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On certain subclasses of analytic functions

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Abstract. In the present paper, we introduce and study certain new subclasses of analytic functions in the open unit disk U. Some inclusion relationships and integral preserving properties have also discussed in particular with reference to a new integral operator.

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1. Introduction

Let A be the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic and normalized in the open unit disk $U = \{z : |z| < 1\}$.

Next we define some well known subclasses such as starlike, convex, close-to-convex and quasi-convex functions of A, denoted by $S^*(\xi)$, $C(\xi)$, $K(\rho, \xi)$ and $K^*(\rho, \xi)$ respectively as follow(cf.[1]-[3]):

$$\begin{aligned} S^*(\xi) &= \left\{ f \in A : \Re\left(\frac{zf'(z)}{f(z)}\right) > \xi, z \in \mathbb{U} \right\}, 0 \leq \xi < 1. \\ C(\xi) &= \left\{ f \in A : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \xi, z \in \mathbb{U} \right\}, 0 \leq \xi < 1. \\ K(\rho, \xi) &= \left\{ f \in A : \exists g(z) \in S^*(\xi) \wedge \Re\left(\frac{zf'(z)}{g(z)}\right) > \rho, z \in \mathbb{U} \right\}, 0 \leq \rho < 1. \\ K^*(\rho, \xi) &= \left\{ f \in A : \exists g(z) \in C(\xi) \wedge \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \rho, z \in \mathbb{U} \right\}. \end{aligned}$$

Note that

$$f(z) \in C(\xi) \Leftrightarrow zf'(z) \in S^*(\xi) \ \wedge \ f(z) \in K^*(\rho, \xi) \Leftrightarrow zf'(z) \in K(\rho, \xi).$$

For $f \in A$ and $\beta, \gamma \geq 0$, we define a new differential operator as follows:

$$\begin{aligned} \Theta^0(\beta,\gamma)f(z) &= f(z) \\ (\gamma+\beta+1)\Theta^1(\beta,\gamma)f(z) &= \beta f(z) + (\gamma+1)(zf'(z)) \\ &\vdots \\ \Theta^n(\beta,\gamma)f(z) &= z + \sum_{k=2}^{\infty}\left(\frac{\beta+k(\gamma+1)}{\gamma+\beta+1}\right)^n a_k z^k. \end{aligned} \tag{1.1}$$

This operator is closely related to the following operators:

1. $\Theta^n(\lambda,0)f(z) = \Theta^n(\lambda)f(z) = z + \sum\limits_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^k$,(see [4, 5]);
2. $\Theta^n(1,0)f(z) = \Theta^n f(z) = z + \sum\limits_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^n a_k z^k$,(see[6]);
3. $\Theta^n(0,0)f(z) = \Theta^n f(z) = z + \sum\limits_{k=2}^{\infty}(k)^n a_k z^k$,(see[7]).

(1.1) $\Rightarrow (\gamma+1)z(\Theta^n(\beta,\gamma)f(z))' = (\gamma+1+\beta)\Theta^{n+1}(\beta,\gamma)f(z) - \beta\Theta^n(\beta,\gamma)f(z)$.

Now for linear operator $\Theta^n(\beta,\gamma)$ we define the following subclasses of A:

$$\begin{aligned} S_n^*(\xi,\beta,\gamma) &= \{f \in A : \Theta^n(\beta,\gamma)f \in S^*(\xi)\}; \\ C_n(\xi,\beta,\gamma) &= \{f \in A : \Theta^n(\beta,\gamma)f \in C(\xi)\}; \\ K_n(\rho,\xi,\beta,\gamma) &= \{f \in A : \Theta^n(\beta,\gamma)f \in K(\rho,\xi)\}; \\ K_n^*(\rho,\xi,\beta,\gamma) &= \{f \in A : \Theta^n(\beta,\gamma)f \in K^*(\rho,\xi)\}. \end{aligned}$$

2. Inclusion relationships

Lemma 2.1. [8, 9] *Let $\varphi(\mu,\upsilon)$ be a complex function such that $\varphi : D \to \mathbb{C}$, $D \subseteq \mathbb{C}\times\mathbb{C}$, and let $\mu = \mu_1 + i\mu_2$, $\upsilon = \upsilon_1 + i\upsilon_2$. Suppose that $\varphi(\mu,\upsilon)$ satisfies the following conditions:*

1. *$\varphi(\mu,\upsilon)$ is continuous in D;*
2. *$(1,0) \in D$ and $\Re\varphi(1,0) > 0$;*
3. *$\Re\varphi(i\mu_2,\upsilon_1) \leq 0$ for all $(i\mu_2,\upsilon_1) \in D$ such that $\upsilon_1 \leq -\frac{1}{2}(1+\mu_2^2)$.*

Let $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in $\mathbb{U}$, such that $(h(z), zh'(z)) \in D$ for all $z \in \mathbb{U}$. If $\Re\{\varphi(h(z), zh'(z))\} > 0 (z \in \mathbb{U})$, then $\Re\{h(z)\} > 0$.

