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Partial sums of harmonic univalent functions

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Abstract. In this paper, authors obtain conditions under which the partial sums of the Libera integral operator of functions in the class $HP(\alpha)$, $(0 \leq \alpha < 1)$, consisting of harmonic univalent functions $f = h+\overline{g}$ for which $Re\{h'(z)+g'(z)\} > \alpha$, belong to the similar class $HP(\beta)$, $(0 \leq \beta < 1)$. Further, we improve a recent result on partial sums of functions of bounded turning in [6].

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1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h+\overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$ (see Clunie and Sheil-Small [2]).

Denote by S_H the class of functions $f = h+\overline{g}$ which are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ g(z) = \sum_{k=1}^{\infty} b_k z^k, \ |b_1| < 1. \tag{1.1}$$

For basic results on harmonic functions one may refer to the following standard introductory text book by Duren [3].

Note that S_H reduces to the class S of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ z \in U. \tag{1.2}$$

For $0 \leq \alpha < 1$, $B(\alpha)$ denote the class of functions of the form (1.2) such that $Re\{f'(z)\} > \alpha$ in U. The functions in $B(\alpha)$ are called functions of bounded turning (cf. [5]).

Recently, Yalcin et al.[13] introduced the subclass $HP(\alpha)$ of S_H consisting of functions f of the form (1.1) satisfying the condition

$$Re\{h'(z) + g'(z)\} > \alpha. \tag{1.3}$$

In [13], $HP^*(\alpha)$ denote the subclass of $HP(\alpha)$ consisting of functions $f = h+\overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \; g(z) = -\sum_{k=1}^{\infty} |b_k| z^k. \tag{1.4}$$

We note that for f of the form (1.2), $HP(\alpha)$ reduces to the class $B(\alpha)$ satisfying the condition $Re\{f'(z)\} > \alpha$ in U.

For f of the form (1.2), the Libera integral operator F is given by

$$F(z) = \frac{2}{z}\int_0^z f(\varsigma)\, d\varsigma = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k. \tag{1.5}$$

For $f = h + \overline{g}$ in S_H, where h and g are given by (1.1), the Libera integral operator led us to define integral operator given by

$$F(z) = \frac{2}{z}\int_0^z h(\varsigma)\, d\varsigma + \overline{\frac{2}{z}\int_0^z g(\varsigma)\, d\varsigma} = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k + \sum_{k=1}^{\infty} \overline{\frac{2}{k+1} b_k z^k}. \tag{1.6}$$

The nth partial sums $F_n(z)$ of the integral operator $F(z)$ for functions f of the form (1.1) are given by

$$\begin{aligned} F_n(z) &= z + \sum_{k=2}^{n} \frac{2}{k+1} a_k z^k + \sum_{k=1}^{n} \overline{\frac{2}{k+1} b_k z^k} \\ &= H_n(z) + \overline{G_n(z)}. \end{aligned} \tag{1.7}$$

The nth partial sums $F_n(z)$ of the Libera integral operator $F(z)$ for analytic univalent functions of the form (1.2) have been studied by various authors in ([6], [8]) (See also [1], [7], [9], [10], [11], [12]), yet analogous results on harmonic univalent functions have not been so far explored. Motivated with the work of Jahangiri and Farahmand [6], an attempt has been made to systematically study the partial sums of harmonic univalent functions.

2. Main results

To derive our first main result, we need the following three lemmas. The first lemma is due to Gasper [4], the second is due to Jahangiri and Farahmand [6] and the third is a well-known and celebrated result (cf. [5]) that can be derived from the Herglotz representation for positive real part functions.

Lemma 2.1. *Let θ be a real number and let m and k be natural numbers. Then*

$$\frac{1}{3}+\sum_{k=1}^{m}\frac{\cos\left(k\theta\right)}{k+2}\geq 0. \tag{2.1}$$

Lemma 2.2. *For $z\in U$,*

$$Re\left(\sum_{k=1}^{m}\frac{z^{k}}{k+2}\right)>-\frac{1}{3}. \tag{2.2}$$

Lemma 2.3. *Let $P\left(z\right)$ be analytic in U, $P\left(0\right)=1$ and $Re\left(P\left(z\right)\right)>\frac{1}{2}$ in U. For functions Q analytic in U, the convolution function $P*Q$ takes values in the convex hull of the image on U under Q.*

The operator "$*$" stands for the Hadamard product or convolution of two power series $f(z)=\sum_{k=0}^{\infty}a_{k}z^{k}$ and $g(z)=\sum_{k=0}^{\infty}b_{k}z^{k}$ is given by

$$\left(f*g\right)\left(z\right)=f\left(z\right)*g\left(z\right)=\sum_{k=0}^{\infty}a_{k}b_{k}z^{k}.$$

Theorem 2.4. *If f of the form (1.1) with $b_1=0$ and $f\in HP\left(\alpha\right)$, then $F_n\in HP\left(\frac{4\alpha-1}{3}\right)$, for $\frac{1}{4}\leq\alpha<1$.*

Proof. Let f be of the form (1.1) and belong to $HP\left(\alpha\right)$ for $\frac{1}{4}\leq\alpha<1$.

