Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 1, 65–73

Multivariate Voronovskaya asymptotic expansions for general singular operators

George A. Anastassiou

Abstract. In this article we continue with the study of approximation properties of smooth general singular integral operators over R^N, $N \geq 1$. We produce multivariate Voronovskaya asymptotic type results and give quantitative results regarding the rate of convergence of multivariate singular integral operators to unit operator. We list specific multivariate singular integral operators that fulfill our theory.

Mathematics Subject Classification (2010): 41A35, 41A60, 41A80, 41A25.

Keywords: Multivariate general singular operator, multivariate Picard, Gauss-Weierstrass, Poisson Cauchy and trigonometric singular integrals, multivariate Voronovskaya type asymptotic expansion, rate of convergence.

1. Introduction

The main motivation for this work comes from [2], [3], [4]. We present here multivariate Voronovskaya type asymptotic expansions regarding the multivariate singular integral operators, see Theorem 2.2 and Corolaries 2.3, 2.4. In Theorem 2.6 we give the simultaneous corresponding Voronovskaya asymptotic expansion for our operators. Our expansions give also the rate of convergence of multivariate general singular integral operators to unit operator. In section 3 we list the multivariate singular Picard, Gauss Weierstrass, Poisson-Cauchy and Trigonometric operators that fulfill our results.

2. Main results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0, \end{cases} \tag{2.1}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2.2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{2.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{2.4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]}(f; x_1, ..., x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j)\, d\mu_{\xi_n}(s), \tag{2.5}$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence of positive real numbers.

The above $\theta_{r,n}^{[m]}$ are not in general positive operators and they preserve constants, see [1].

We make

Remark 2.1. Here $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$. Let $l = 0, 1, ..., m$. The lth order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$ and $|\alpha| := \sum_{i=1}^{N} \alpha_i = l$.

Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$; $x_0, z \in \mathbb{R}^N$.

Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right](x_{01} + t(z_1 - x_{01}), ..., x_{0N} + t(z_N - x_{0N})), \tag{2.6}$$

for all $j = 0, 1, ..., m$.

In particular we choose $z = (z_1, ..., z_N) = (x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) = x + sj$, and $x_0 = (x_{01}, ..., x_{0N}) = (x_1, x_2, ..., x_N) = x$, to get $g_{x+sj}(t) := f(x + t(sj))$.

Notice $g_{x+sj}(0) = f(x)$.

Also for $\widetilde{j} = 0, 1, ..., m-1$ we have

$$g_{x+sj}^{(\widetilde{j})}(0) = \sum_{|\alpha|=\widetilde{j}} \left(\frac{\widetilde{j}!}{\prod_{i=1}^{N} \alpha_i!}\right) \left(\prod_{i=1}^{N} (s_i j)^{\alpha_i}\right) f_\alpha(x). \tag{2.7}$$

Furthermore we get

$$\frac{g_{x+sj}^{(m)}(\theta)}{m!}=\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(s_ij)^{\alpha_i}\right)f_\alpha\left(x+\theta\left(sj\right)\right), \tag{2.8}$$

$0\leq\theta\leq1$.

For $\widetilde{j}=1,...,m-1$, and $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\alpha|:=\sum_{i=1}^{N}\alpha_i=\widetilde{j}$, we define

$$c_{\alpha,n,\widetilde{j}}:=c_{\alpha,n}:=\int_{\mathbb{R}^N}\prod_{i=1}^{N}s_i^{\alpha_i}d\mu_{\xi_n}\left(s_1,...,s_N\right). \tag{2.9}$$

Consequently we obtain

$$\sum_{\widetilde{j}=1}^{m}\frac{\int_{\mathbb{R}^N}g_{x+sj}^{(\widetilde{j})}(0)\,d\mu_{\xi_n}(s)}{\widetilde{j}!}$$

$$=\sum_{\widetilde{j}=1}^{m-1}j^{\widetilde{j}}\left(\sum_{|\alpha|=\widetilde{j}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)c_{\alpha,n}f_\alpha(x)\right). \tag{2.10}$$

