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Inclusion results for four dimensional Cesàro submethods

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Abstract. We define submethods of four dimensional Cesàro matrix. Comparisons between these submethods are established.

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1. Introduction

Some equivalance results for Cesàro submethods have been studied by Goffman and Petersen [2], Armitage and Maddox [1] and Osikiewicz [5]. In this paper we consider the same concept for four dimensional Cesàro method $C_1 := (C, 1, 1)$. First we recall some definitions.

A double sequence $[x] = (x_{jk})$ is said to be $P-$*convergent* (i.e., it is convergent in Pringsheim sense) to L if for all $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that $|x_{nm} - L| < \varepsilon$ for all $n, m \geq n_0$ [7]. In this case we write $P - \lim_{j,k} x_{jk} = L$. Recall that $[x]$ is bounded if and only if

$$\|x\|_{(\infty,2)} := \sup_{j,k} |x_{jk}| < \infty.$$

By $l_{(\infty,2)}$ we denote the set of all bounded double sequences.
Note that a $P-$*convergent* double sequence need not be in $l_{(\infty,2)}$. Let

$$P - l_{(\infty,2)} := \left\{ [x] = (x_{jk}) : \sup_{n \geq h_1, m \geq h_2} |x_{jk}| < \infty, \ for\ some\ h_1, \ h_2 \in \mathbb{N} \right\}$$

and call it the space of all $P-$*bounded double sequences* where $\mathbb{N}$ denotes the set of all positive integers. If a double sequence is $P-$*convergent* then it is $P-$*bounded* and it is easy to see that $P - \lim [x] [y] = 0$ whenever $P - \lim [x] = 0$ and $[y]$ is $P-$*bounded*.

Let $A = (a_{jk}^{nm})$ be a four dimensional summability matrix and $[x] = (x_{jk})$ be a double sequence. If $[Ax] := \{(Ax)_{nm}\}$ is $P-convergent$ to L then we say $[x]$ is $A-summable$ to L where

$$(Ax)_{nm} := \sum_{j,k} a_{jk}^{nm} x_{jk},\ for\ all\ n, m \in \mathbb{N}.$$

A is said to be $RH-regular$ if it maps every bounded $P-convergent$ sequence into a $P-convergent$ sequence with the same $P-$limit [3]. Some recent developments concerning the summability by four dimensional matrices may be found in [6].

Recall that four dimensional Cesàro matrix $C_1 = (c_{jk}^{nm})$ is defined by

$$c_{jk}^{nm} = \begin{cases} \frac{1}{nm}, & j \le n\ and\ k \le m \\ 0, & otherwise. \end{cases}$$

The double index sequence $\beta = \beta(n,m)$ is defined as $\beta(n,m) = (\lambda(n), \mu(m))$ where $\lambda(n)$ and $\mu(m)$ are strictly increasing single sequences of positive integers. Let $[x] = (x_{jk})$ be a double sequence. We say $[y] = (y_{jk})$ is a subsequence of $[x]$ if $y_{jk} = x_{\beta(j,k)}$ for all $j, k \in \mathbb{N}$.

Let $\beta(n,m) = (\lambda(n), \mu(m))$ be a double index sequence and $[x] = (x_{jk})$ be a double sequence. Then the Cesàro submethod $C_\beta := (C_\beta, 1, 1)$ is defined to be

$$(C_\beta x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{(j,k)=(1,1)}^{(\lambda(n),\mu(m))} x_{jk}$$

where $\sum_{(j,k)=(1,1)}^{(\lambda(n),\mu(m))} x_{jk} = \sum_{j=1}^{\lambda(n)} \sum_{k=1}^{\mu(m)} x_{jk}$. Since $\{(C_\beta x)_{nm}\}$ is a subsequence of $\{(Cx)_{nm}\}$, the method C_β is $RH-regular$ for any β.

Let $x = (x_k)$ be a single sequence and $[x^c] = (x_{jk}^c)$, $[x^r] = (x_{jk}^r)$ be two double sequences such that

$$x_{jk}^c = x_j,\ \text{for all } k \in \mathbb{N}$$
$$x_{jk}^r = x_k,\ \text{for all } j \in \mathbb{N}.$$

It easy to see that the following statements are equivalent:

(a) $\lim x = L$; (b) $P-\lim [x^c] = L$; (c) $P-\lim [x^r] = L$.

The next result follows easily.

Proposition 1.1. *Let $[x] = (x_{jk})$ be a double sequence such that $x_{jk} = y_j z_k$ for all $j, k \in \mathbb{N}$ where $y = (y_j)$ and $z = (z_k)$ are single sequences (we call such a double sequence as a factorable double sequence). If y, z are convergent to L_1, L_2 respectively then $[x]$ is $P-$convergent to $L_1 L_2$.*

2. Inclusion results

Let A and B two four dimensional summability matrix methods. If every double sequence which is A summable is also B summable to the same limit, then we say B includes A and we write $A \subseteq B$.

