], Theorem 3.6 of [10]).

Lemma 3.12. *If a function $f : (X, \mu) \to (Y, \lambda)$ is weakly (μ, λ)-continuous, then f is slightly (μ, λ)-continuous.*

Proof. This follows easily from Theorem 3.9. $\square$

The following implications are hold:

DIAGRAM

$$\begin{tikzcd}
 & \text{contra-}(\mu,\lambda)\text{-continuity} \arrow[r] \arrow[d] & \text{weak } (\mu,\lambda)\text{-continuity} \arrow[d] \\
(\mu,\lambda)\text{-continuity} \arrow[r] & \text{weak contra-}(\mu,\lambda)\text{-continuity} \arrow[r] & \text{slightly } (\mu,\lambda)\text{-continuity}
\end{tikzcd}$$

Remark 3.13. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ be a GT on X. If $\mu = \tau$ (resp. $\mathrm{PO}(X)$, $\beta(X)$), then by DIAGRAM we obtain the diagram constructed in [9] (resp. [11], [10]).

Definition 3.14. *A generalized topological space (Y, λ) is said to be* 0-λ-dimensional *if each point of Y has a neighborhood base consisting of λ-clopen sets.*

Theorem 3.15. *Let (Y, λ) be 0-λ-dimensional. Then for a function $f : (X, \mu) \to (Y, \lambda)$, the following properties are equivalent:*

1. *f is (μ, λ)-continuous;*
2. *f is weakly contra-(μ, λ)-continuous;*
3. *f is slightly (μ, λ)-continuous.*

Proof. The proofs of the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow from DIAGRAM.
(3) $\Rightarrow$ (1): Let $x \in X$ and let V be a λ-open subset of Y containing $f(x)$. Since Y is 0-λ-dimensional, there exists a λ-clopen set U containing $f(x)$ such that $U \subseteq V$. Since f is slightly (μ, λ)-continuous, then there exists a μ-open subset G in X containing x such that $f(G) \subseteq U \subseteq V$. Thus, f is (μ, λ)-continuous. $\square$

Definition 3.16. A topological space (Y, σ) is said to be *extremally disconnected* [61] (briefly E.D.) if the closure of each open set of Y is open in Y.

Theorem 3.17. *If $f : (X, \mu) \to (Y, \sigma)$ is weakly contra-(μ, σ)-continuous and (Y, σ) is E.D., then f is weakly (μ, σ)-continuous.*

Proof. Let V be an open set of Y. Since (Y, σ) is E.D., $\mathrm{Cl}(V)$ is clopen. Since f is weakly contra-(μ, σ)-continuous, $c_\mu(f^{-1}(V)) \subset c_\mu(f^{-1}(\mathrm{Cl}(V))) \subset f^{-1}(\mathrm{Cl}(V))$. Hence $c_\mu(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y. We claim that f is weakly (μ, σ)-continuous if $c_\mu(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y. Now let $x \in X$ and V be any open set containing $f(x)$. Since $V \cap (Y - Cl(V)) = \phi$, clearly $f(x) \notin Cl(Y - Cl(V))$ and hence $x \notin f^{-1}(Cl(Y - Cl(V)))$. Since $Y - Cl(V)$ is open, we have $x \notin c_\mu(f^{-1}(Y - Cl(V))$. Therefore, there exists a μ-open set U containing x such that $U \cap f^{-1}(Y - Cl(V)) = \phi$; hence $f(U) \cap (Y - Cl(V)) = \phi$. This shows that $f(U) \subseteq Cl(V)$. Therefore, f is weakly (μ, σ)-continuous. $\square$

Remark 3.18. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ a GT on X. If $\mu = \tau$ (resp. $\mathrm{PO}(X)$, $\beta(X)$), then by Theorem 3.17 we obtain the results established in Corollary 3.9 of [9] (resp. Corollary 3.10 of [11], Corollary 3.9 of [10]).

4. Weak contra-(μ, λ)-continuity and $(g\mu, \lambda)$-continuity

Definition 4.1. Let (X, τ) be a topological space. A subset A of X is said to be

1. *g-closed* [41] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
2. *αg-closed* [27] if $\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
3. *gs-closed* [26] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
4. *gp-closed* [6] if $\mathrm{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
5. *gsp-closed* [28] if $\mathrm{spCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,
6. *γg-closed* [33] if $\mathrm{bCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

Definition 4.2. *Let μ be a GT on a topological space (X, τ). Then a subset A of X is said to be generalized μ-closed (briefly $g\mu$-closed)* [52] *if $c_\mu(A) \subset U$ whenever $A \subset U$ and $U \in \tau$. The complement of a $g\mu$-closed set is called a generalized μ-open (or simply $g\mu$-open) set.*

Remark 4.3. [52] Let (X, τ) be a topological space and μ be a GT on X. Then every $g\mu$-closed set reduces to a g-closed (resp. gs-closed, gp-closed, αg-closed, gsp-closed, γg-closed) set if one takes μ to be τ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$).