Next we observe by multivariate Taylor's formula that

$$f(x+js)=g_{x+js}(1)=\sum_{\widetilde{j}=0}^{m-1}\frac{g_{x+js}^{(\widetilde{j})}(0)}{\widetilde{j}!}+\frac{g_{x+js}^{(m)}(\theta)}{m!}, \tag{2.11}$$

where $\theta\in(0,1)$. Which leads to

$$\int_{\mathbb{R}^N}\left(f\left(x+sj\right)-f\left(x\right)\right)d\mu_{\xi_n}(s) \tag{2.12}$$

$$=\sum_{\widetilde{j}=1}^{m-1}j^{\widetilde{j}}\left(\sum_{|\alpha|=\widetilde{j}}\frac{1}{\left(\prod_{i=1}^{N}\alpha_i!\right)}c_{\alpha,n,\widetilde{j}}f_\alpha(x)\right)$$

$$+j^m\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\prod_{i=1}^{N}s_i^{\alpha_i}f_\alpha\left(x+\theta sj\right)d\mu_{\xi_n}(s).$$

Hence

$$\theta_{r,n}^{[m]}(f;x)-f(x)=\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}(f(x+sj)-f(x))\,d\mu_{\xi_n}(s)$$

$$=\sum_{\widetilde{j}=1}^{m-1}\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^{\widetilde{j}}\left(\sum_{|\alpha|=\widetilde{j}}\frac{1}{\left(\prod_{i=1}^{N}\alpha_i!\right)}c_{\alpha,n,\widetilde{j}}f_\alpha(x)\right) \tag{2.13}$$

$$+\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^m\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\prod_{i=1}^{N}s_i^{\alpha_i}f_\alpha(x+\theta sj)\,d\mu_{\xi_n}(s)$$

$$=\sum_{\widetilde{j}=1}^{m-1}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\alpha|=\widetilde{j}}\frac{c_{\alpha,n,\widetilde{j}}f_\alpha(x)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right) \tag{2.14}$$

$$+\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f_\alpha(x+\theta sj)\right)d\mu_{\xi_n}(s).$$

Thus we have

$$\psi:=\theta_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m-1}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\alpha|=\widetilde{j}}\frac{c_{\alpha,n,\widetilde{j}}f_\alpha(x)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right) \tag{2.15}$$

$$=\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f_\alpha(x+\theta sj)\right)d\mu_{\xi_n}(s). \tag{2.16}$$

Call

$$\phi_\alpha(x,s):=\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f_\alpha(x+\theta sj). \tag{2.17}$$

Thus

$$\psi = \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right) \phi_\alpha (x, s) \, d\mu_{\xi_n} (s) . \tag{2.18}$$

Consider

$$\Delta_{\xi_n} := \frac{\psi}{\xi_n^m}. \tag{2.19}$$

Assume f_α is bounded for all $\alpha : |\alpha| = m$, by $M > 0$. I.e. $\|f_\alpha\|_\infty \leq M$. Therefore

$$|\phi_\alpha (x, s)| \leq \left(\sum_{j=1}^{r} \binom{r}{j} \right) M = (2^r - 1) M. \tag{2.20}$$

Consequently

$$|\Delta_{\xi_n}| \leq \frac{(2^r - 1) M}{\xi_n^m} \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n} (s) \right) . \tag{2.21}$$

Assume for $|\alpha| = m$ that

$$\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n} (s) \leq \rho, \text{ for any } (\xi_n)_{n\in\mathbb{N}} . \tag{2.22}$$

Therefore

$$|\Delta_{\xi_n}| \leq (2^r - 1) M\rho \left(\sum_{|\alpha|=m} \frac{1}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) =: \lambda, \quad \lambda > 0. \tag{2.23}$$

Hence

$$\frac{|\psi|}{\xi_n^m} \leq \lambda \quad \text{and} \quad \frac{|\psi| \, \xi_n^\gamma}{\xi_n^m} \leq \lambda \xi_n^\gamma \to 0, \tag{2.24}$$

where $0 < \gamma \leq 1$, as $\xi_n \to 0+$.

I.e.

$$\frac{|\psi|}{\xi_n^{m-\gamma}} \to 0, \quad \text{as} \;\; \xi_n \to 0+, \tag{2.25}$$

which means $\psi = 0 \left(\xi_n^{m-\gamma} \right)$.