In [1] Armitage and Maddox have given an inclusion theorem for submethods of ordinary Cesàro method. Now, we give an analog of that result for four dimensional Cesàro submethods.

Theorem 2.1. *Let* $\beta_1(n,m) = \left(\lambda^{(1)}(n), \mu^{(1)}(m)\right)$ *and* $\beta_2(n,m) = \left(\lambda^{(2)}(n), \mu^{(2)}(m)\right)$ *be two double index sequences.*

i) If $E(\lambda^{(2)})\backslash E\left(\lambda^{(1)}\right)$ *and* $E(\mu^{(2)})\backslash E(\mu^{(1)})$ *are finite sets then* $C_{\beta_1} \subseteq C_{\beta_2}$.

ii) If $C_{\beta_1} \subseteq C_{\beta_2}$ *then* $E(\lambda^{(2)})\backslash E\left(\lambda^{(1)}\right)$ *or* $E(\mu^{(2)})\backslash E(\mu^{(1)})$ *is finite set, where*

$$E\left(\lambda^{(i)}\right) := \left\{\lambda^{(i)}(n) : n \in \mathbb{N}\right\} \text{ and } E\left(\mu^{(i)}\right) := \left\{\mu^{(i)}(m) : m \in \mathbb{N}\right\};\ i=1,2.$$

Proof. i) If $E(\lambda^{(2)})\backslash E\left(\lambda^{(1)}\right)$ and $E(\mu^{(2)})\backslash E(\mu^{(1)})$ are finite then there exists n_0 such that $\left\{\lambda^{(2)}(n) : n \geq n_0\right\} \subset E\left(\lambda^{(1)}\right)$ and $\left\{\mu^{(2)}(m) : m \geq n_0\right\} \subset E(\mu^{(1)})$. Let $n(j)$ and $m(k)$ be two increasing index sequences such that for all $n, m \geq n_0$

$$\lambda^{(2)}(n) = \lambda^{(1)}(n(j)) \text{ and } \mu^{(2)}(m) = \mu^{(1)}(m(k)).$$

Then $P - \lim(C_{\beta_1}x)_{nm} = L$ implies $P - \lim(C_{\beta_1}x)_{n(j),m(k)} = L$. Hence this implies $P - \lim(C_{\beta_2}x)_{nm} = L$.

ii) Suppose that C_{β_1} implies C_{β_2} but that $E(\lambda^{(2)})\backslash E\left(\lambda^{(1)}\right)$ and $E(\mu^{(2)})\backslash E(\mu^{(1)})$ are infinite sets. Then there are strictly increasing sequences $\lambda^{(2)}(n(j))$ and $\mu^{(2)}(m(k))$ such that for all $j,\ k \in \mathbb{N}$ $\lambda^{(2)}(n(j)) \notin E\left(\lambda^{(1)}\right)$ and $\mu^{(2)}(m(k)) \notin E(\mu^{(1)})$. Define $[t] = (t_{nm})$ by

$$t_{nm}= \begin{cases} jk, & \text{if } n = \lambda^{(2)}(n(j)) \text{ and } m = \mu^{(2)}(m(k)) \\ 0, & \text{otherwise} \end{cases}.$$

Let $(Cs)_{nm} = t_{nm}$, i.e. $\frac{1}{nm}\sum_{(j,k)=(1,1)}^{(n,m)} s_{jk} = t_{nm}$. If $n \in E\left(\lambda^{(1)}\right)$ and $m \in E(\mu^{(1)})$ then $t_{nm} = 0$ which implies the sequence $[s]$ is $C_{\beta_1} -$ *summable* to zero. Now we define a double index sequence β_3 as

$$\beta_3 = (\lambda^{(2)}(n(j)), \mu^{(2)}(m(k))).$$

Since

$$\frac{1}{\lambda^{(2)}(n(j))\mu^{(2)}(m(k))} \sum_{(p,q)=(1,1)}^{\left((\lambda^{(2)}(n(j))\mu^{(2)}(m(k)))\right)} s_{pq} = C_{\lambda^{(2)}(n(j)),\mu^{(2)}(m(k))}$$

and $t_{nm} = jk$ for $n \in \left\{\lambda^{(2)}(n(j))\right\}$ and $m \in \left\{\mu^{(2)}(m(k))\right\}$ we have $[s] \notin C_{\beta_3}$ which implies $[s] \notin C_{\beta_2}$. □

Osikiewicz [5] has given a characterization for equivalence of Cesàro method and its submethods. The following theorem is an analog for four dimensional Cesàro method and its submethods.

