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A kind of bilevel traveling salesman problem

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Abstract. The present paper highlights a type of a bilevel optimization problem on a graph. It models a real practical problem. Let N be a finite set, $G = (N, E)$ be a weighted graph and let $I \subset N$. Let $\mathcal{C}_1$, respectively $\mathcal{C}_2$, be the set of those subgraphs $G_1 = (N_1, E_1)$, respectively $G_2 = (N_2, E_2)$, of G which fulfill some given conditions in each case. Let a and b be positive numbers and let g be a natural value function defined on the set of subgraphs of G. We study the following bilevel programming problem:

$$\begin{cases} ah(G_1) + bh(G_2) \ \rightarrow \ \min \\ \text{such that} \\ G_1 \in \mathcal{C}_1, \\ G_2 \in S^*(G_1), \end{cases}$$

where $h(G_i)$ represents the value of a Hamiltonian circuit of minimum value corresponding to the subgraph $G_i, i = 1, 2$, and

$$S^*(G_1) \ = \ \text{argmin}\{g(G_2) \,|\, G_2 \in \mathcal{C}_2 \, \text{and} \, N_1 \cap N_2 \cap I = \emptyset\}.$$

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1. Introduction

Multilevel programming and, subsequently bilevel programming, have lately become important areas in optimization. The investigations of such types of problems are strongly motivated by their actual real-life applications in areas such as economics, medicine, engineering etc. The increasing number of these applications have led mathematicians to develop new theories and mathematical models.

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In mathematical terms, the bilevel programming problem is an optimization problem where a subset of the variables is constrained to be an optimal solution of a given optimization problem parameterized by the remaining variables. References [2], [3] and [6] are useful papers for studies concerning the bilevel and multilevel programming.

Also, as it is well known, the traveling salesman problem (TSP) is one of the oldest and most studied combinatorial problem. From the mathematical point of view, researches concerning the TSP had an important role in development of the graph theory. Reference [1] presents various aspects of the TSP, especially the ones related to the methods and algorithms of solving it. References [4] and [5] provide some of the problems known under the generic name of *The Vehicle Routing Problem*, which represent a generalization of the TSP.

In the present paper, a problem of generating new types of routes is studied, using the bilevel optimization problem as a mathematical tool.

2. The practical problem

The problem studied in the present paper is based on an actual practical problem, named by us *The Milk Collection Problem*:[1] A dairy products manufacturing company collects twice a day the milk from a certain area. Collection points are located only on roads linking villages in the area. The milk is brought to the collection points by the owners. The quantity of milk delivered depends on the time when the collection is scheduled. Some providers can bring the milk to the collection points only in the morning. Others only in the evening, and some of them both in the morning and in the evening. There exists the possibility for some providers, who deliver milk in the morning, to store it (in conditions that do not impair the milk quality) and to offer it for delivery only in the evening. The others do not have this possibility. The providers impose that, either, the entire quantity of milk offered will be collected by the dairy products manufacturing company, or nothing. The milk is collected by the dairy products manufacturing company, in the morning and in the evening, using a collector tank, which has a capacity denoted by $\overline{Q}$.

The problem that arises is that of planning the providers:

- those who bring milk to the collection points in the morning, and the milk is collected by the collector tank in the morning;
- those who bring milk to the collection points in the morning, but it is necessary to store it until evening, when it will be collected by the collector tank;
- those who bring milk to the collection points in the evening, and the milk is collected by the collector tank in the evening,

such that the total cost required for milk collection in a day to be minimum and, a collection point to be visited by the collector tank at most once in the morning and at most once in the evening.

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The providers planning must satisfy the following requirements:
a) the quantity of milk collected in the morning not to exceed the capacity $\overline{Q}$ of the collector tank;
b) the quantity of milk collected in the evening (which may be from the evening milk or from the stored one) not to exceed the capacity $\overline{Q}$ of the collector tank;
c) the quantity of milk collected in the morning, and the quantity collected in the evening, must be greater than a specified quantity, denoted by $\underline{Q}$, in order to ensure the continuity in the production process;
d) the quantity of stored milk to be minimum and in the same time fulfilling the conditions a)-c).