Since

$$Re\left\{h'\left(z\right)+g'\left(z\right)\right\}>\alpha,$$

we have

$$Re\left\{1+\frac{1}{2\left(1-\alpha\right)}\left(\sum_{k=2}^{\infty}ka_{k}z^{k-1}+\sum_{k=2}^{\infty}kb_{k}z^{k-1}\right)\right\}>\frac{1}{2}. \tag{2.3}$$

Applying the convolution properties of power series to $H_n'\left(z\right)+G_n'\left(z\right)$, we may write

$$H_n'\left(z\right)+G_n'\left(z\right)=1+\sum_{k=2}^{n}\frac{2k}{k+1}a_{k}z^{k-1}+\sum_{k=2}^{n}\frac{2k}{k+1}b_{k}z^{k-1}$$

$$=\left(1+\frac{1}{2\left(1-\alpha\right)}\left(\sum_{k=2}^{\infty}k\left(a_k+b_k\right)z^{k-1}\right)\right)*\left(1+\left(1-\alpha\right)\sum_{k=2}^{n}\frac{4}{k+1}z^{k-1}\right)$$

$$=P(z)*Q(z). \tag{2.4}$$

From Lemma 2.2 for $m=n-1$, we obtain

$$Re\left(\sum_{k=2}^{n}\frac{z^{k-1}}{k+1}\right)>-\frac{1}{3}. \tag{2.5}$$

By applying a simple algebra to inequality (2.5) and $Q(z)$ in (2.4)), one may obtain

$$Re\left(Q\left(z\right)\right)=Re\left\{1+\left(1-\alpha\right)\sum_{k=2}^{n}\frac{4}{k+1}z^{k-1}\right\}>\frac{4\alpha-1}{3}.$$

On the other hand, the power series $P(z)$ in (2.4) in conjunction with the condition (2.3) yields

$$Re\left(P\left(z\right)\right) > \frac{1}{2}.$$

Therefore, by Lemma 2.3, $Re\left\{H_n'\left(z\right) + G_n'\left(z\right)\right\} > \frac{4\alpha-1}{3}$.
This completes the proof of Theorem 2.4. □

If f of the form (1.2) in Theorem 2.4, we obtain the following result of Jahangiri and Farahmand in [6].

Corollary 2.5. *If f of the form (1.2) and $f \in B\left(\alpha\right)$, then $F_n \in B\left(\frac{4\alpha-1}{3}\right)$, for $\frac{1}{4} \leq \alpha < 1$.*

To prove our next theorem, we need the following Lemma due to Yalcin et al. [13].

Lemma 2.6. *Let $f = h + \overline{g}$ be given by (1.4). Then $f \in HP^*\left(\alpha\right)$ if and only if*

$$\sum_{k=2}^{\infty} k\left|a_k\right| + \sum_{k=1}^{\infty} k\left|b_k\right| \leq 1-\alpha,\ 0 \leq \alpha < 1.$$

Theorem 2.7. *Let f be of the form (1.4) with $b_1 = 0$ and $f \in HP^*\left(\alpha\right)$, then the functions $F\left(z\right)$ defined by (1.6) belongs to $HP^*\left(\rho\right)$, where $\rho = \frac{1+2\alpha}{3}$. The result is sharp. Further, the converse need not to be true.*

Proof. Since $f \in HP^*\left(\alpha\right)$, Lemma 2.6 ensures that

$$\sum_{k=2}^{\infty} \frac{k}{1-\alpha}\left(\left|a_k\right| + \left|b_k\right|\right) \leq 1. \tag{2.6}$$

Also, from (1.6) we have

$$F\left(z\right) = z - \sum_{k=2}^{\infty} \frac{2}{k+1}\left|a_k\right| z^k - \sum_{k=2}^{\infty} \frac{2}{k+1}\left|b_k\right| \bar{z}^k.$$

Let $F\left(z\right) \in HP^*\left(\sigma\right)$, then, by Lemma 2.6, we have

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\sigma}\right)\left(\frac{2}{k+1}\left|a_k\right| + \frac{2}{k+1}\left|b_k\right|\right) \leq 1.$$

Thus we have to find largest value of σ so that the above inequality holds. Now this inequality holds if

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\sigma}\right)\left(\frac{2}{k+1}\left|a_k\right| + \frac{2}{k+1}\left|b_k\right|\right) \leq \sum_{k=2}^{\infty} \frac{k}{1-\alpha}\left(\left|a_k\right| + \left|b_k\right|\right).$$

or, if

$$\left(\frac{k}{1-\sigma}\right)\frac{2}{k+1} \leq \frac{k}{1-\alpha},\ \text{for each}\ \ k = 2, 3, 4......$$

which is equivalent to

$$\sigma \leq \frac{k-1+2\alpha}{k+1} = \rho_k,\ k = 2, 3, 4.......$$

It is easy to verify that ρ_k is an increasing function of k. Therefore, $\rho = \inf_{k\geq 2} \rho_k = \rho_2$ and, hence

$$\rho = \frac{1+2\alpha}{3}.$$

To show the sharpness, we take the function $f(z)$ given by

$$f(z) = z - \frac{(1-\alpha)}{2}|x|z^2 - \frac{(1-\alpha)}{2}|y|\bar{z}^2, \text{ where } |x|+|y| = 1.$$