Theorem 2.2. *Let $\beta = (\lambda(n), \mu(m))$ be a double index sequence.*

i) If

$$\lim_n \frac{\lambda(n+1)}{\lambda(n)} = \lim_m \frac{\mu(m+1)}{\mu(m)} = 1 \tag{2.1}$$

then C_1 and C_β are equivalent for bounded double sequences.

ii) If C_1 and C_β are equivalent for bounded double sequences then

$$\lim_n \frac{\lambda(n+1)}{\lambda(n)} = 1 \text{ or } \lim_m \frac{\mu(m+1)}{\mu(m)} = 1.$$

Proof. i) By Theorem 2.1 we have $C_1 \subseteq C_\beta$. Let $[x] = (x_{jk})$ be a bounded double sequence that is C_β summable to L and assume

$$\lim_n \frac{\lambda(n+1)}{\lambda(n)} = \lim_m \frac{\mu(m+1)}{\mu(m)} = 1.$$

Consider the sets $F_1 = \mathbb{N} \backslash E(\lambda) =: \{\alpha_1(n)\}$ and $F_2 = \mathbb{N} \backslash E(\mu) =: \{\alpha_2(m)\}$.
Case I. If the sets F_1 and F_2 are finite, then Theorem 2.1 implies that $C_\beta \subseteq C_1$.
Case II. Assume F_1 and F_2 are both infinite sets. Then there exists an n_0 such that for $n, m \geq n_0$, $\alpha_1(n) > \lambda(1)$ and $\alpha_2(m) > \mu(1)$. Since $E(\lambda) \cap F_1 = \varnothing$ and $E(\mu) \cap F_2 = \varnothing$, for all $n, m \geq n_0$, there exist $p, q \in \mathbb{N}$ such that $\lambda(p) < \alpha_1(n) < \lambda(p+1)$ and $\mu(q) < \alpha_2(m) < \mu(q+1)$. It can be written that $\alpha_1(n) = \lambda(p) + a$ and $\alpha_2(m) = \mu(q) + b$, where

$$0 < a < \lambda(p+1) - \lambda(p) \text{ and } 0 < b < \mu(q+1) - \mu(q). \tag{2.2}$$

Now define a double index sequence β' as

$$\beta'(n, m) = (\alpha_1(n), \alpha_2(m)).$$

Then for $n, m \geq n_0$,

$$\begin{aligned}
\left|(C_{\beta'}x)_{nm} - (C_\beta x)_{pq}\right| &= \left| \frac{1}{\alpha_1(n)\alpha_2(m)} \sum_{(j,k)=(1,1)}^{(\alpha_1(n),\alpha_2(m))} x_{jk} - \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} \right| \\
&= \left| \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \sum_{(j,k)=(1,1)}^{(\lambda(p)+a,\mu(q)+b)} x_{jk} - \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} \right| \\
&= \left| \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} - \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} \right. \\
&\quad + \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \left\{ \sum_{(j,k)=(1,\mu(q)+1)}^{(\lambda(p),\mu(q)+b)} x_{jk} + \sum_{(j,k)=(\lambda(p)+1,1)}^{(\lambda(p)+a,\mu(q))} x_{jk} \right. \\
&\quad \left. \left. + \sum_{(j,k)=(\lambda(p)+1,\mu(q)+1)}^{(\lambda(p)+a,\mu(q)+b)} x_{jk} \right\} \right| \\
&\leq \|x\|_{(\infty,2)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} \left| \frac{1}{(\lambda(p)+a)(\mu(q)+b)} - \frac{1}{\lambda(p)\mu(q)} \right|
\end{aligned}$$

$$+\|x\|_{(\infty,2)}\frac{b\lambda(p)+a\mu(q)+ab}{(\lambda(p)+a)(\mu(q)+b)}$$
$$\leq 2\|x\|_{(\infty,2)}\frac{b\lambda(p)+a\mu(q)+ab}{\lambda(p)\mu(q)}.$$

By 2.2 we have

$$\begin{aligned}\left|(C_{\beta'}x)_{nm}-(C_{\beta}x)_{pq}\right| &\leq 2\|x\|_{(\infty,2)}\frac{b\lambda(p)+a\mu(q)+ab}{\lambda(p)\mu(q)}\\ &\leq 2\|x\|_{(\infty,2)}\left(\frac{\lambda(p+1)\mu(q+1)}{\lambda(p)\mu(q)}-1\right).\end{aligned}\tag{2.3}$$

Since

$$\left|(C_{\beta'}x)_{nm}-L\right|\leq\left|(C_{\beta'}x)_{nm}-(C_{\beta}x)_{pq}\right|+\left|(C_{\beta}x)_{pq}-L\right|$$

it follows from 2.1, 2.3 and Proposition 1.1 that $P-\lim_{n,m}(C_{\beta'}x)_{nm}=L$.

As the double sequence $\{(C_1x)_{nm}\}$ may be partitioned into two subsequences $\left\{(C_{\beta'}x)_{nm}\right\}$ and $\left\{(C_{\beta}x)_{nm}\right\}$, each having the common P-limit L, $[x]$ must be C_1- *summable* to L. Hence $C_\beta\subseteq C_1$.