3. The mathematical model of the milk collection problem

Let n be the number of the providers. Let us denote by $N = \{1, ..., n\}$. As well, let us denote by L_i, $i \in \{0, 1, ..., n+1\}$, the collection point where the provider i brings the milk. Let L_0 be the location where the collector tank starts and L_{n+1} be the location where the collector tank must return. We agree that they coincide, so we have $L_0 = L_{n+1}$. The collector tank transportation cost between each two locations $i, j \in \{0, 1, \dots, n+1\}$ is known, and it is denoted by c_{ij}. By q_i^1, respectively q_i^2, we denote the quantity of milk that can be delivered by the provider $i \in N$ in the morning, respectively in the evening.

Let I_1 be the set of indices corresponding to the collection points where providers can deliver milk only in the morning, but can not store it. Let I_2 be the set of indices corresponding to the collection points where providers can deliver milk both in the morning and in the evening, but in case they deliver milk in the morning, do not accept to store it. Let I_3 be the set of indices corresponding to the collection points where providers can deliver milk both in the morning and in the evening, and accept to store the morning milk in case it is required. Let I_4 be the set of indices corresponding to the collection points where providers can deliver milk only in the evening.

It is obvious that

$$I_1 \cap I_2 = \emptyset,\ I_1 \cap I_3 = \emptyset,\ I_1 \cap I_4 = \emptyset,\ I_2 \cap I_3 = \emptyset,\ I_2 \cap I_4 = \emptyset,\ I_3 \cap I_4 = \emptyset,$$

$$I_1 \cup I_2 \cup I_3 \cup I_4 = N.$$

Now, let us consider the complete undirected graph $G = (\tilde{N}; E)$, where

$$N = \{1, \dots, n\},\ \ \tilde{N} = N \cup \{0\} \cup \{n+1\} = \{0, 1, ..., n, n+1\}$$

and

$$E = \{\{ij\} \,|\, i \in \tilde{N},\, j \in \tilde{N},\, j \neq i\}.$$

The graph vertices correspond to the locations L_i, $i \in \{0, 1, \dots, n+1\}$. Let us denote by Λ the set of subgraphs $\Gamma = (N_\Gamma, E_\Gamma)$ of $G = (\tilde{N}; E)$. We weight the graph G using the cost matrix $C = [c_{ij}]_{i,j \in \tilde{N}}$, where $c_{i,j}$, for $i \neq j$, is the minimum transport cost (of the collector tank) from the location i to location j and $c_{ii} = +\infty$, for each $i \in \tilde{N}$. As well, we attach to each node $i \in \{1, \dots, n\}$ two positive weights, q_i^1 and q_i^2.

In order to elaborate the mathematical model for this problem, we consider two subgraphs $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$ of G as variables. The set of nodes of the first subgraph, N_1, corresponds to the indices of the collection points where the milk is collected in the morning. The set of nodes of the second subgraph, N_2, corresponds to the indices of the collection points where the milk is collected in the evening.

The main objective is to determine the sets N_1 and N_2, such that the total transport cost in one day to be minimum and the problem restrictions to be satisfied.

Let us note that, for a fixed set N_1, it is obtained a minimum cost of the morning collection if it is followed a Hamiltonian circuit of minimum value in G_1. Analogous, for a fixed set N_2, it is obtained a minimum cost of the evening collection (stored milk or delivered to the collection points only in the evening) if it is followed a Hamiltonian circuit of minimum value in G_2. Therefore, if for a subgraph Γ of G, we denote by $h(\Gamma)$ the value of a Hamiltonian circuit of minimum value corresponding to the subgraph Γ, then the minimum cost of milk collection in one day it is equal to $h(G_1) + h(G_2)$.

The graphs G_1 and G_2 can not be chosen randomly; they must fulfill the problem restrictions. Thus, in the morning the collector tank can collect only from the nodes in which the providers deliver milk only in the morning. Therefore, $N_1 \subseteq I_1 \cup I_2 \cup I_3$. In the evening the collector tank can collect milk only from the nodes in which the providers deliver milk only in the evening or in which the morning milk was stored until evening. Therefore, $N_2 \subseteq I_2 \cup I_3 \cup I_4$.