We proved

Theorem 2.2. *Let* $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, *with all* $\|f_\alpha\|_\infty \le M$, $M > 0$, *all* $\alpha : |\alpha| = m$. *Let* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence,* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$. *Call* $c_{\alpha,n,\widetilde{j}} = \int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i}\right) d\mu_{\xi_n}(s)$, *all* $|\alpha| = \widetilde{j} = 1, ..., m-1$. *Assume* $\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \le \rho$, *all* $\alpha : |\alpha| = m$, $\rho > 0$, *for any such* $(\xi_n)_{n\in\mathbb{N}}$. *Also* $0 < \gamma \le 1$, $x \in \mathbb{R}^N$. *Then*

$$\theta_{r,n}^{[m]}(f;x) - f(x) = \sum_{\widetilde{j}=1}^{m-1} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i!\right)} \right) + 0\left(\xi_n^{m-\gamma}\right). \tag{2.26}$$

When $m = 1$ *the sum collapses.*

Above we assume $\theta_{r,n}^{[m]}(f;x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$.

Corollary 2.3. *Let* $f \in C^1(\mathbb{R}^N)$, $N \ge 1$, *with all* $\left\|\frac{\partial f}{\partial x_i}\right\|_\infty \le M$, $M > 0$, $i = 1, ..., N$. *Let* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence,* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$. *Assume*

$$\xi_n^{-1} \int_{\mathbb{R}^N} |s_i|\, d\mu_{\xi_n}(s) \le \rho, \quad \text{all } \ i = 1, ..., N, \tag{2.27}$$

$\rho > 0$, *for any such* $(\xi_n)_{n\in\mathbb{N}}$. *Also* $0 < \gamma \le 1$, $x \in \mathbb{R}^N$. *Then*

$$\theta_{r,n}^{[1]}(f;x) - f(x) = 0\left(\xi_n^{1-\gamma}\right). \tag{2.28}$$

Above we assume $\theta_{r,n}^{[1]}(f;x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$.

Corollary 2.4. *Let* $f \in C^2(\mathbb{R}^2)$, *with all* $\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_\infty, \left\|\frac{\partial^2 f}{\partial x_2^2}\right\|_\infty, \left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\|_\infty \le M$, $M > 0$. *Let* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence,* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^2$.

Call

$$c_1 = \int_{\mathbb{R}^2} s_1 d\mu_{\xi_n}(s), \quad c_2 = \int_{\mathbb{R}^2} s_2 d\mu_{\xi_n}(s). \tag{2.29}$$

Assume

$$\xi_n^{-2} \int_{\mathbb{R}^2} s_1^2 d\mu_{\xi_n}(s), \quad \xi_n^{-2} \int_{\mathbb{R}^2} s_2^2 d\mu_{\xi_n}(s), \quad \xi_n^{-2} \int_{\mathbb{R}^2} |s_1|\,|s_2|\, d\mu_{\xi_n}(s) \le \rho,$$

$\rho > 0$, *for any such* $(\xi_n)_{n\in\mathbb{N}}$. *Also* $0 < \gamma \le 1$, $x \in \mathbb{R}^2$. *Then*

$$\theta_{r,n}^{[2]}(f;x) - f(x) = \tag{2.30}$$

$$\left(\sum_{j=1}^r \alpha_{j,r}^{[2]} j\right) \left(c_1 \frac{\partial f}{\partial x_1}(x) + c_2 \frac{\partial f}{\partial x_2}(x)\right) + 0\left(\xi_n^{2-\gamma}\right).$$

We continue with

Theorem 2.5. *Let $f \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$, $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ a bounded sequence. Let $\beta := (\beta_1, ..., \beta_N)$, $\beta_i \in \mathbb{Z}^+$, $i = 1, ..., N$; $|\beta| := \sum_{i=1}^{N} \beta_i = l$. Here $f(x+sj)$, $x, s \in \mathbb{R}^N$, is μ_{ξ_n}-integrable wrt s, for $j = 1, ..., r$. There exist μ_{ξ_n}-integrable functions $h_{i_1,j}$, $h_{\beta_1,i_2,j}$, $h_{\beta_1,\beta_2,i_3,j}, ..., h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j} \geq 0$ $(j = 1, ..., r)$ on $\mathbb{R}^N$ such that*

$$\left|\frac{\partial^{i_1} f(x+sj)}{\partial x_1^{i_1}}\right| \leq h_{i_1,j}(s), \quad i_1 = 1, ..., \beta_1, \tag{2.31}$$

$$\left|\frac{\partial^{\beta_1+i_2} f(x+sj)}{\partial x_2^{i_2} \partial x_1^{\beta_1}}\right| \leq h_{\beta_1,i_2,j}(s), \quad i_2 = 1, ..., \beta_2,$$

$$\vdots$$

$$\left|\frac{\partial^{\beta_1+\beta_2+...+\beta_{N-1}+i_N} f(x+sj)}{\partial x_N^{i_N} \partial x_{N-1}^{\beta_{N-1}} ... \partial x_2^{\beta_2} \partial x_1^{\beta_1}}\right| \leq h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j}(s), \quad i_N = 1, ..., \beta_N,$$