Case III. Assume F_1 is infinite set and F_2 is finite set and define a double index sequence β' as

$$\beta'(n,m)=(\alpha_1(n),\mu(m)).$$

Now using the same argument in Case II with taking $b=0$ we have

$$\left|(C_{\beta'}x)_{nm}-(C_{\beta}x)_{pq}\right|\leq 2\|x\|_{(\infty,2)}\left(\frac{\lambda(p+1)}{\lambda(p)}-1\right).\tag{2.4}$$

Since

$$\left|(C_{\beta'}x)_{nm}-L\right|\leq\left|(C_{\beta'}x)_{nm}-(C_{\beta}x)_{pq}\right|+\left|(C_{\beta}x)_{pq}-L\right|$$

it follows from 2.1, 2.4 and Proposition 1.1 that $P-\lim_{n,m}(C_{\beta'}x)_{nm}=L$.

As the double sequence $\{(C_1x)_{nm}\}$ may be partitioned into two subsequences $\left\{(C_{\beta'}x)_{nm}\right\}$ and $\left\{(C_{\beta}x)_{nm}\right\}$, each having the common P-limit L, $[x]$ must be C_1- *summable* to L. Hence $C_\beta\subseteq C_1$.

Case IV. If F_1 is finite set and F_2 is infinite set, then we can get the proof as in Case III by changing the roles of F_1 and F_2.

Hence for all cases we get $C_\beta\subseteq C_1$.

ii) Assume that $\limsup_n\frac{\lambda(n+1)}{\lambda(n)}>1$ and $\limsup_m\frac{\mu(m+1)}{\mu(m)}>1$. Then, we choose two strictly increasing sequences of positive integers $n(j)$ and $m(k)$ such that

$$\lim_j\frac{\lambda(n(j)+1)}{\lambda(n(j))}=L_1>1\text{ and }\lim_k\frac{\mu(m(k)+1)}{\mu(m(k))}=L_2>1\tag{2.5}$$

with $\lambda(n(j)+1)-\lambda(n(j))$ and $\mu(m(k)+1)-\mu(m(k))$ are odd. Let I_j and S_k be the intervals $[\lambda(n(j))+1,\lambda(n(j)+1)-1]$ and $[\mu(m(k))+1,\mu(m(k)+1)-1]$, respectively. $|I_j|$ and $|S_k|$ will always be even by the choice of $n(j)$ and $m(k)$, where $|E|$ is the number of the integers in E. If we define a double sequence $[x]$ by $x_{pq}=0$

if $p \in \left[\lambda(n(j))+1, \lambda(n(j))+\frac{|I_j|}{2}\right]$ or $q \in \left[\mu(m(k))+1, \mu(m(k))+\frac{|S_k|}{2}\right]$, $x_{pq}=1$ if $p \in \left(\lambda(n(j))+\frac{|I_j|}{2}, \lambda(n(j)+1)-1\right]$ and $q \in \left(\mu(m(k))+\frac{|S_k|}{2}, \mu(m(k)+1)-1\right]$, $x_{pq}=0$ if $p \notin |I_j|$ or $q \notin |S_k|$ and p or q is odd, $x_{pq}=1$ if $p \notin |I_j|$ or $q \notin |S_k|$ and p and q are even, for $j,k=1,2,\ldots$. Then for given j,k we have $\sum_{(p,q)\in I_j\times S_k} x_{pq} = \frac{|I_j|\,|S_k|}{4}$ and for given n,m we have

$$(C_\beta x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{(p,q)=(1,1)}^{(\lambda(n),\mu(m))} x_{pq} = \frac{1}{\lambda(n)\mu(m)} \left[\left|\frac{\lambda(n)}{2}\right|\right] \left[\left|\frac{\mu(m)}{2}\right|\right]$$

where $[|K|]$ denotes the greatest integer that is not greater than K. Hence, we have $P-\lim_{n,m}(C_\beta x)_{nm} = \frac{1}{4}$. Now define a double index sequence $\sigma(j,k)$ by

$$\sigma(j,k) = (a(j), b(k))$$

where $a(j) = \lambda(n(j)) + \frac{|I_j|}{2}$ and $b(k) = \mu(m(k)) + \frac{|S_k|}{2}$. For all j we get