In each collection point, where the providers can deliver milk both in the morning and in the evening, and where there exists the possibility to store the morning milk until evening, the quantity of milk delivered in the morning is collected just once: in the morning or in the evening. Therefore, the following condition occurs:

$$N_1 \cap N_2 \cap I_3 = \emptyset.$$

The quantity of milk collected in the morning, equal to $\sum_{i\in N_1} q_i^1$, must be greater than, or equal to, $\underline{Q}$, and can not exceed the collector tank capacity $\overline{Q}$. Also, the quantity of milk collected in the evening, equal to $\sum\limits_{i\in N_2\cap I_3} q_i^1 + \sum\limits_{i\in N_2} q_i^2$, can not exceed the collector tank capacity $\overline{Q}$ and must be greater than, or equal to, $\underline{Q}$.

Let S be the set of all pairs (G_1, G_2) of subgraphs of the weighted graph G, $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$, satisfying the conditions (3.1)-(3.5):

$$N_1 \subseteq I_1 \cup I_2 \cup I_3, \tag{3.1}$$

$$N_2 \subseteq I_2 \cup I_3 \cup I_4, \tag{3.2}$$

$$N_1 \cap N_2 \cap I_3 = \emptyset, \tag{3.3}$$

$$\underline{Q} \leq \sum_{i\in N_1} q_i^1 \leq \overline{Q}, \tag{3.4}$$

$$\underline{Q} \leq \sum_{i\in N_2\cap I_3} q_i^1 + \sum_{i\in N_2} q_i^2 \leq \overline{Q}. \tag{3.5}$$

In order to plan the evening collection, it is necessary that the planning of the morning collection and of the stored milk, to be done with respect to the capacity type restrictions (3.4) and (3.5). For a given N_2, the quantity of stored milk is equal to $\sum_{i\in N_2\cap I_3} q_i^1$.

Let g be the real function defined on the set Λ of subgraphs of G, given by

$$g(\Gamma) \;=\; \sum_{i\in N_\Gamma\cap I_3} q_i^1,\; \forall\, \Gamma=(N_\Gamma,E_\Gamma)\in\Lambda. \tag{3.6}$$

For each $G_1\in\Lambda$, let us denote by

$$S(G_1)=\{G_2\in\Lambda \,|\, (G_1,G_2)\in S\}.$$

If $S(G_1)\neq\emptyset$, then the minimum quantity of stored milk (which can be determined taking into account the morning planning, i.e. knowing N_1) is obtained solving the following problem:

$$\begin{cases} g(G_2) \;\to\; \min \\ G_2\in S(G_1). \end{cases}$$

Let us denote by $S^*(G_1)$ the set of the optimal solutions of this problem; so, $S^*(G_1) \;=\; \operatorname{argmin}\{g(G_2)\,|\,G_2\in S(G_1)\}$.

Under these circumstances, the milk collection problem is reduced to solve the following bilevel programming problem:

$$(BP)\quad \begin{cases} h(G_1)\,+\,h(G_2) \;\to\; \min \\ \text{such that} \\ (G_1,G_2)\in S, \\ G_2\in S^*(G_1). \end{cases}$$

Furthermore, a method for solving the problem (BP) is given, in a little more general context.

4. Generalization of the mathematical model for the milk collection problem

Let N be a finite set, $G=(N,E)$ be a weighted graph and let $I\subset N$. Let Λ be the set of subgraphs $\Gamma=(N_\Gamma,E_\Gamma)$ of G with $N_\Gamma\neq\emptyset$ and $E_\Gamma\neq\emptyset$.