Definition 4.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be g-continuous* [12] *or weakly g-continuous* [46] *(resp. gs-continuous* [26]*, gp-continuous* [6]*, αg-continuous* [27]*, gsp-continuous* [28]*, γg-continuous* [33]*) if $f^{-1}(K)$ is g-closed (resp. gs-closed, gp-closed, αg-closed, gsp-closed, γg-closed) in X for every closed set K of Y.*

Definition 4.5. *Let (X, τ) be a topological space and μ be a GT on X. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be $(g\mu, \lambda)$-continuous if $f^{-1}(K)$ is $g\mu$-closed for every λ-closed set K of Y.*

Remark 4.6. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, μ, λ be GTS's on X, Y and $\sigma = \lambda$. If $\mu = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$) and $f : (X, \mu) \to (Y, \lambda)$ is $(g\mu, \lambda)$-continuous, then f is g-continuous (resp. gs-continuous, gp-continuous, αg-continuous, gsp-continuous, γg-continuous).

Definition 4.7. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *approximately continuous* [8] (resp. *approximately semi-continuous*, *approximately precontinuous* [11], *approximately α-continuous*, *approximately β-continuous* [10], *approximately b-continuous*) if $\mathrm{Cl}(A) \subset f^{-1}(V)$ (resp. $\mathrm{sCl}(A) \subset f^{-1}(V)$, $\mathrm{pCl}(A) \subset f^{-1}(V)$, $\alpha\mathrm{Cl}(A) \subset f^{-1}(V)$, $\mathrm{spCl}(A) \subset f^{-1}(V)$, $\mathrm{bCl}(A) \subset f^{-1}(V)$) whenever V is open in Y and A is g-closed (resp. gs-closed, gp-closed, αg-closed, gsp-closed, γg-closed) in X such that $A \subset f^{-1}(V)$.

Definition 4.8. *Let (X, τ) be a topological space and μ be a GT on X. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be approximately (μ, λ)-continuous if $c_\mu(A) \subset f^{-1}(V)$ whenever V is λ-open in Y and A is $g\mu$-closed in X such that $A \subset f^{-1}(V)$.*

Remark 4.9. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, μ, λ be GTS's on X, Y and $\sigma = \lambda$. If $\mu = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$) and $f : (X, \mu) \to (Y, \lambda)$ is approximately (μ, λ)-continuous, then f is approximately continuous (resp. approximately semi-continuous, approximately precontinuous, approximately α-continuous, approximately β-continuous, approximately b-continuous).

Theorem 4.10. *If $f : (X, \mu) \to (Y, \lambda)$ is a $(g\mu, \lambda)$-continuous and approximately (μ, λ)-continuous function, then f is weakly contra-(μ, λ)–continuous.*

Proof. Let V be a λ-open set of Y and A a λ-closed set of Y such that $A \subset V$. Since f is $(g\mu, \lambda)$-continuous, $f^{-1}(A)$ is $g\mu$-closed. Since $f^{-1}(A) \subset f^{-1}(V)$ and f is approximately (μ, λ)-continuous, $c_\mu(f^{-1}(A)) \subset f^{-1}(V)$. This shows that f is weakly contra-(μ, λ)-continuous. □

Remark 4.11. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, μ, λ be GTS's on X, Y and $\sigma = \lambda$. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by Theorem 4.10 we obtain the results established in Theorem 3.11 of [9] (resp. Theorem 3.11 of [11], Theorem 3.10 of [10]).

Theorem 4.12. *If $f : (X, \mu) \to (Y, \lambda)$ is weakly contra-(μ, λ)-continuous and $f(A)$ is λ-closed in Y for every $g\mu$-closed set A of X, then f is approximately (μ, λ)-continuous.*

Proof. Let V be any λ-open set of Y and A any $g\mu$-closed set of X such that $A \subset f^{-1}(V)$. Then $f(A)$ is λ-closed and $f(A) \subset V$. Since f is weakly contra-(μ, λ)-continuous, $c_\mu(f^{-1}(f(A))) \subset f^{-1}(V)$ and hence $c_\mu(A) \subset c_\mu(f^{-1}(f(A))) \subset f^{-1}(V)$. This shows that f is approximately (μ, λ)-continuous. □

Remark 4.13. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ, λ be GT's on X, Y. If $\mu = \tau$ (resp. PO(X), $\beta(X)$) and $\lambda = \sigma$, then by Theorem 4.12 we obtain the results established in Theorem 3.12 of [9] (resp. Theorem 3.12 of [11], Theorem 3.11 of [10]).