$$\begin{aligned}
(C_\sigma x)_{jk} &= \frac{1}{a(j)b(k)} \sum_{(p,q)=(1,1)}^{(a(k),b(k))} x_{pq} \\
&= \frac{1}{\left(\lambda(n(j))+\frac{|I_j|}{2}\right)} \frac{1}{\left(\mu(m(k))+\frac{|S_k|}{2}\right)} \sum_{(p,q)=(1,1)}^{\left(\lambda(n(j))+\frac{|I_j|}{2}, \mu(m(k))+\frac{|S_k|}{2}\right)} x_{pq} \\
&= \frac{1}{\left(\lambda(n(j))+\frac{|I_j|}{2}\right)} \frac{1}{\left(\mu(m(k))+\frac{|S_k|}{2}\right)} \sum_{(p,q)=(1,1)}^{(\lambda(n(j)),\mu(m(k)))} x_{pq} \\
&\approx \frac{1}{\left(\lambda(n(j))+\frac{|I_j|}{2}\right)} \frac{1}{\left(\mu(m(k))+\frac{|S_k|}{2}\right)} \frac{\lambda(n(j))}{2} \frac{\mu(m(k))}{2} \\
&= \frac{\lambda(n(j))}{2\lambda(n(j))+|I_j|} \frac{\mu(m(k))}{2\mu(m(k))+|S_k|} \\
&= \frac{\lambda(n(j))}{2\lambda(n(j))+\lambda(n(j)+1)-\lambda(n(j))-1} \frac{\mu(m(k))}{2\mu(m(k))+\mu(m(k)+1)-\mu(m(k))-1} \\
&= \frac{1}{\frac{\lambda(n(j)+1)}{\lambda(n(j))}+1-\frac{1}{\lambda(n(j))}} \frac{1}{\frac{\mu(m(k)+1)}{\mu(m(k))}+1-\frac{1}{\mu(m(k))}}.
\end{aligned}$$

From 2.5 and Proposition 1.1 we have

$$P-\lim_{j,k}(C_\sigma x)_{jk} = \frac{1}{L_1+1}\frac{1}{L_2+1} < \frac{1}{4}.$$

Since $\left\{(C_\sigma x)_{jk}\right\}$ and $\left\{(C_\beta x)_{nm}\right\}$ are two subsequences of $\left\{(C_1 x)_{nm}\right\}$ with $P-\lim\limits_{j,k}(C_\sigma x)_{jk} < \frac{1}{4}$ and $P-\lim\limits_{n,m}(C_\beta x)_{nm} = \frac{1}{4}$, $[x]$ cannot be $C_1-summable$.On the other hand, we may choose $n(j)$ and $m(k)$ such that

$$\lambda(n(j)+1)-\lambda(n(j)) \text{ and } \mu(m(k)+1)-\mu(m(k)) \text{ are even}$$

or

$$\lambda(n(j)+1)-\lambda(n(j)) \text{ is odd and } \mu(m(k)+1)-\mu(m(k)) \text{ is even}$$

or

$$\lambda(n(j)+1)-\lambda(n(j)) \text{ is even and } \mu(m(k)+1)-\mu(m(k)) \text{ is odd}$$

and we will continue the proof in the same way. Hence, we have C_1 and C_β are not equivalent for bounded sequences. □

Osikiewicz [5] has given an inclusion result between submethods of the ordinary Cesàro method. The following theorem gives similar results for four dimensional Cesàro submethods.

Theorem 2.3. *Let $\beta_1(n,m) = (\lambda^{(1)}(n), \mu^{(1)}(m))$ and $\beta_2(n,m) = (\lambda^{(2)}(n), \mu^{(2)}(m))$ be two double index sequences such that*

$$P-\lim_{nm}\frac{\lambda^{(1)}(n)\mu^{(1)}(m)}{\lambda^{(2)}(n)\mu^{(2)}(m)} = 1$$

then C_{β_1} and C_{β_2} are equivalent for bounded double sequences.

Proof. Let $[x]$ be a bounded double sequence, and define two double sequences $T(n,m)$ and $t(n,m)$ by

$$T(n,m) = \max\left\{\lambda^{(1)}(n)\mu^{(1)}(m), \lambda^{(2)}(n)\mu^{(2)}(m)\right\}$$

and

$$t(n,m) = \min\left\{\lambda^{(1)}(n)\mu^{(1)}(m), \lambda^{(2)}(n)\mu^{(2)}(m)\right\}.$$

It is easy to see that $P-\lim\limits_{nm}\frac{t(n,m)}{T(n,m)} = 1$. Now define two double index sequences $T^*(n,m) = (T_1(n), T_2(m))$ and $t^*(n,m) = (t_1(n), t_2(m))$ by

$$T^*(n,m) = \begin{cases} \left(\lambda^{(1)}(n), \mu^{(1)}(m)\right), & \lambda^{(1)}(n)\mu^{(1)}(m) = T(n,m) \\ \left(\lambda^{(2)}(n), \mu^{(2)}(m)\right), & \lambda^{(2)}(n)\mu^{(2)}(m) = T(n,m) \end{cases}$$

and

$$t^*(n,m) = \begin{cases} \left(\lambda^{(1)}(n), \mu^{(1)}(m)\right), & \lambda^{(1)}(n)\mu^{(1)}(m) = t(n,m) \\ \left(\lambda^{(2)}(n), \mu^{(2)}(m)\right), & \lambda^{(2)}(n)\mu^{(2)}(m) = t(n,m). \end{cases}$$