$\forall\, x, s \in \mathbb{R}^N$.

Then, both of the next exist and

$$\left(\theta_{r,n}^{[m]}(f;x)\right)_\beta = \theta_{r,n}^{[m]}(f_\beta; x). \tag{2.32}$$

Proof. By H. Bauer [5], pp. 103-104. □

We finish with

Theorem 2.6. *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l, N \in \mathbb{N}$. Assumptions of Theorem 2.5 are valid. Call $\gamma = 0, \beta$. Assume $\|f_{\gamma+\alpha}\|_\infty \leq M$, $M > 0$, for all $\alpha : |\alpha| = m$. Let $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence, μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$. Call $c_{\alpha,n,\widetilde{j}} = \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) d\mu_{\xi_n}(s)$, all $|\alpha| = \widetilde{j} = 1, ..., m-1$. Assume $\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \leq \rho$, all $\alpha : |\alpha| = m$, $\rho > 0$, for any such $(\xi_n)_{n\in\mathbb{N}}$. Also $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. Then*

$$\left(\theta_{r,n}^{[m]}(f;x)\right)_\gamma - f_\gamma(x) = \sum_{\widetilde{j}=1}^{m-1} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}}\, f_{\gamma+\alpha}(x)}{\left(\prod_{i=1}^{N} \alpha_i!\right)}\right) + 0\left(\xi_n^{m-\gamma}\right). \tag{2.33}$$

When $m = 1$ the sum collapses.

3. Applications

Let all entities as in section 2. We define the following specific operators:

i) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}(f;x_1,...,x_N) := \frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{3.1}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1 j, x_2+s_2 j,...,x_N+s_N j)\, e^{-\frac{\left(\sum_{i=1}^{N}|s_i|\right)}{\xi_n}} ds_1...ds_N.$$

ii) The general multivariate Gauss-Weierstrass singular integral operators:

$$W_{r,n}^{[m]}(f;x_1,...,x_N) := \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{3.2}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1 j, x_2+s_2 j,...,x_N+s_N j)\, e^{-\frac{\left(\sum_{i=1}^{N}s_i^2\right)}{\xi_n}} ds_1...ds_N.$$

iii) The general multivariate Poisson-Cauchy singular integral operators:

$$U_{r,n}^{[m]}(f;x_1,...,x_N) := W_n^N \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{3.3}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1 j,...,x_N+s_N j) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^\beta} ds_1...ds_N,$$

with $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma(\beta)\,\alpha\xi_n^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}. \tag{3.4}$$

iv) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}(f;x_1,...,x_N) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{3.5}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1 j,...,x_N+s_N j) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1...ds_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta}\pi(-1)^\beta \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!\,(\beta+k)!}. \tag{3.6}$$

One can apply the results of this article to the operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and derive interesting results. We intend to do that in a future article.

References

[1] Anastassiou, G., *General uniform Approximation theory by multivariate singular integral operators*, submitted, 2010.

[2] Anastassiou, G., Mezei, R., *A Voronovskaya type theorem for Poisson-Cauchy type singular operators*, J. Math. Anal. Appl., **366**(2010), 525-529.

[3] Anastassiou, G., Mezei, R., *Voronovskaya type theorems for smooth Picard and Gauss-Weierstrass singular operators*, Studia Mathematica, Babes-Bolyai, **55**(2010), no. 4, 175-184.

[4] Anastassiou, G., Mezei, R., *A Voronovskaya type theorem for general singular operators with applications*, submitted, 2010.

[5] Bauer, H., *M$\alpha\beta$-und Integrations theorie*, de Gruyter, Berlin, 1990.

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
e-mail: ganastss@memphis.edu