Let $\mathcal{C}_1$ be the set of those elements $G_1=(N_1,E_1)$ of Λ which fulfill some given conditions. Also, let $\mathcal{C}_2$ be the set of those elements $G_2=(N_2,E_2)$ of Λ which fulfill other given conditions. In both cases, the conditions are some restrictions imposed to be fulfilled by the set of nodes N_1, respectively N_2. It can be defined, for example, by inequalities or equalities, which are generated by some given functions, or by some inclusions. For example, regarding the milk collection problem, $\mathcal{C}_1$ it is the set of those subgraphs which verify the conditions (3.1) and (3.4), while $\mathcal{C}_2$ it is the set of those subgraphs which verify the conditions (3.2) and (3.5).

Furthermore, for a subgraph $\Gamma\in\Lambda$, we denote by $h(\Gamma)$ the value of a Hamiltonian circuit of minimum value corresponding to it.

Now, let a and b be positive numbers and let $F:\Lambda\times\Lambda\to\mathbb{R}_+$ be the function given by

$$F(G_1,G_2)\;=\;a\cdot h(G_1)+b\cdot h(G_2),\;\forall\,(G_1,G_2)\in\Lambda\times\Lambda. \tag{4.1}$$

Also, let $g : \Lambda \to \mathbb{N}$ be a given function. For the milk collection problem, the function g returns the quantity of stored morning milk which it is collected in the evening.

The bilevel problem proposed to be solved is

$$(PBG) \quad \begin{cases} F(G_1, G_2) \ \to \ \min \\ G_1 \in \mathcal{C}_1, \\ G_2 \in S^*(G_1), \end{cases}$$

where $S^*(G_1)$ it is the set of the optimal solutions of the problem

$$(P(G_1)) \quad \begin{cases} g(G_2) \ \to \ \min \\ G_2 \in \mathcal{C}_2, \\ N_1 \cap N_2 \cap I = \emptyset. \end{cases}$$

Let us denote by

$$S = \{(G_1, G_2) \in \Lambda \times \Lambda \,|\, G_1 \in \mathcal{C}_1,\ G_2 \in \mathcal{C}_2,\ N_1 \cap N_2 \cap I = \emptyset\},$$

and

$$S_1 = \{G_1 \in \Lambda \mid \exists\, G_2 \in \Lambda \text{ s. t. } (G_1, G_2) \in S\},$$

i.e.

$$S_1 = \{G_1 \in \mathcal{C}_1 \mid \exists\, G_2 \in \mathcal{C}_2 \text{ s. t. } N_1 \cap N_2 \cap I = \emptyset\}.$$

For each $G_1 \in S_1$ we consider the set

$$S(G_1) \ = \ \{G_2 \in \mathcal{C}_2 \mid (G_1, G_2) \in S\} \ = \ \{G_2 \in \mathcal{C}_2 \mid N_1 \cap N_2 \cap I = \emptyset\}.$$

It is easy to see that $S(G_1)$ it is the set of feasible solutions of the problem $(P(G_1))$.

In what follows, we will use the lexicographic ordering relation in $\mathbb{R}^2$, denoted by $<_{\text{lex}}$, namely:

If we consider the points $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$, then we have $x <_{\text{lex}} y$ if: (i) $x_1 < y_1$; or (ii) $x_1 = y_1$ and $x_2 < y_2$.

We denote $x \leq_{\text{lex}} y$ if $x <_{\text{lex}} y$ or $x = y$. The relation $\leq_{\text{lex}}$ is a total order relation on $\mathbb{R}^2$.

If A it is a subset of $\mathbb{R}^2$, $f = (f_1, f_2) : A \to \mathbb{R}^2$ it is a given function and $B \subseteq A$, then the requirement to determine the minimum point of the set $f(B)$ in relation to the lexicographic ordering relation and to determine the points of B for which this minimum is reached, is denoted by

$$(P) \quad \begin{cases} f(x) \ \to \ \text{lex} - \min \\ x \in B. \end{cases}$$

The minimum point $(f_1^*, f_2^*) \in \mathbb{R}^2$ of the set $f(B)$ is called optimum of f on B or optimal value of the problem (P); the set of points $b \in B$, such that $f(b) = (f_1^*, f_2^*)$, is called optimal solution of the problem (P).