Definition 4.14. *A topological space (X, τ) is said to be* strongly S-closed [29] *(resp.* strongly S-semi-closed, strongly S-preclosed [11]*,* strongly S-α-closed*,* strongly S_β-closed [10]*,* strongly S-b-closed*) if every cover of X by closed (resp. semi-closed, preclosed, α-closed, β-closed, b-closed) sets of (X, τ) has a finite subcover.*

Definition 4.15. *A GTS (X, μ) is said to be* strongly S-μ-closed *if every cover of X by μ-closed sets of (X, μ) has a finite subcover.*

Remark 4.16. Let (X, τ) be a topological space and $\mu = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, SPO(X), BO(X)). If (X, μ) is strongly S-μ-closed, then (X, τ) is strongly S-closed (resp. strongly S-semi-closed, strongly S-preclosed, strongly S-α-closed, strongly S_β-closed, strongly S-b-closed).

Definition 4.17. A topological space (X, τ) is called a P_Σ*-space* [9] or *C-space* [10], [11] if for every open set U and each $x \in U$, there exists a closed set A such that $x \in A \subset U$.

Theorem 4.18. *Let $f : (X, \mu) \to (Y, \sigma)$ be a weakly contra-(μ, σ)-continuous function, μ a GT on X and (Y, σ) a C-space. If (X, μ) is strongly S-μ-closed, then $f(X)$ is compact.*

Proof. Let (X, μ) be strongly S-μ-closed and $\{V_\alpha : \alpha \in \Delta\}$ any cover of $f(X)$ by open sets of (Y, σ). For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since Y is a C-space, there exists a closed set $A_{\alpha(x)}$ such that $f(x) \in A_{\alpha(x)} \subset V_{\alpha(x)}$. Since f is weakly contra-(μ, σ)-continuous, $c_\mu(f^{-1}(A_{\alpha(x)})) \subset f^{-1}(V_{\alpha(x)})$. The family $\{c_\mu(f^{-1}(A_{\alpha(x)})) : x \in X\}$ is a μ-closed cover of X. Since X is strongly S-μ-closed, there exist a finite number of points, say, $x_1, x_2, ..., x_n$ in X such that $X = \bigcup\{c_\mu(f^{-1}(A_{\alpha(x_k)})) : x_k \in X, 1 \leq k \leq n\}$. Therefore, we obtain

$$f(X) = \bigcup\{f(c_\mu(f^{-1}(A_{\alpha(x_k)}))) : x_k \in X, 1 \leq k \leq n\} \subset \bigcup\{V_{\alpha(x_k)} : x_k \in X, 1 \leq k \leq n\}.$$

This shows that $f(X)$ is compact. □

Remark 4.19. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and μ be a GT on X. If $\mu = \tau$ (resp. PO(X), $\beta(X)$), then by Theorem 4.18 we obtain the results established in Theorem 4.1 of [9] (resp. Theorem 4.1 of [11], Theorem 4.11 of [10]).

Lemma 4.20. [3] *For a function $f : (X, \mu) \to (Y, \lambda)$, the following properties are equivalent:*

1. *f is contra-(μ, λ)-continuous;*
2. *for every λ-closed subset F of Y, $f^{-1}(F)$ is μ-open in X.*

We will denote by $\mathcal{M}_\mu$ the union of all μ-open sets in a GTS (X, μ).

Definition 4.21. *A generalized topological space (X, μ) is said to be μ-compact if every μ-open cover of $\mathcal{M}_\mu$ has a finite subcover.*

Theorem 4.22. *Let $f : (X, \mu) \to (Y, \sigma)$ be a contra-(μ, σ)-continuous surjection and μ a GT on X. If (X, μ) is μ-compact, then (Y, σ) is strongly S-closed.*

Proof. Let (X, μ) be μ-compact and $\{V_\alpha : \alpha \in \Delta\}$ any cover of Y by closed sets of (Y, σ). Since f is contra-(μ, σ)-continuous, by Lemma 4.20 the family $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a μ-open cover of $\mathcal{M}_\mu$. Since (X, μ) is μ-compact, there exists a finite subset Δ_0 of Δ such that $\mathcal{M}_\mu = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Therefore, $Y = f(\mathcal{M}_\mu) = \cup\{V_\alpha : \alpha \in \Delta_0\}$. This shows that (Y, σ) is strongly S-clsoed. $\square$

Remark 4.23. Let (X, τ) be a topological space. If $\mu = \tau$ (resp. SO(X), PO(X), α(X)), then by Theorem 4.22 we obtain the results established in Theorem 4.2 of [30] (resp. Theorem 4.2 of [30], Corollary 5.1 of [36], Corollary 5.1 of [35]).

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