Theorem 2.2. *Let $f \in A$, $0 \leq \xi < 1$, $\beta, \gamma \geq 0, n \in \mathbb{N}$ then*

$$S_{n+1}^*(\xi,\beta,\gamma) \subseteq S_n^*(\xi,\beta,\gamma) \subseteq S_{n-1}^*(\xi,\beta,\gamma).$$

Proof. Let $f \in S_{n+1}^*(\xi,\beta,\gamma)$, and suppose that

$$\frac{z(\Theta^n(\beta,\gamma)f(z))'}{\Theta^n(\beta,\gamma)f(z)} = \xi + (1-\xi)h(z).$$

Since

$$(1+\frac{\beta}{\gamma+1})\frac{\Theta^{n+1}(\beta,\gamma)f(z)}{\Theta^n(\beta,\gamma)f(z)} = \xi + (1-\xi)h(z) + \frac{\beta}{\gamma+1},$$

therefore

$$\frac{z(\Theta^{n+1}(\beta,\gamma)f(z))'}{\Theta^{n+1}(\beta,\gamma)f(z)}-\xi \quad = \quad (1-\xi)h(z)+\frac{(\gamma+1)(1-\xi)zh'(z)}{\beta+(\gamma+1)\xi+(1-\xi)h(z)}.$$

Taking $h(z)=\mu=\mu_1+i\mu_2$ and $zh'(z)=\upsilon=\upsilon_1+i\upsilon_2$, we define $\varphi(\mu,\upsilon)$ by:

$$\varphi(\mu,\upsilon) \quad = \quad (1-\xi)\mu+\frac{(\gamma+1)(1-\xi)\upsilon}{\beta+(\gamma+1)\xi+(1-\xi)\mu}.$$

$$\Rightarrow \Re\{\varphi(i\mu_2,\upsilon_1)\} \quad = \quad \frac{[1+(\gamma+1)\xi](\gamma+1)(1-\xi)\upsilon_1}{(1+(\gamma+1)\xi)^2+(1-\xi)^2\mu_2^2},$$

$$\Re\{\varphi(i\mu_2,\upsilon_1)\} \quad \leq \quad -\frac{[1+(\gamma+1)\xi](\gamma+1)(1-\xi)(1+\mu_2^2)}{(1+(\gamma+1)\xi)^2+(1-\xi)^2\mu_2^2}<0.$$

Clearly $\varphi(\mu,\upsilon)$ satisfies the conditions of Lemma 2.1. Hence $\Re\{h(z)\}>0(z\in\mathbb{U})$, implies $f\in S_n^*(\xi,\beta,\gamma)$. □

Theorem 2.3. *Let $f\in A$, $0\leq\xi<1$, $\beta,\gamma\geq 0, n\in\mathbb{N}_0$ then*

$$C_{n+1}(\xi,\beta,\gamma)\subseteq C_n(\xi,\beta,\gamma)\subseteq C_{n-1}(\xi,\beta,\gamma).$$

Proof. Let $f\in C_{n+1}(\xi,\beta,\gamma)\Rightarrow\Theta^{n+1}(\beta,\gamma)f\in C(\xi)\Leftrightarrow z(\Theta^{n+1}(\beta,\gamma)f)'\in S^*(\xi)\Rightarrow \Theta^{n+1}(\beta,\gamma)(zf')\in S^*(\xi)\Rightarrow zf'\in S_{n+1}^*(\xi,\beta,\gamma)\subseteq S_n^*(\xi,\beta,\gamma)\Rightarrow zf'\in S_n^*(\xi,\beta,\gamma)\Rightarrow \Theta^n(\beta,\gamma)(zf')\in S^*(\xi)\Rightarrow z(\Theta^n(\beta,\gamma)f)'\in S^*(\xi)\Leftrightarrow\Theta^n(\beta,\gamma)f\in C(\xi)\Rightarrow f\in C_n(\xi,\beta,\gamma)$. □

Theorem 2.4. *Let $f\in A$, $0\leq\xi<1$, $\beta,\gamma\geq 0, 0\leq\rho<1, n\in\mathbb{N}_0$ then*

$$K_{n+1}(\rho,\xi,\beta,\gamma)\subseteq K_n(\rho,\xi,\beta,\gamma)\subseteq K_{n-1}(\rho,\xi,\beta,\gamma).$$