Recalling our problem, let $H \in 2^I$. We consider the problems

$$(P_1(H)) \quad \begin{cases} h(G_1) \ \to \ \min \\ G_1 \in \mathcal{C}_1, \\ N_1 \cap I = H, \end{cases}$$

and

$$(P_2(H)) \quad \begin{cases} \begin{pmatrix} g(G_2) \\ h(G_2) \end{pmatrix} \to \text{lex} - \min \\ G_2 \in \mathcal{C}_2, \\ N_2 \cap H = \emptyset. \end{cases}$$

Let us denote by h_1^H the optimal value of the problem $(P_1(H))$ and by (g_2^H, h_2^H) the optimal value of the problem $(P_2(H))$.

Also, we define the function $\tilde{F} : 2^I \to \mathbb{R}$,

$$\tilde{F}(H) = a \cdot h_1^H + b \cdot h_2^H, \ \forall\, H \in 2^I, \tag{4.2}$$

and we consider the problem

$$(PP) \quad \begin{cases} \tilde{F}(H) \to \min \\ H \in 2^I. \end{cases}$$

Furthermore, we establish relations between the feasible solutions, and then between the optimal solutions, of the problems (PBG) and (PP).

Lemma 4.1. If $G_1^0 \in \mathcal{C}_1$, G_2^0 it is a feasible solution of the problem $(P(G_1^0))$ and $H^0 = N_1^0 \cap I$, then G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.

Proof. As $G_2^0 \in S(G_1^0)$ we deduce that $G_2^0 \in \mathcal{C}_2$ and $N_1^0 \cap N_2^0 \cap I = \emptyset$. From the last equality, considering that $H^0 = N_1^0 \cap I$, we get that

$$N_2^0 \cap H^0 = N_2^0 \cap N_1^0 \cap I = \emptyset.$$

So, G_2^0 is a feasible solution of the problem $(P_2(H^0))$.$\diamond$

Lemma 4.2. If (G_1^0, G_2^0) it is a feasible solution of the problem (PBG) and $H^0 = N_1^0 \cap I$, then

i) G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and

ii) G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.

Proof. Since (G_1^0, G_2^0) it is a feasible solution of the problem (PBG) and $H^0 = N_1^0 \cap I$, immediately results that: i) G_1^0 it is a feasible solution of the problem $(P_1(H^0))$; and ii) G_2^0 it is an optimal solution of the problem $(P(G_1^0))$. Because any optimal solution is a feasible one, applying Lemma 4.1, we deduce that G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.$\diamond$.

Lemma 4.3. If $H^0 \in 2^I$, G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and G_2^0 it is an optimal solution of the problem $(P_2(H^0))$, then (G_1^0, G_2^0) it is a feasible solution of the problem (PBG).

Proof. Since G_1^0 it is a feasible solution of the problem $(P_1(H^0))$, we have

$$G_1^0 \in \mathcal{C}_1, \tag{4.3}$$

$$N_1^0 \cap I = H^0. \tag{4.4}$$

Because any optimal solution it is a feasible one, we have

$$G_2^0 \in \mathcal{C}_2, \tag{4.5}$$

$$N_2^0 \cap H^0 = \emptyset. \tag{4.6}$$

From (4.4) and (4.6), it results that

$$N_1^0 \cap N_2^0 \cap I \ = \ N_2^0 \cap H^0 = \emptyset. \tag{4.7}$$

In view of (4.5) and (4.7), we deduce that G_2^0 it is a feasible solution of the problem $(P(G_1^0))$, i.e. $G_2^0 \in S(G_1^0)$. We must prove that G_2^0 it is an optimal solution of this problem. Let us suppose the opposite. Then, there exists $G_2 \in S(G_1^0)$ such that $g(G_2) < g(G_2^0)$, which implies

$$\begin{pmatrix} g(G_2) \\ h(G_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^0) \\ h(G_2^0) \end{pmatrix}. \tag{4.8}$$