Then

$$\begin{aligned} F(z) &= z - \frac{(1-\alpha)}{3}|x|z^2 - \frac{(1-\alpha)}{3}|y|\bar{z}^2 \\ &= H(z) + \overline{G(z)} \end{aligned}$$

and therefore

$$\begin{aligned} H'(z) + G'(z) &= 1 - \frac{2(1-\alpha)}{3}|x|z - \frac{2(1-\alpha)}{3}|y|z \\ &= \frac{3-2(1-\alpha)(|x|+|y|)z}{3} \\ &= \frac{1+2\alpha}{3}, \text{ for } z \to 1. \end{aligned}$$

Hence, the result is sharp.

We now show that the converse of above theorem need not to be true. To this end, we consider the function

$$F(z) = z - \frac{(1-\sigma)}{3}|x|z^3 - \frac{(1-\sigma)}{3}|y|\bar{z}^3,$$

where

$$|x|+|y| = 1, \ \sigma = \frac{2\alpha+1}{3}.$$

Lemma 2.6 guarantees that $F(z) \in HP^*(\sigma)$.

But the corresponding function

$$f(z) = z - \frac{2(1-\sigma)}{3}|x|z^3 - \frac{2(1-\alpha)}{3}|y|\bar{z}^3,$$

does not belong to $HP^*(\alpha)$, since, for this $f(z)$ the coefficient inequality of Lemma 2.6 is not satisfied. □

In next theorem, we improve the result of Theorem 2.4 for functions f of the form (1.4) for this we need the following Lemma due to Yalcin et al. [13].

Lemma 2.8. *If* $0 \leq \alpha_1 \leq \alpha_2 < 1$ *, then*

$$HP^*(\alpha_2) \subseteq HP^*(\alpha_1).$$

Theorem 2.9. *Let f of the form* (1.4) *with* $b_1 = 0$ *and* $f \in HP^*(\alpha)$. *Then the function* $F_n(z)$ *defined by* (1.7) *belong to* $HP^*\left(\frac{2\alpha+1}{3}\right)$.

Proof. Since

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=2}^{\infty} |b_k| \bar{z}^k.$$

Then

$$F(z) = z - \sum_{k=2}^{\infty} \frac{2}{k+1} |a_k| z^k - \sum_{k=2}^{\infty} \frac{2}{k+1} |b_k| \bar{z}^k.$$

By using Theorem 2.7, we have

$$F(z) \in HP^*(\sigma), \text{ where } \sigma = \frac{2\alpha+1}{3}.$$

Now

$$F_n(z) = z - \sum_{k=2}^{n} \frac{2}{k+1} |a_k| z^k - \sum_{k=2}^{n} \frac{2}{k+1} |b_k| \bar{z}^k.$$

To show that $F_n(z) \in HP^*(\sigma)$, we have

$$\begin{aligned}
&\sum_{k=2}^{n} \left(\frac{k}{1-\sigma}\right)\left(\frac{2}{k+1}|a_k| + \frac{2}{k+1}|b_k|\right) \\
\leq &\sum_{k=2}^{\infty} \left(\frac{k}{1-\sigma}\right)\left(\frac{2}{k+1}|a_k| + \frac{2}{k+1}|b_k|\right) \\
\leq &1.
\end{aligned}$$

Thus $F_n(z) \in HP^*(\sigma)$.

In next theorem, we improve a result of Jahangiri and Farahmand in [6] when f has form $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$, for this we need the following Lemma. □

Lemma 2.10. *If* $0 \leq \alpha_1 \leq \alpha_2 < 1$, *then*

$$B(\alpha_2) \subseteq B(\alpha_1).$$

Proof. The proof of the above lemma is straightforward, so we omit the details. □

Theorem 2.11. *Let* $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$. *If* $f(z) \in B(\alpha)$, *then*

$$F_n(z) = z - \sum_{k=2}^{n} \frac{2}{k+1} |a_k| z^k$$

belongs to $B\left(\frac{2\alpha+1}{3}\right)$.

Proof. The proof of this theorem is much akin to that of Theorem 2.9 and therefore we omit the details. □

Remark 2.12. For $\frac{1}{4} \leq \alpha < 1$, $f(z) \in B(\alpha)$ Jahangiri and Farahmand [6] shows that $F_n(z) \in B\left(\frac{4\alpha-1}{3}\right)$ and our result states that $F_n(z) \in B\left(\frac{2\alpha+1}{3}\right)$.
Since $\frac{2\alpha+1}{3} > \frac{4\alpha-1}{3}$, for $\frac{1}{4} \leq \alpha < 1$, and using Lemma 2.10, we have

$$B\left(\frac{2\alpha+1}{3}\right) \subset B\left(\frac{4\alpha-1}{3}\right).$$

Hence our result provides a smaller class in comparison to the class given by Jahangiri and Farahmand [6].

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