Note that $T(n,m)=T_1(n)T_2(m)$ and $t(n,m)=t_1(n)t_2(m)$. Then for fixed n,m we get

$$\begin{aligned}
\left|(C_{\beta_1}x)_{nm}-(C_{\beta_2}x)_{nm}\right| &= \left|\frac{1}{\lambda^{(1)}(n)\mu^{(1)}(m)}\sum_{(j,k)=(1,1)}^{\left(\lambda^{(1)}(n),\mu^{(1)}(m)\right)} x_{jk} - \right.\\
&\quad \left.\frac{1}{\lambda^{(2)}(n)\mu^{(2)}(m)}\sum_{(j,k)=(1,1)}^{\left(\lambda^{(2)}(n),\mu^{(2)}(m)\right)} x_{jk}\right| \\
&= \left|\frac{1}{T(n,m)}\sum_{(j,k)=(1,1)}^{T^*(n,m)} x_{jk} - \frac{1}{t(n,m)}\sum_{(j,k)=(1,1)}^{t^*(n,m)} x_{jk}\right| \\
&= \left|\sum_{(j,k)=(1,1)}^{t^*(n,m)}\left(\frac{1}{T(n,m)}-\frac{1}{t(n,m)}\right)x_{jk}\right.\\
&\quad +\frac{1}{T(n,m)}\left\{\sum_{(j,k)=(t_1(n)+1,1)}^{(T_1(n),t_2(m))} x_{jk} + \sum_{(j,k)=(1,t_2(m)+1)}^{(t_1(n),T_2(m))} x_{jk}\right.\\
&\quad \left.\left.+\sum_{(j,k)=(t_1(n)+1,t_2(m)+1)}^{(T_1(n),T_2(m))} x_{jk}\right\}\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|x\|_{(\infty,2)}\sum_{(j,k)=(1,1)}^{t^*(n,m)}\frac{T_1(n)T_2(m)-t_1(n)t_2(m)}{T_1(n)T_2(m)t_1(n)t_2(m)} \\
&\quad +\|x\|_{(\infty,2)}\frac{1}{T_1(n)T_2(m)}\left\{\left(T_1(n)-t_1(n)\right)t_2(m)\right.\\
&\quad \left.+t_1(n)\left(T_2(m)-t_2(m)\right)+\left(T_1(n)-t_1(n)\right)\left(T_2(m)-t_2(m)\right)\right\} \\
&= 2\|x\|_{(\infty,2)}\frac{T_1(n)T_2(m)-t_1(n)t_2(m)}{T_1(n)T_2(m)} \\
&= 2\|x\|_{(\infty,2)}\left(1-\frac{t_1(n)t_2(m)}{T_1(n)T_2(m)}\right) \\
&= 2\|x\|_{(\infty,2)}\left(1-\frac{t(n,m)}{T(n,m)}\right). \qquad (2.6)
\end{aligned}$$

Since

$$\left|(C_{\beta_1}x)_{nm}-L\right| \leq \left|(C_{\beta_1}x)_{nm}-(C_{\beta_2}x)_{nm}\right| + \left|(C_{\beta_2}x)_{nm}-L\right|,$$

2.6 implies that $[x]$ is C_{β_1} summable to L provided that $[x]$ is C_{β_2} summable to L. Hence, C_{β_1} is equivalent to C_{β_2} for bounded double sequences. □

We have compared C_β and C_1 for bounded double sequences in Theorem 2.2. Next, replacing the convergence condition in 2.1 by $P-$*boundedness*, we show that

C_β is equivalent to C_1 for nonnegative double sequences that are C_β – *summable* to 0.

Theorem 2.4. *Let $\beta = (\lambda(n), \mu(m))$ be a double index sequence. Then the following statements are equivalent:*

i) The double sequence $[y] = (y_{nm})$ defined by

$$y_{nm} = \left(\frac{\lambda(n+1)\mu(m+1)}{\lambda(n)\mu(m)}\right), \ for\ all\ n, m \in \mathbb{N} \tag{2.7}$$

is P – bounded.

ii) $[x]$ is C_1 summable to 0 where $[x]$ is a nonnegative double sequence that is C_β summable to 0.

Proof. Let $[x]$ is a nonnegative double sequence that is C_β summable to 0 and assume that the double sequence $[y]$ defined by 2.7 is P – *bounded*. Consider the sets

$$F_1 = \mathbb{N}\backslash E(\lambda) =: \{\alpha_1(n)\} \text{ and } F_2 = \mathbb{N}\backslash E(\mu) =: \{\alpha_2(m)\}.$$

Case I. If the sets F_1 and F_2 are finite then Theorem 2.1 implies that $C_\beta \subseteq C_1$.
Case II. Assume F_1 and F_2 are both infinite sets. Then there exists an n_0 such that for $n, m \geq n_0$, $\alpha_1(n) > \lambda(1)$ and $\alpha_2(m) > \mu(1)$. Since $E(\lambda) \cap F_1 = \varnothing$ and $E(\mu) \cap F_2 = \varnothing$, for all $n, m \geq n_0$, there exist $p, q \in \mathbb{N}$ such that $\lambda(p) < \alpha_1(n) < \lambda(p+1)$ and $\mu(q) < \alpha_2(m) < \mu(q+1)$. It can be written that $\alpha_1(n) = \lambda(p)+a$ and $\alpha_2(m) = \mu(q)+b$, where

$$0 < a < \lambda(p+1) - \lambda(p) \ and\ 0 < b < \mu(q+1) - \mu(q). \tag{2.8}$$