Proof. Let $f\in K_{n+1}(\rho,\xi,\beta,\gamma)$ and suppose that

$$\left(\frac{z(\Theta^n(\beta,\gamma)f(z))'}{\Theta^n(\beta,\gamma)g(z)}\right) \quad = \quad \rho+(1-\rho)h(z), z\in\mathbb{U}.$$

Using (1.1) we have

$$\frac{z(\Theta^{n+1}(\beta,\gamma)f(z))'}{\Theta^{n+1}(\beta,\gamma)g(z)} \quad = \quad \frac{\frac{(\gamma+1)z(\Theta^n(\beta,\gamma)f'(z))'}{\Theta^n(\beta,\gamma)g(z)}+\frac{\beta z(\Theta^n(\beta,\gamma)zf'(z))}{\Theta^n(\beta,\gamma)g(z)}}{\frac{(\gamma+1)z(\Theta^n(\beta,\gamma)g(z))'}{\Theta^n(\beta,\gamma)g(z)}+\beta}.$$

Since $\frac{(\Theta^n(\beta,\gamma)zf'(z))}{\Theta^n(\beta,\gamma)g(z)}=\rho+(1-\rho)h(z)$ and $g(z)\in S_{n+1}^*(\xi,\beta,\gamma)\subseteq S_n^*(\xi,\beta,\gamma)$. Therefore

$$\frac{z(\Theta^{n+1}(\beta,\gamma)f(z))'}{\Theta^{n+1}(\beta,\gamma)g(z)}-\rho \quad = \quad (1-\rho)h(z)+\frac{(\gamma+1)(1-\rho)zh'(z)}{(\gamma+1)(\xi+(1-\xi)H(z))+\beta}.$$

Taking $h(z)=\mu=\mu_1+i\mu_2$ and $zh'(z)=\upsilon=\upsilon_1+i\upsilon_2$, we define $\varphi(\mu,\upsilon)$ by

$$\varphi(\mu,\upsilon) \quad = \quad (1-\rho)\mu+\frac{(\gamma+1)(1-\rho)\upsilon}{(\gamma+1)(\xi+(1-\xi)H(z))+\beta}.$$

This implies

$$\Re[\varphi(i\mu_2,\upsilon_1)]=-\frac{(\gamma+1)(1+\mu_2^2)(1-\rho)[\beta+\xi(\gamma+1)+(\gamma+1)(1-\xi)h_1(x_1,y_1)]}{[\beta+\xi(\gamma+1)+(\gamma+1)(1-\xi)h_1(x_1,y_1)]^2+[(\gamma+1)(1-\xi)h_2(x_1,y_1)]^2}<0.$$

Hence, the function $\varphi(\mu, \upsilon)$ satisfies the conditions of Lemma 2.1. Implies $\Re\{h(z)\} > 0 (z \in \mathbb{U})$ gives $f \in K_n(\rho, \xi, \beta, \gamma)$. □

Similarly we proved the following theorem.

Theorem 2.5. *Let $f \in A$, $0 \le \xi < 1$, $0 \le \rho < 1$, $\beta, \gamma \ge 0, n \in \mathbb{N}_0$ then*

$$K^*_{n+1}(\rho, \xi, \beta, \gamma) \subseteq K^*_n(\rho, \xi, \beta, \gamma) \subseteq K^*_{n-1}(\rho, \xi, \beta, \gamma).$$

3. Integral operator

For $c > -1$ and $f \in A$, the integral operator $L_c(f) : A \to A$ is defined by

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt. \tag{3.1}$$

The operator $L_c(f)$ was introduced by Bernardi [10].

Theorem 3.1. *Let $c > -1$, $0 \le \xi < 1$. If $f \in S^*_n(\xi, \beta, \gamma)$, then $L_c(f) \in S^*_n(\xi, \beta, \gamma)$.*

Proof. By using (3.1) we get

$$\frac{z(\Theta^n(\beta,\gamma)L_c f(z))'}{\Theta^n(\beta,\gamma)L_c f(z)} = (c+1)\frac{\Theta^n(\beta,\gamma)f(z)}{\Theta^n(\beta,\gamma)L_c f(z)} - c.$$

$$\text{Let } \frac{z(\Theta^n(\beta,\gamma)L_c f(z))'}{\Theta^n(\beta,\gamma)L_c f(z)} = \xi + (1-\xi)h(z),\ h(z) = 1 + c_1 z + c_2 z^2 + \cdots.$$

$$\frac{z(\Theta^n(\beta,\gamma)L_c f(z))'}{\Theta^n(\beta,\gamma)L_c f(z)} - \xi = (1-\xi)h(z) + \frac{(1-\xi)zh'(z)}{\xi + (1-\xi)h(z) + c}.$$