Recalling Lemma 4.1, since $G_2 \in S(G_1^0)$, we deduce that G_2 it is a feasible solution of the problem $(P_2(H^0))$; the inequality (4.8) contradicts the fact that G_2^0 it is an optimal solution of the problem $(P_2(H^0))$. So, $G_2^0 \in S^*(G_1^0)$. Considering now the fact that $G_1^0 \in \mathcal{C}_1$, it results that (G_1^0, G_2^0) it is a feasible solution of the problem (PBG).$\diamond$

Theorem 4.4. If (G_1^0, G_2^0) it is an optimal solution of the problem (PBG), then taking $H^0 = N_1^0 \cap I$, the following sentences are true:
i) G_1^0 it is an optimal solution of the problem $(P_1(H^0))$;
ii) G_2^0 it is an optimal solution of the problem $(P_2(H^0))$;
iii) H^0 it is an optimal solution of the problem (PP).
Proof. i) Based on Lemma 4.2, we get that G_1^0 it is a feasible solution of the problem $(P_1(H^0))$. Let us suppose that G_1^0 it is not an optimal solution of the problem $(P_1(H^0))$. Then, there exists a feasible solution G_1 of the problem $(P_1(H^0))$, such that

$$h(G_1) < h(G_1^0). \tag{4.9}$$

As G_1 it is a feasible solution of the problem $(P_1(H^0))$, we have $G_1 \in \mathcal{C}_1$ and $N_1 \cap I = H^0$. Then

$$N_1 \cap N_2^0 \cap I = H^0 \cap N_2^0 = N_1^0 \cap I \cap N_2^0 = \emptyset.$$

Therefore, $G_2^0 \in S(G_1)$. Two cases can occur:
1) $G_2^0 \in S^*(G_1)$; or 2) $G_2^0 \notin S^*(G_1)$.

If $G_2^0 \in S^*(G_1)$, then (G_1, G_2^0) it is a feasible solution of the problem (PBG). Since $a > 0$, from (4.9), we deduce that

$$ah(G_1) + bh(G_2^0) < ah(G_1^0) + bh(G_2^0),$$

which contradicts the optimality of (G_1^0, G_2^0).

Now, let us suppose that $G_2^0 \notin S^*(G_1)$. Then, there exists $G_2 \in S(G_1)$ such that

$$g(G_2) < g(G_2^0). \tag{4.10}$$

Because $G_2 \in S(G_1)$, we have $G_2 \in \mathcal{C}_2$ and $N_2 \cap N_1 \cap I = \emptyset$. On the other hand, G_1 being a feasible solution of $(P_1(H^0))$, we have $N_1 \cap I = H^0$. It results that

$$N_1^0 \cap N_2 \cap I = H^0 \cap N_2 = N_1 \cap I \cap N_2 = \emptyset.$$

So, $G_2 \in S(G_1^0)$. Therefore, (4.10) contradicts the fact that $G_2^0 \in S^*(G_1^0)$.

Hence, G_1^0 it is an optimal solution of the problem $(P_1(H^0))$.

ii) From Lemma 4.2, ii), G_2^0 it is a feasible solution of $(P_2(H^0))$. Let us suppose that G_2^0 it is not an optimal solution of $(P_2(H^0))$. Since the set of solutions of the problem $(P_2(H^0))$ is a finite and nonempty set, the problem will have optimal solutions. Let $\tilde{G}_2 = (\tilde{N}_2, \tilde{E}_2)$ be an optimal solution of this problem. Based on Lemma 4.3,

$(G_1^0, \tilde{G}_2)$ it is a feasible solution of the problem (PBG). As we supposed the contrary, we have

$$\begin{pmatrix} g(\tilde{G}_2) \\ h(\tilde{G}_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^0) \\ h(G_2^0) \end{pmatrix}. \tag{4.11}$$

Two cases can occur:
1) $g(\tilde{G}_2) < g(G_2^0)$; or 2) $g(\tilde{G}_2) = g(G_2^0)$ and $h(\tilde{G}_2) < h(G_2^0)$.

In the first case, we deduce that $G_2^0 \notin S^*(G_1^0)$, which contradicts the fact that (G_1^0, G_2^0) it is a feasible solution of (PBG).