Now define a double index sequence β' as

$$\beta'(n, m) = (\alpha_1(n), \alpha_2(m)).$$

Then for $n, m \geq n_0$ we have,

$$\begin{aligned}
(C_{\beta'}x)_{nm} &= \frac{1}{\alpha_1(n)\alpha_2(m)} \sum_{(j,k)=(1,1)}^{(\alpha_1(n),\alpha_2(m))} x_{jk} = \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} \\
&+ \frac{1}{(\lambda(p)+a)(\mu(q)+b)} \left\{ \sum_{(j,k)=(1,\mu(q)+1)}^{(\lambda(p),\mu(q)+b)} x_{jk} + \sum_{(j,k)=(\lambda(p)+1,1)}^{(\lambda(p)+a,\mu(q))} x_{jk} + \right. \\
&\left. \sum_{(j,k)=(\lambda(p)+1,\mu(q)+1)}^{(\lambda(p)+a,\mu(q)+b)} x_{jk} \right\} \\
&\leq \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk} + \frac{3}{(\lambda(p)+a)(\mu(q)+b)} \sum_{(j,k)=(1,1)}^{(\lambda(p+1),\mu(q+1))} x_{jk} \\
&\leq \frac{1}{\lambda(p)\mu(q)} \sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(q))} x_{jk}
\end{aligned}$$

$$+3\frac{\lambda(p+1)\mu(q+1)}{(\lambda(p)+a)\,(\mu(q)+b)}\frac{1}{\lambda(p+1)\mu(q+1)}\sum_{(j,k)=(1,1)}^{(\lambda(p+1),\mu(q+1))}x_{jk}$$

$$\leq (C_{\beta}x)_{pq}+3\frac{\lambda(p+1)\mu(q+1)}{\lambda(p)\mu(q)}\,(C_{\beta}x)_{p+1,q+1}\,. \qquad (2.9)$$

Since $P-\lim[x]=0$ and $[y]$ is $P-$*bounded*, from 2.9 we get

$$P-\lim_{n,m}\left(C_{\beta'}x\right)_{nm}=0.$$

As the double sequence $\{(C_1x)_{nm}\}$ may be partitioned into two subsequences $\left\{(C_{\beta'}x)_{nm}\right\}$ and $\left\{(C_{\beta}x)_{nm}\right\}$, each having the common P-limit 0, $[x]$ must be C_1-*summable* to 0.

Case III. Assume F_1 is infinite set and F_2 is finite set and define a double index sequence β' as

$$\beta'(n,m)=(\alpha_1(n),\mu(m))\,.$$

Then for all $n\geq n_0$ and for all $m\in\mathbb{N}$

$$\begin{aligned}
(C_{\beta'}x)_{nm} &= \frac{1}{\alpha_1(n)\mu(m)}\sum_{(j,k)=(1,1)}^{(\alpha_1(n),\mu(m))}x_{jk}\\
&= \frac{1}{(\lambda(p)+a)\,\mu(m)}\left\{\sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(m))}x_{jk}+\sum_{(j,k)=(\lambda(p)+1,1)}^{(\lambda(p)+a,\mu(m))}x_{jk}\right\}\\
&\leq \frac{1}{\lambda(p)\mu(m)}\sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(m))}x_{jk}+\frac{1}{(\lambda(p)+a)\,\mu(m)}\sum_{(j,k)=(1,1)}^{(\lambda(p+1),\mu(m+1))}x_{jk}\\
&\leq \frac{1}{\lambda(p)\mu(m)}\sum_{(j,k)=(1,1)}^{(\lambda(p),\mu(m))}x_{jk}\\
&\quad +\frac{\lambda(p+1)\mu(m+1)}{(\lambda(p)+a)\,\mu(m)}\frac{1}{\lambda(p+1)\mu(m+1)}\sum_{(j,k)=(1,1)}^{(\lambda(p+1),\mu(m+1))}x_{jk}\\
&\leq (C_{\beta}x)_{pq}+\frac{\lambda(p+1)\mu(m+1)}{\lambda(p)\mu(m)}\,(C_{\beta}x)_{p+1,m+1}\,.
\end{aligned}$$

Then as in Case II we have $P-\lim_{n,m}(C_1x)_{nm}=0$.

Case IV. If F_1 is finite set and F_2 is infinite set, then we can get the proof as in Case III by interchanging the roles of F_1 and F_2.