This implies

$$\varphi(\mu, \upsilon) = (1-\xi)\mu + \frac{(1-\xi)\upsilon}{\xi + c + (1-\xi)\mu}, \text{ (same as Theorem 2.2)},$$

and

$$\Re[\varphi(i\mu_2, \upsilon_1)] = \frac{(\xi+c)(1-\xi)\upsilon_1}{[\xi+c]^2 + [(1-\xi)\mu_2]^2} \le \frac{-(\xi+c)(1-\xi)(1+\mu_2^2)}{2[\xi+c]^2 + 2[(1-\xi)\mu_2]^2} < 0.$$

After using Theorem 2.1 and Lemma 2.1., we have $L_c(f) \in S^*_n(\xi, \lambda, \alpha, \beta, \mu)$. □

Theorem 3.2. *Let $c > -1$, $0 \le \xi < 1$. If $f(z) \in C_n(\xi, \beta, \gamma)$, then $L_c(f) \in C_n(\xi, \beta, \gamma)$.*

Proof. Proof is same as that of Theorem 2.3. □

Theorem 3.3. *Let $c > -1$, $0 \le \xi < 1$, $0 \le \rho < 1$. If $f(z) \in K_n(\rho, \xi, \beta, \gamma)$, then $L_c(f) \in K_n(\rho, \xi, \beta, \gamma)$.*

Proof. Since $f \in K_n(\xi, \beta, \gamma) \Rightarrow \Theta^n(\beta, \gamma) f \in K(\rho, \xi)$. Let

$$\frac{z(\Theta^n(\beta,\gamma)L_c f(z))'{}'}{\Theta^n(\beta,\gamma)L_c g(z)} = \rho + (1-\rho).$$

Using (3.1) we have

$$\left(\frac{z(\Theta^n(\beta,\gamma)f(z))'}{\Theta^n(\beta,\gamma)g(z)}\right) = \frac{\frac{z(\Theta^n(\beta,\gamma)L_c(zf'(z))'}{\Theta^n(\beta,\gamma)L_c g(z)} + c\frac{(\Theta^n(\beta,\gamma)L_c(zf'(z))}{\Theta^n(\beta,\gamma)L_c g(z)}}{\frac{z(\Theta^n(\beta,\gamma)L_c(g(z))'}{\Theta^n(\beta,\gamma)L_c g(z)} + c}.$$

$$\text{Since}\, g(z) \in S_n^*(\xi,\beta,\gamma) \quad \Rightarrow L_c(g(z)) \in S_n^*(\xi,\beta,\gamma).\,\text{Let}$$

$$\frac{z(\Theta^n(\beta,\gamma)L_c(g(z))'}{\Theta^n(\beta,\gamma)L_c g(z)} = \xi + (1-\xi)H(z),\ \Re(H(z)) > 0.$$

$$\Rightarrow \left(\frac{z(\Theta^n(\beta,\gamma)f(z))'}{\Theta^n(\beta,\gamma)g(z)}\right) - \rho = (1-\rho)h(z) + \frac{(1-\rho)zh'(z)}{\xi + (1-\xi)H(z) + c}.$$

Using method of Theorem 2.3, we get

$$\varphi(\mu,\upsilon) = (1-\rho)\mu + \frac{(1-\rho)\upsilon}{\xi + (1-\xi)H(z) + c}.$$

Taking $h(z) = \mu = \mu_1 + i\mu_2$ and $zh'(z) = \upsilon = \upsilon_1 + i\upsilon_2$, we define the function $\varphi(\mu,\upsilon)$ by:

$$\Re[\varphi(i\mu_2,\upsilon_1)] = -\frac{1}{2}\frac{(1+\mu_2^2)(1-\rho)[(\xi+c) + (1-\xi)h_1(x_1,y_1)]}{[(\xi+c) + (1-\xi)h_1(x_1,y_1)]^2 + [(1-\xi)h_2(x_2,y_2)]^2} < 0.$$

Hence, by using Lemma 2.1. we have $L_c(f) \in K_n(\rho,\xi,\beta,\gamma)$. □

Similarly we proved the following theorem.

Theorem 3.4. *Let $c > -1$, $0 \le \xi < 1$, and $0 \le \rho < 1$. If $f(z) \in K_n^*(\rho,\xi,\beta,\gamma)$, then $L_c(f) \in K_n^*(\rho,\xi,\beta,\gamma)$.*

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