In the second case, we have

$$F(G_1^0, \tilde{G}_2) = ah(G_1^0) + bh(\tilde{G}_2) \; < \; ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0),$$

which contradicts the optimality of (G_1^0, G_2^0).

iii) Since the set 2^I is a nonempty and finite set, the problem (PP) has an optimal solution. Let us suppose that H^0 it is not an optimal solution of the problem (PP), and let H^* be the optimal solution of the problem (PP). Under these circumstances, we have

$$\tilde{F}(H^*) < \tilde{F}(H^0). \tag{4.12}$$

Let us notice that, from i) and ii), taking into account the way in which the functions F and $\tilde{F}$ are defined (see (4.1), (4.2)), we have

$$\tilde{F}(H^0) \; = \; ah_1^{H^0} + bh_2^{H^0} \; = \; ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0). \tag{4.13}$$

Now, let G_1^* be an optimal solution of the problem $(P_1(H^*))$ and G_2^* be an optimal solution of the problem $(P_2(H^*))$. Then,

$$\tilde{F}(H^*) \; = \; ah_1^{H^*} + bh_2^{H^*} \; = \; ah(G_1^*) + bh(G_2^*) \; = \; F(G_1^*, G_2^*). \tag{4.14}$$

From (4.12)-(4.14), it results that

$$F(G_1^*, G_2^*) \; < \; F(G_1^0, G_2^0). \tag{4.15}$$

On the other hand, applying Lemma 4.3, we deduce that (G_1^*, G_2^*) it is a feasible solution of the problem (PBG). Hence, (4.15) contradicts the optimality of (G_1^0, G_2^0).$\diamond$

Theorem 4.5. If H^0 it is an optimal solution of the problem (PP) and G_1^0, respectively G_2^0, it is an optimal solution of the problem $(P_1(H^0))$, respectively $(P_2(H^0))$, then (G_1^0, G_2^0) it is an optimal solution of the problem (PBG).

Proof. As G_1^0, respectively G_2^0, it is an optimal solution of $(P_1(H^0))$, respectively $(P_2(H^0))$, we get that $h_1^{H^0} = h(G_1^0)$ and $h_2^{H^0} = h(G_2^0)$. Therefore,

$$\tilde{F}(H^0) = ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0). \tag{4.16}$$

On the other hand, applying Lemma 4.3, we get that (G_1^0, G_2^0) it is a feasible solution of (PBG). If (G_1^0, G_2^0) it is not an optimal solution of the problem (PBG), then there exists $(\tilde{G}_1 = (\tilde{N}_1, \tilde{E}_1), \tilde{G}_2 = (\tilde{N}_2, \tilde{E}_2))$ feasible solution of (PBG), such that

$$ah(\tilde{G}_1) + bh(\tilde{G}_2) \; = \; F(\tilde{G}_1, \tilde{G}_2) < F(G_1^0, G_2^0) \; = \; ah(G_1^0) + bh(G_2^0). \tag{4.17}$$

From (4.17) we deduce that: $h(\tilde{G}_1) < h(G_1^0)$ or $h(\tilde{G}_2) < h(G_2^0)$.

Let $\tilde{H} = \tilde{N}_1 \cap I$. As $(\tilde{G}_1, \tilde{G}_2)$ it is a feasible solution of (PBG), we have $\tilde{G}_2 \in S^*(\tilde{G}_1)$; this implies that $h_2^{\tilde{H}} = F(\tilde{G}_1, \tilde{G}_2)$. Hence, (4.17) implies $\tilde{F}(\tilde{H}) < \tilde{F}(H^0)$. This contradicts the optimality of H^0.$\diamond$

Let

$$\lambda \geq 1 + \max\{F(G_1, G_2),\ \forall\, (G_1, G_2) \in \Lambda\}. \tag{4.18}$$

Let $G_1 \in S_1$ and $H \in 2^I$, fulfilling the following condition

$$N_1 \cap I = H. \tag{4.19}$$

Let us consider the problem

$$(PL_2(H)) \quad \begin{cases} \lambda \cdot g(G_2) + F(G_1, G_2) \ \rightarrow \ \min \\ G_2 \in \mathcal{C}_2, \\ H \cap N_2 = \emptyset. \end{cases}$$

Theorem 4.6. If $G_1 \in S_1$ and $H \in 2^I$ such that the condition (4.19) is fulfilled, then an element G_2 it is an optimal solution of the problem $(PL_2(H))$ if and only if it is an optimal solution of the problem $(P_2(H))$.