Conversely assume that $[y]$ is not $P-$*bounded*. Then there exist two index sequences $n(j)$ and $m(k)$ such that

$$P-\lim\frac{\lambda(n(j)+1)\mu(m(k)+1)}{\lambda(n(j))\mu(m(k))}=\infty, \qquad (2.10)$$

$$\lambda(n(j)+1)>2\lambda\,(n(j))\ \text{ and }\ \mu(m(k)+1)>2\mu(m(k)), \qquad (2.11)$$

for all a, $b \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$$\lim_j \frac{\lambda(n(j))}{\lambda(n(j+a))} = 1 \text{ and } \lim_k \frac{\mu(m(k))}{\mu(m(k+b))} = 1 \tag{2.12}$$

and for all $s \in \mathbb{N}$

$$\lambda(n(1))\mu(m(1)) + ... + \lambda(n(s-1))\mu(m(s-1)) < \lambda(n(s))\mu(m(s)).$$

Now define the double sequence $[x] = (x_{pq})$ by

$$x_{pq} = \begin{cases} 1, & \begin{cases} p \in (\lambda(n(t)), 2\lambda\,(n(t))] \\ \quad and \\ q \in (\mu(m(t)), 2\mu(m(t))] \end{cases} \quad t = 1, 2, ... \\ 0, & \quad otherwise \end{cases} .$$

For fixed n, m such that $\lambda(n(1)+1) \leq \lambda(n)$ and $\mu(m(1)+1) \leq \mu(m)$, there exist j, k such that $\lambda(n(j)+1) \leq \lambda(n) \leq \lambda(n(j+1))$ and $\mu(m(k)+1) \leq \mu(m) \leq \mu(m(k+1))$. Then we have,

$$(C_\beta x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{(p,q)=(1,1)}^{(\lambda(n),\mu(m))} x_{pq}$$

$$= \frac{1}{\lambda(n)\mu(m)} \left\{ \sum_{(p,q)=(\lambda(n(1))+1,\mu(m(k))+1)}^{(2\lambda(n(1)),2\mu(m(1)))} 1 + ... + \sum_{(p,q)=(\lambda(n(i))+1,\mu(m(i))+1)}^{(2\lambda(n(i)),2\mu(m(i)))} \right\}$$

$$= \frac{\lambda(n(i)+1)\mu(m(i)+1)}{\lambda(n)\mu(m)} \frac{\lambda(n(1))\mu(m(1)) + ... + \lambda(n(i))\mu(m(i))}{\lambda(n(i)+1)\mu(m(i)+1)}$$

$$\leq \frac{\lambda(n(i)+1)\mu(m(i)+1)}{\lambda(n)\mu(m)} \frac{2\lambda(n(i))\mu(m(i))}{\lambda(n(i)+1)\mu(m(i)+1)}$$

$$\leq \frac{2\lambda(n(i))\mu(m(i))}{\lambda(n(i)+1)\mu(m(i)+1)} \tag{2.13}$$

where $i = \min\{j, k\}$. Hence, from 2.10 and 2.13 we get

$$P - \lim_{n,m} (C_\beta x)_{nm} = 0. \tag{2.14}$$

Now let $\beta'(j,k) = (\alpha(j), \gamma(k))$ be a double index sequences where

$$\alpha(j) = 2\lambda(n(j)) \text{ and } \gamma(k) = 2\mu(m(k)).$$

Then we get

$$\begin{aligned}
(C_{\beta'}x)_{jk} &= \frac{1}{\alpha(j)\gamma(k)} \sum_{(j,k)=(1,1)}^{(\alpha(j),\gamma(k))} x_{jk} \\
&= \frac{1}{4\lambda(n(j))\mu(m(k))} \sum_{(j,k)=(1,1)}^{(2\lambda(n(j)),2\mu(m(k)))} x_{jk} \\
&= \frac{1}{4\lambda(n(j))\mu(m(k))} \left\{ \sum_{(j,k)=(1,1)}^{(\lambda(n(j)),\mu(m(k)))} x_{jk} + \sum_{(j,k)=(\lambda(n(j))+1,1)}^{(2\lambda(n(j)),\mu(m(k)))} x_{jk} \right. \\
&\quad \left. + \sum_{(j,k)=(1,\mu(m(k))+1)}^{(\lambda(n(j)),2\mu(m(k)))} x_{jk} + \sum_{(j,k)=(\lambda(n(j))+1,\mu(m(k))+1)}^{(2\lambda(n(j)),2\mu(m(k)))} x_{jk} \right\} \\
&= \frac{1}{4}(C_\beta x)_{jk} + \frac{3}{4}\frac{\lambda(n(i))\mu(m(i))}{\lambda(n(j))\mu(m(k))}.
\end{aligned}$$

Since $i = \min\{j,k\}$, there exist nonnegative integers a, b such that $i = j + a$ and $i = k + b$. Then 2.12, 2.14 and Proposition 1.1 imply that $P - \lim_{j,k} (C_{\beta'}x)_{jk} = \frac{3}{4}$.

Hence $[x]$ is not C_1 summable. □

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