Proof. First, let us remark that both problems have the same set of feasible solutions.

Necessity. Let G_2 be an optimal solution of the problem $(PL_2(H))$. Let us suppose that G_2 it is not an optimal solution of the problem $(P_2(H))$. Two cases can occur:

1) there exists G_2^* a feasible solution of the problem $(P_2(H))$, such that $g(G_2^*) < g(G_2)$; or

2) there exists G_2^* a feasible solution of the problem $(P_2(H))$, such that $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) < F(G_1, G_2)$.

As $\lambda > 0$, in the first case we obtain that

$$\lambda \cdot g(G_2^*) \ < \ \lambda \cdot g(G_2). \tag{4.20}$$

If $F(G_1, G_2^*) \leq F(G_1, G_2)$, then $\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2)$. This contradicts the hypothesis that (G_1, G_2) it is an optimal solution of the problem $(PL_2(H))$.

Let us now suppose that $F(G_1, G_2^*) > F(G_1, G_2)$. As $g(G_2) \in \mathbb{N}$ and $g(G_2^*) \in \mathbb{N}$, based on $g(G_2^*) < g(G_2)$, we have $g(G_2) - g(G_2^*) \geq 1$. Therefore,

$$\frac{F(G_1, G_2) - F(G_1, G_2^*)}{g(G_2) - g(G_2^*)} \ \leq \ \frac{F(G_1, G_2) - F(G_1, G_2^*)}{1} \ \leq \ F(G_1, G_2) \ < \ \lambda. \tag{4.21}$$

It follows that $\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2)$, which contradicts the hypothesis that (G_1, G_2) it is an optimal solution of the problem $(PL_2(H))$.

If we consider case 2), then we immediately get that $\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2)$, which contradicts the optimality of (G_1, G_2) for the problem $(PL_2(H))$.

Sufficiency. Let G_2 be an optimal solution of the problem $(P_2(H))$ and let G_2^* be a feasible solution of the problem $(PL(H))$. Since G_2^* it is a feasible solution of the problem $(P_2(H))$, we get that

$$\begin{pmatrix} g(G_2) \\ F(G_1, G_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^*) \\ F(G_1, G_2^*) \end{pmatrix}.$$

Three cases are possible:
1) $g(G_2) < g(G_2^*)$;
2) $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) < F(G_1, G_2)$;
3) $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) = F(G_1, G_2)$.

Since $g(G_2) \in \mathbb{N}$ and $g(G_2^*) \in \mathbb{N}$, in the first case we have

$$g(G_2) - g(G_2^*) \leq -1.$$

Therefore,

$$\begin{aligned} &\lambda \cdot g(G_2) + F(G_1, G_2) - (\lambda g(G_2^*) + F(G_1, G_2^*)) \\ &= \lambda \cdot (g(G_2) - g(G_2^*)) + F(G_1, G_2) - F(G_1, G_2^*) \\ &\leq -\lambda + F(G_1, G_2) - F(G_1, G_2^*) \\ &\leq -1 - F(G_1, G_2^*) < 0. \end{aligned}$$

In the cases 2) and 3) it results that

$$\lambda \cdot g(G_2) + F(G_1, G_2) \leq \lambda g(G_2^*) + F(G_1, G_2^*).$$

Since G_2^* it is a feasible solution chosen arbitrary, it results that G_2 it is an optimal solution of the problem $(PL_2(H))$.$\diamond$

Based on Theorem 4.6, we can reduce the solving of the problem (PBG) to solving $2^{|I|}$ couples of problems $(P_1(H), P_2(H))$, where the parameter H belongs to the set 2^I. Based on Theorem 4.6, the solving of the problem $(P_2(H))$ can be replaced by solving the problem $(PL_2(H))$.

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